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On the existence of solutions of some non-linear Dirichlet problems

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Abstract. In this paper we study the existence and multiplicity of solutions of non-linear elliptic equations of the form

$$\Delta u+\lambda u-u|u|^{p-1}=g \quad \text{in } \Omega,$$
$$u=0 \quad \text{on } \partial\Omega.$$

Here, Ω is a smooth and bounded domain, $N \geqslant 2$ and $\lambda \in R$. We shall prove the existence problem in case $\lambda \in R$ and the uniqueness problem in case $\lambda \leqslant \lambda_1$. Assuming that $p>1$ for $N \leqslant 4$ or $1<p<N/(N-4)$ in case $N>4$.

Introduction. A non-linear elliptic problem of the form

$$\begin{aligned} \Delta u+f(u) &= g \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a smooth and bounded domain, $f\colon R \to R$, has been studied by many authors. For example, this problem for asymptotically linear f has been studied by Ambrosetti and Prodi [1], Dancer [5], Berger and Podolak [4], Thews [12], Lazer and McKenna [8] and others.

Plastock [11] has proved existence and uniqueness of solutions of problem (1) for $N=3$ and $f(t)=t-t^3$.

Berestycki and Bahri [2] have shown that problem (1) has infinitely many distinct solutions in case $N \geqslant 2$,

$$f(t)=t|t|^{p-1}, \quad 1<p<\frac{N+2+\sqrt{9N^2-4N+4}}{4N-4} < (N+1)/(N-1) \quad \text{and} \quad g \in L_2(\Omega).$$

We shall consider the following non-linear Dirichlet problem

$$\begin{aligned} \Delta u+\lambda u-|u|^{p-1}u &= g \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where Ω denotes a bounded domain in R^N with smooth boundary $\partial\Omega$, Δ denotes the Laplace operator, $g \in L_2(\Omega)$ and $\lambda \in R$.

Additionally we assume that $p > 1$ for $N \leqslant 4$ or $1 < p < N/(N-4)$ for $N > 4$.

The following spaces will be used:

$L_p(\Omega)$, $p \geqslant 1$ – Lebesgue spaces with norms $\|u\|_{0p} = (\int_\Omega |u|^p dx)^{1/p}$,

$W_{mp}(\Omega)$, $\mathring{W}_{12}(\Omega)$ – Sobolev spaces with norms,

$\|u\|_{mp} = (\sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{0p}^p)^{1/p}$, $\|u\|_{12} = (\sum_{|\alpha| = 1} \|D^\alpha u\|_{12}^2)^{1/2}$, respectively.

Let $E = \{u \in W_{22}(\Omega);\ u = 0 \text{ on } \partial\Omega\}$. It is known that if $\partial\Omega$ is a smooth manifold, then $E \subset \mathring{W}_{12}(\Omega)$, $E = \overline{C_0^2(\Omega)}$ in $W_{22}(\Omega)$, E is the Hilbert space with the scalar product $(u, v)_E = \int_\Omega \Delta u \Delta v dx$ and the problem

$$\Delta u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega$$

has exactly one solution in E for each $f \in L_2(\Omega)$.

I. For each $u \in E$, let us define the elements Lu, Tu as follows:

$$(1.1) \qquad (Lu, w)_{12} = \int_\Omega uwdx, \quad (Tu, w)_{12} = \int_\Omega |u|^{p-1} uwdx$$

for every $w \in \mathring{W}_{12}(\Omega)$.

LEMMA 1.1. *If $u \in E$, then Lu, $Tu \in E$.*

Proof. Let $u \in E$. Since the imbeddings $\mathring{W}_{12}(\Omega) \subset L_2(\Omega)$ and $W_{22}(\Omega) \subset L_{2N/(N-4)}(\Omega)$ are continuous, we have

$$\left|\int_\Omega uwdx\right| \leqslant \|u\|_{02}\|w\|_{02} \leqslant c\|u\|_{02}\|w\|_{12} \leqslant c_1\|w\|_{12}$$

and

$$\left|\int_\Omega |u|^{p-1} uwdx\right| \leqslant \left\||u|^{p-1} u\right\|_{02}\|w\|_{02} \leqslant c_2\|w\|_{12}$$

for every $w \in \mathring{W}_{12}(\Omega)$. This means that these functions are linear continuous functionals defined on the Hilbert space $\mathring{W}_{12}(\Omega)$. The Riesz theorem implies that Lu, $Tu \in \mathring{W}_{12}(\Omega)$. It follows from the definition of the generalized derivative that $-\Delta Lu = u$ and $-\Delta Tu = |u|^{p-1} u$, where u, $|u|^{p-1} u \in L_2(\Omega)$. Therefore, Lu, $Tu \in E$. ■

Thus we can define the operators L, T: $E \to E$. Now we shall prove some properties of L and T.

