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Estimations of the coefficients of quasi-starlike functions

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Abstract. In the earlier paper (see [1], p. 175) it was introduced the class $\mathfrak{G}^M$ of quasi-starlike functions g determined by the equation

$$G(g) = \frac{1}{M} G(z), \quad |z| < 1,$$

where G is a starlike function and M is an arbitrary real number in the interval $(1, \infty)$.

In this paper the sharp estimations of the form:

$$|a_2| \leqslant 2a(1-a), \qquad 0 < a < 1,$$

$$|a_3| \leqslant \begin{cases} a(1-a)(3-5a), & 0 < a \leqslant \frac{1}{3}, \\ a(1-a^2), & \frac{1}{3} \leqslant a < 1, \end{cases}$$

$$|a_4| \leqslant \begin{cases} 2a(1-a)(7a^2-8a+2), & 0 < a \leqslant x_3, \\ \dfrac{2a(1-a-4a^2)^{3/2}(8a+7)^{3/2}(7a^2-8a+2)}{\sqrt{5}(16a^2+64a+1)(1-4a)^{1/2}(a^2+a+1)^{1/2}}, & x_3 \leqslant a \leqslant x_4, \\ a(2-3a^2)^{3/2}, & x_4 \leqslant a \leqslant x_6, \\ a(1-a^2), & x_6 \leqslant a < 1, \end{cases}$$

where $a = 1/M$ and x_3, x_4, x_6 are roots of equations

$$48x^3+24x^2+61x-13 = 0,$$
$$528x^4+360x^3-377x^2-200x+58 = 0,$$
$$30x^6-54x^4-6x^3+36x^2-5 = 0,$$

was obtained in the class $\mathfrak{G}^M$.

1. Introduction. In [1] a class $\mathfrak{G}^M$ of quasi-starlike functions was introduced

(1) $$g(z) = az+a_2z^2+\ldots, \quad a = 1/M, \quad |z|<1,$$

determined by the equation

(2) $$G(g) = \frac{1}{M} G(z), \quad |z|<1,$$

where G is a starlike function belonging to class S, and M is an arbitrary real number in the interval $(1, \infty)$. In the class $\mathfrak{G}^M$ subclasses $\mathfrak{G}^M_m$ of quasi-starlike functions were singled out for which function G from (2) is given by the formula

$$G(z) = \frac{z}{\prod_{k=1}^{m}(1-\sigma_k)^{\beta_k}}, \quad |z|<1, \tag{3}$$

where

$$\sum_{k=1}^{m}\beta_k = 2, \quad \beta_k > 0, \tag{4}$$

$$\sigma_k = e^{i\varphi_k}, \quad \varphi_k - \text{real}, \quad k = 1, \ldots, m, \quad \sigma_l \neq \sigma_j \quad \text{for} \quad l \neq j, \quad l, j = 1, \ldots, m. \tag{5}$$

In the above-mentioned paper ([1]) it was proved in Theorem 2 that the extremum of the functional $H_g = H(a_2, \ldots, a_N)$ in the class $\mathfrak{S}^M$ is attained by a certain function of the class $\mathfrak{S}^M_m$, m being smaller than or equal to $m \leqslant N-1$. In Theorem 1, at same time, it was proved that numbers $\bar{\sigma}_k$, $k = 1, \ldots, m$, in (3) and (5) are double roots of the function

$$\tilde{\mathscr{R}}(z) = \sum_{p=1}^{N-1}\frac{1}{p}\left(\frac{\mathscr{D}_p}{z^p}+\overline{\mathscr{D}}_p z^p\right)+\lambda, \tag{6}$$

where

$$\mathscr{D}_p = \sum_{l=1}^{N-p} c_l \sum_{k=l}^{N-p}(a_{k+p}^{(l+p)}-a_k^{(l)})H_{k+p} \quad (p = 0, 1, \ldots, N-1), \tag{7}$$

$$H_p = \frac{\partial H}{\partial a_p}+\overline{\left(\frac{\partial H}{\partial \bar{a}_p}\right)} \quad (p = 2, \ldots, N), \tag{8}$$

$$c_1 = 1, \quad c_p = -\sum_{k=1}^{p-1} c_k d_{p-k} \quad (p = 2, \ldots), \tag{9}$$

$$d_0 = 1, \quad d_p = \sum_{k=1}^{m}\beta_k\sigma_k^p \quad (p = 2, \ldots), \tag{10}$$

$$\big(g(z)\big)^p = a_p^{(p)}z^p + a_{p+1}^{(p)}z^{p+1}+\ldots \quad (p = 1, 2, \ldots). \tag{11}$$

The main purpose of the paper is to obtain on the basis of the general results in [1] the estimations of the coefficients a_2, a_3, a_4, of quasi-starlike functions.