LEMMA 1.2. *The operator L is linear and compact.*

Proof. It is obvious that L is linear. Let us consider a bounded sequence $\{u_n\}$ in E. Since the imbedding $E \subset L_2(\Omega)$ is compact, there exists a subsequence $\{u_{nk}\}$ such that $u_{nk} \xrightarrow[k\to+\infty]{} u$ in $L_2(\Omega)$. Hence $\|Lu_{nk}-Lu\|_E \leqslant c\|u_{nk}-u\|_{02} \xrightarrow[k\to+\infty]{} 0$. This means that $Lu_{nk} \xrightarrow[k\to\infty]{} Lu$ in E and that the operator L is compact. ■

LEMMA 1.3. *The map T is continuous and compact.*

Proof. Let $u_n \xrightarrow[n\to\infty]{} u$ in E. The continuity of the imbedding $W_{22}(\Omega) \subset L_{2N/(N-4)}(\Omega)$ implies that $u_n \xrightarrow[n\to\infty]{} u$ in $L_{2N/(N-4)}(\Omega)$. The map $v \mapsto |v|^{p-1}v$ from $L_{2N/(N-4)}(\Omega)$ to $L_2(\Omega)$ is continuous, so that $|u_n|^{p-1}u_n \xrightarrow[n\to\infty]{} |u|^{p-1}u$ in $L_2(\Omega)$. Hence $\|Tu_n-Tu\|_E \leqslant c\big\||u_n|^{p-1}u_n-|u|^{p-1}u\big\|_{02} \xrightarrow[n\to\infty]{} 0$. This proves the continuity of T.

Now let $\{u_n\}$ be a bounded sequence in E. From the compactness of the imbedding $W_{22}(\Omega) \subset L_{2p}(\Omega)$ we conclude that there exists a subsequence $\{u_{nk}\}$ such that $u_{nk} \xrightarrow[k\to\infty]{} v$ in $L_{2p}(\Omega)$. The continuity of the map $w \to |w|^{p-1}w$ implies that $|u_{nk}|^{p-1}u_{nk} \xrightarrow[k\to\infty]{} |v|^{p-1}v$ in $L_2(\Omega)$. Hence

$$\|Tu_{nk}-Tv\|_E \leqslant c\big\||u_{nk}|^{p-1}u_{nk}-|v|^{p-1}v\big\|_{02} \xrightarrow[k\to\infty]{} 0$$

Consequently, T is a compact. ■

Let $A\colon E \to E$ be a map defined by

$$A = I - \lambda L + T.$$

We shall prove some properties of the map A.

LEMMA 1.4. *The map A is odd.*

Proof. $A(-u) = (-u)-\lambda L(-u)+T(-u) = -u+\lambda Lu-Tu = -(u-\lambda Lu + Tu) = Au$. ■

THEOREM 1.5. *The map A is proper.*

Proof. $A(-u) = (-u)-\lambda L(-u)+T(-u) = -u+\lambda Lu-Tu = -(u-\lambda Lu + Tu) = Au$.

Let $\{u_n\}$ be any sequence of elements of X, so that $\{Au_n\} \subset Y$. Since Y is compact, there exists a subsequence $\{Au_{nk}\}$, denoted by $\{Au_k\}$ for brevity, such that $Au_k \xrightarrow[k\to\infty]{} v$ in Y. Let us assume that there exists a subsequence $\{u_{km}\}$ of the sequence $\{u_k\}$ such that $\|u_{km}\|_E \leqslant c$ for each $m \in N$. Then the compactness of the operators L and T implies that there exists a subsequence $\{u_{kml}\}$ such that $Lu_{kml} \xrightarrow[l\to\infty]{} w$, $Tu_{kml} \xrightarrow[l\to\infty]{} h$ and $Au_{kml} \xrightarrow[l\to\infty]{} v$. Hence

$$\begin{aligned}\|u_{kml}-(v+\lambda w-h)\|_E &= \|Au_{kml}+\lambda Lu_{kml}-Tu_{kml}-v-\lambda w+h\|_E \\ &\leqslant \|Au_{kml}-v\|_E+|\lambda|\,\|Lu_{kml}-w\|_E+\|Tu_{kml}-h\|_E \xrightarrow[l\to\infty]{} 0.\end{aligned}$$

Thus the sequence $\{u_n\}$ has a convergent subsequence.