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Since the function $g(z)$ of (1) belongs to $\mathfrak{G}_m^M$ and so does $e^{i\theta}g(ze^{-i\theta})$ (where θ is an arbitrary real parameter), the maximum of the functional $|a_n|$ under consideration equals the maximum of $\mathrm{re}\, a_n$.

So let us consider the functional $H_g = \mathrm{re}\, a_n = 1/2\,(a_n + \bar{a}_n)$, $n = 2, 3, 4$. We may, without loss of generality, assume that for the extremal function with respect to H_g we have

$$\mathrm{re}\, a_n = a_n \geqslant 0, \quad n = 2, 3, 4. \tag{12}$$

2. Estimating a_2. Consider the functional

$$H_g = \mathrm{re}\, a_2. \tag{13}$$

From Theorem 2 [1] we infer that functional (13) attains its maximum for some function $g \in \mathfrak{G}_1^M$. Then function g in view of (2) and (3) is determined by the equation

$$\frac{g}{(1-\sigma g)^2} = \frac{1}{M}\frac{z}{(1-\sigma z)^2}. \tag{14}$$

From (14) we immediately have

$$a_2 = \frac{2}{M}\left(1 - \frac{1}{M}\right)\sigma. \tag{15}$$

Hence, by (12), for the extremal function we deduce the following result:

THEOREM 1. *For any quasi-starlike function g of the form* (1) *the following inequality is true*:

$$|a_2| \leqslant 2a(1-a), \quad a = 1/M, \tag{16}$$

where for the function g determined by the equation

$$\frac{g}{(1-\sigma g)^2} = \frac{1}{M}\frac{z}{(1-\sigma z)^2}$$

we have

$$|a_2| = 2a(1-a), \quad a = 1/M.$$

3. Estimating a_3. Consider the functional

$$H_g = \mathrm{re}\, a_3. \tag{17}$$

From Theorem 2 [1] we infer that functional (17) attains its maximum for some function $g \in \bigcup_{k=1}^{2} \mathfrak{G}_k^M$. We consider separately the cases $g \in \mathfrak{G}_1^M$ and $g \in \mathfrak{G}_2^M$.

3.1. Suppose first that $g \in \mathfrak{G}_1^M$. Then the function g is determined by equation (14) and its coefficient a_3 is of the form

$$a_3 = a(1-a)(3-5a)\sigma^2, \quad a = 1/M.$$

Hence, by (12), we get

$$a_3 = -a(1-a)(3-5a)\sigma^2 \quad \text{for} \quad 3/5 \leqslant a < 1, \tag{18}$$

$$a_3 = a(1-a)(3-5a)\sigma^2 \quad \text{for} \quad 0 < a \leqslant 3/5. \tag{18'}$$

3.2. Suppose next that $g \in \mathfrak{G}_2^M$. Then the function g is determined by the equation

$$\frac{g}{(1-\sigma_1 g)^{\beta_1}(1-\sigma_2 g)^{\beta_2}} = \frac{1}{M}\,\frac{z}{(1-\sigma_1 z)^{\beta_1}(1-\sigma_2 z)^{\beta_2}} \tag{19}$$

and its coefficient a_3 is of the form

$$a_3 = \tfrac{1}{2}a(1-a)\big((1-3a)(\beta_1\sigma_1+\beta_2\sigma_2)^2+(1+a)(\beta_1\sigma_1^2+\beta_2\sigma_2^2)\big). \tag{20}$$

From Theorem 1 [1] we infer in our case that

$$\tilde{\mathscr{R}}(z) = \sum_{p=1}^{2}\frac{1}{p}\left(\frac{\mathscr{D}_p}{z_p}+\overline{\mathscr{D}}_p z^p\right),$$

where, by (7)–(11), we have

$$\mathscr{D}_1 = a(1-a)(3a-1)(\beta_1\sigma_1+\beta_2\sigma_2),$$
$$\mathscr{D}_2 = -a(1-a^2).$$