Let us suppose that the sequence $\{u_n\}$ does not have a bounded subsequence.

We shall now prove some properties of the function $f\colon E \to R$ defined by

$$f(v) = \int_\Omega \Delta v \Delta Tv dx.$$

PROPERTY 1. *f is continuous.*

Proof. Let $v_n \xrightarrow[n\to\infty]{} v$ in E. Then

$$\begin{aligned} |f(v_n)-f(v)| &= \Big|\int_\Omega \Delta v_n \Delta Tv_n dx - \int_\Omega \Delta v \Delta Tv dx\Big| \\ &\leqslant \Big|\int_\Omega \Delta v_n \Delta Tv_n dx - \int_\Omega \Delta v \Delta Tv_n dx\Big| + \Big|\int_\Omega \Delta v \Delta Tv_n dx - \int_\Omega \Delta v \Delta Tv dx\Big| \\ &\leqslant \|\Delta v\|_{02} \|\Delta Tv_n - \Delta Tv\|_{02} + \|\Delta Tv_n\|_{02} \|\Delta v_n - \Delta v\|_{02} \\ &\leqslant \|v\|_E \|Tv_n - Tv\|_E + \|Tv_n\|_E \|v_n - v\|_E \\ &\leqslant c\|Tv_n - Tv\|_E + c_1 \|v_n - v\|_E \xrightarrow[n\to\infty]{} 0. \ \blacksquare \end{aligned}$$

PROPERTY 2. *$f(v) \geqslant 0$ for every $v \in E$.*

Proof. For $v \in C_0^2(\Omega)$ we obtain

$$f(v) = \int_\Omega \Delta v \Delta Tv dx = \int_\Omega (-\Delta v)|v|^{p-1} v dx = \int_\Omega \nabla v \nabla(|v|^{p-1} v) dx$$

$$= \int_\Omega \nabla v (|v|^{p-1} \nabla v + (p-1)|v|^{p-1} \nabla v) dx = p\int_\Omega |\nabla v|^2 |v|^{p-1} dx \geqslant 0.$$

The density of the set $C_0^2(\Omega)$ in E implies that $f(v) \geqslant 0$ for every $v \in E$. $\blacksquare$

To complete our main proof, we assume that $\|u_k\|_{12} \xrightarrow[k\to\infty]{} +\infty$. Then

$$\begin{aligned} (Au_k, u_k)_{12} &= (u_k, u_k)_{12} - \lambda(Lu_k, u_k)_{12} + (Tu_k, u_k)_{12} \\ &= \|u_k\|_{12}^2 - \lambda\|u_k\|_{02}^2 + \|u_k\|_{0p+1}^{p+1} \geqslant \|u_k\|_{12}^2 - \lambda\|u_k\|_{02}^2 + c\|u_k\|_{02}^{p+1} \\ &= \|u_k\|_{12}^2 + \|u_k\|_{02}^2(c\|u_k\|_{02}^{p-1} - \lambda) \end{aligned}$$

and

$$(Au_k, u_k)_{12} \leqslant \|Au_k\|_{12} \|u_k\|_{12}.$$

Consequently,

$$\|Au_k\|_{12} \geqslant \|u_k\|_{12} + \|u_k\|_{02}^2/\|u_k\|_{12}(c\|u_k\|_{02}^{p-1} - \lambda) \xrightarrow[k\to\infty]{} +\infty$$

and, since $\|Au_k\|_E \geqslant c\|Au_k\|_{12}$, we get $\|Au_k\|_E \xrightarrow[k\to\infty]{} +\infty$. This is contrary to the assumption that $\lim_{k\to\infty} Au_k = v$ and therefore $\|u_k\|_{12} \leqslant C$ for all $k \in N$. Then

$$(Au_k, u_k)_E = (u_k, u_k)_E - \lambda(Lu_k, u_k)_E + (Tu_k, u_k)_E$$
$$= \|u_k\|_E^2 - \lambda\|u_k\|_{12}^2 + f(u_k).$$