Moreover, $\tilde{\mathscr{R}}(z)$ has double roots at the points σ_k $(k=1,2)$. Hence and from (4) we obtain the set of equations

$$\begin{aligned} \sigma_1^2\sigma_2^2 &= 1, \\ \frac{3a-1}{a+1}(\beta_1\sigma_1+\beta_2\sigma_2) &= \sigma_1+\sigma_2, \\ \beta_1+\beta_2 &= 2. \end{aligned} \tag{21}$$

From the second and third equation of the set (21) we have

$$\begin{vmatrix} \frac{3a-1}{a+1}\sigma_1 & \frac{3a-1}{a+1}\sigma_2 & \sigma_1+\sigma_2 \\ \frac{3a-1}{a+1}\bar{\sigma}_1 & \frac{3a-1}{a+1}\bar{\sigma}_2 & \bar{\sigma}_1+\bar{\sigma}_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Hence, after a direct calculation,

$$\frac{3a-1}{a+1}(\sigma_1\bar{\sigma}_2-\bar{\sigma}_1\sigma_2) = 0. \tag{22}$$

Suppose first that $a \neq 1/3$. Then from (22) and the first equation of (21) we get

$$\sigma_1^2 - \sigma_2^2 = 0, \quad \sigma_1^2\sigma_2^2 = 1 \tag{23}$$

and, in view of $\sigma_1 \neq \sigma_2$, we get further

$$\sigma_1 = i, \quad \sigma_2 = -i, \tag{24}$$

$$\sigma_1 = 1, \quad \sigma_2 = -1. \tag{24'}$$

Substituting the numbers obtained for $\sigma_1\sigma_2$ in (24) and (24′) into (21) we get at all times $\beta_1 = \beta_2 = 1$. Then we have two solutions

$$\sigma_1 = i, \quad \sigma_2 = -i, \quad \beta_1 = \beta_2 = 1, \tag{25}$$

$$\sigma_1 = 1, \quad \sigma_2 = -1, \quad \beta_1 = \beta_2 = 1. \tag{25'}$$

Substituting in succession (25) and (25′) into (20) we have

$$a_3 = -a(1-a^2) < 0, \tag{26}$$

$$a_3 = a(1-a^2). \tag{26'}$$

By (12) it follows from (26) that the function g determined by the system of numbers (25) is not an extremal function. Then we omit the solution (25).

Suppose next that $a = 1/3$. From (21) we have

$$\sigma_1 = i, \quad \sigma_2 = -i, \quad \beta_1+\beta_2 = 2, \quad \beta_1 > 0, \quad \beta_2 > 0, \tag{27}$$

$$\sigma_1 = 1, \quad \sigma_2 = -1, \quad \beta_1+\beta_2 = 2, \quad \beta_1 > 0, \quad \beta_2 > 0. \tag{27'}$$

The solution (27) is omitted, as before. Substituting (27′) into formulas (20), we get

$$a_3 = 8/27 \quad \text{for } M = 3. \tag{28}$$

It is worth noticing that for any function g determined by the equation

$$\frac{g}{(1-\sigma g)^{\beta_1}(1-\sigma g)^{\beta_2}} = \frac{1}{M}\frac{z}{(1-\sigma z)^{\beta_1}(1-\sigma z)^{\beta_2}}, \tag{29}$$

where $\beta_1+\beta_2 = 2$, $\beta_1 > 0$, $\beta_2 > 0$, $|\sigma| = 1$, we have (28).

Comparing (18), (18′), (20) and (28), we obtain

THEOREM 2. *For any quasi-starlike function g of the form* (1) *the following inequalities are true:*

$$|a_3| \leqslant \begin{cases} a(1-a)(3-5a) & \text{for } 0 < a \leqslant \frac{1}{3}, \\ a(1-a^2) & \text{for } \frac{1}{3} \leqslant a < 1 \ (a = 1/M), \end{cases}$$

where for the function g determined by the equation

$$\frac{g}{1-\sigma^2 g} = \frac{1}{M}\frac{z}{1-\sigma^2 z}$$

we have

$$|a_3| = a(1-a^2), \quad a = 1/M,$$

for g determined by (29) *we have* $a_3 = 8/27$, *and for g determined by* (14) *we have*

$$|a_3| = a(1-a)(3-5a), \quad a = 1/M.$$

4. Estimating a_4. Consider the functional

$$H_g = \operatorname{re} a_4. \tag{30}$$

From Theorem 1 [1] we infer that functional (30) attains its maximum for some function $g \in \bigcup_{k=1}^{3} \mathfrak{G}_k^M$. We consider, separately, the cases $g \in \mathfrak{G}_k^M$ $(k = 1, 2, 3)$ (¹).