But $\|u_k\|_{12} \leqslant C$, $f(u_k) \geqslant 0$ and so $(Au_k, u_k)_E \xrightarrow[k\to\infty]{} +\infty$. Further, $(Au_k, u_k)_E \leqslant \|Au_k\|_E \|u_k\|_E$. This gives

$$\|Au_k\|_E \geqslant \|u_k\|_E - \lambda\|u_k\|_{12}^2/\|u_k\|_E + f(u_k)/\|u_k\|_E \xrightarrow[k\to\infty]{} +\infty.$$

This is contrary to the assumption that $\lim_{k\to\infty} Au_k = v$. Thus X is compact and A is proper. ∎

THEOREM 1.6. *The map A is surjective.*

Proof. It follows from the fact that a continuous proper and odd map of the form $I+C$ (where C is compact) is surjective.

THEOREM 1.7. *For $\lambda \leqslant \lambda_1$, A is a homeomorphism, where λ_1 is the first eigenvalue of the operator $-\Delta$.*

Proof. We shall prove that A is injective. Let us assume that there exist $u, v \in E$ such that $Au = Av$. Then

$$(Au, u-v)_{12} = (Av, u-v)_{12},$$
$$(u-v, u)_{12} - \lambda(Lu, u-v)_{12} + (Tu, u-v)_{12} - (v, u-v)_{12} +$$
$$+\lambda(Lv, u-v)_{12} - (Tv, u-v)_{12}$$
$$= (u-v, u-v)_{12} - \lambda(Lu-Lv, u-v)_{12} + (Tu-Tv, u-v)_{12}$$
$$= \|u-v\|_{12}^2 - \lambda\|u-v\|_{02}^2 + \int_\Omega (|u|^{p-1}u - |v|^{p-1}v)(u-v)\,dx = 0.$$

Hence

$$\|u-v\|_{12}^2 - \lambda\|u-v\|_{02}^2 = -\int_\Omega (|u|^{p-1}u - |v|^{p-1}v)(u-v)\,dx.$$

Since $\lambda \leqslant \lambda_1$, the left-hand side of this equation is non-negative, the right-hand side is non-positive, so $u = v$ and therefore A is injective. From Schauder theorem and Theorem 1.5 we conclude that A is a homeomorphism. ∎

II. An element $u \in \mathring{W}_{12}(\Omega)$ is called a *weak solution of problem* (2) if the following conditions are satisfied:

(2.1) $$|u|^{p-1}uw \in L_1(\Omega),$$

(2.2) $$\int_{\Omega} \nabla u \nabla w dx - \lambda \int_{\Omega} uw dx + \int_{\Omega} |u|^{p-1} uw dx = -\int_{\Omega} gw dx$$

for every $w \in \mathring{W}_{12}(\Omega)$.

We shall now consider the weak solutions $u \in E$. From (1.1), the left-hand side of (2.2) can be written in the form

$$(u, w)_{12} - \lambda(Lu, w)_{12} + (Tu, w)_{12} = (u - \lambda Lu + Tu, w)_{12}.$$

Let $V = \{v \in W_{-12}(\Omega);$ there exists $h \in E$ such that $\int_{\Omega} vw dx = (h, w)_{12}$ for every $w \in \mathring{W}_{12}(\Omega)\}$. Let $-g \in V$. Then (2.2) can be written in the form

(2.3) $$(u - \lambda Lu + Tu, w)_{12} = (h, w)_{12} \quad \text{for every } w \in \mathring{W}_{12}(\Omega).$$

The equation is equivalent to

(2.4) $$u - \lambda Lu + Tu = h$$

or

(2.5) $$Au = h.$$

THEOREM 2.1. $L_2(\Omega) \subset V$.

Proof. Let $g \in L_2(\Omega)$. Then $f_1(w) = \int_{\Omega} gw dx$ is a linear continuous functional on $\mathring{W}_{12}(\Omega)$, and so $f_1(w) = \int_{\Omega} gw dx = (h, w)_{12}$ where $h \in \mathring{W}_{12}(\Omega)$ for every $w \in \mathring{W}_{12}(\Omega)$ and $-\Delta h = g$. The above problem has exactly one solution in E. It means that $h \in E$ and $g \in V$. ■

We may state the main result of this paper.

THEOREM 2.2. *For each $g \in L_2(\Omega)$ problem* (2) *has a weak solution in E. If $\lambda \leqslant \lambda_1$, then* (2) *has exactly one weak solution for each $g \in L_2(\Omega)$.*

Proof follows from Theorems 1.6, 1.7 and 2.1.

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