4.1. Suppose first that $g \in \mathfrak{G}_1^M$. Then the function g is determined by equation (14) and its coefficient $a_{4(1)}$ is of the form

$$a_{4(1)} = 2a(1-a)(7a^2-8a+2)\sigma^3, \quad a = 1/M.$$

Hence, by (12), we get

$$a_{4(1)}^{(1)} = 2a(1-a)(7a^2-8a+2) \quad \text{for } 0 < a \leqslant \frac{4-\sqrt{2}}{7},\ \frac{4+\sqrt{2}}{7} \leqslant a < 1, \tag{31}$$

$$a_{4(1)}^{(2)} = -2a(1-a)(7a^2-8a+2) \quad \text{for } \frac{4-\sqrt{2}}{7} \leqslant a \leqslant \frac{4+\sqrt{2}}{7}. \tag{31'}$$

4.2. Suppose next that $g \in \mathfrak{G}_2^M$. Then the function g is determined by equation (19) and its coefficient $a_{4(2)}$ is of the form

$$a_{4(2)} = p\big(\tfrac{1}{3}q(\beta_1\sigma_1+\beta_2\sigma_2)^3 - 2\varkappa(\beta_1\sigma_1+\beta_2\sigma_2)(\beta_1\sigma_1^2+\beta_2\sigma_2^2) + (\beta_1\sigma_1^3+\beta_2\sigma_2^3)\big), \tag{32}$$

where

$$p = \tfrac{1}{3}a(1-a^3), \quad q = \frac{3}{2}\cdot\frac{16a^2-11a+1}{a^2+a+1}, \quad \varkappa = \frac{3}{4}\cdot\frac{4a^2+a-1}{a^2+a+1}. \tag{33}$$

(¹) This maximum will be denoted by $a_{4(k)}$.

From Theorem 1 [1] we have

$$\tilde{\mathscr{R}}(z) = \sum_{p=1}^{3} \frac{1}{p}\left(\frac{\mathscr{D}_p}{z^p} + \bar{\mathscr{D}}_p z^p\right) + \lambda, \tag{34}$$

where

$$\mathscr{D}_1 = p\big(2\varkappa(\beta_1\sigma_1^2+\beta_2\sigma_2^2) - q(\beta_1\sigma_1+\beta_2\sigma_2)^2\big),$$
$$\mathscr{D}_2 = 4p\varkappa(\beta_1\sigma_1+\beta_2\sigma_2),$$
$$\mathscr{D}_3 = -3p,$$

has double roots at the points $\bar{\sigma}_k$ $(k = 1, 2)$. The remaining roots of $\tilde{\mathscr{R}}(z)$ may, in view of (34), be denoted by $\varrho_1\bar{\sigma}$ and $1/\varrho_1\bar{\sigma}$ $(|\sigma| = 1$ and $0 < \varrho_1 \leqslant 1)$. Hence and from (4) follows the equation set:

$$\begin{aligned} \sigma_1^2\sigma_2^2\sigma^2 &= 1 \\ \varkappa(\beta_1\sigma_1+\beta_2\sigma_2) &= s+\varrho\sigma, \\ 2\varkappa(\beta_1\sigma_1^2+\beta_2\sigma_2^2) - q(\beta_1\sigma_1+\beta_2\sigma_2)^2 &= -s^2-\sigma^2-2\sigma_1\sigma_2-4\varrho\sigma s, \\ \beta_1+\beta_2 &= 2, \end{aligned} \tag{35}$$

where

$$\varrho = \tfrac{1}{2}\left(\varrho_1 + \frac{1}{\varrho_1}\right) \geqslant 1, \tag{36}$$

$$s = \sigma_1+\sigma_2. \tag{37}$$

Let us therefore consider two systems of equations:

$$\begin{aligned} \sigma_1\sigma_2\sigma &= 1, \\ \varkappa(\beta_1\sigma_1+\beta_2\sigma_2) &= s+\varrho\sigma, \\ 2\varkappa(\beta_1\sigma_1^2+\beta_2\sigma_2^2) - q(\beta_1\sigma_1+\beta_2\sigma_2)^2 &= -s^2-\sigma^2-2\sigma_1\sigma_2-4\varrho\sigma s, \\ \beta_1+\beta_2 &= 2 \end{aligned} \tag{38}$$

and

$$\begin{aligned} \sigma_1\sigma_2\sigma &= -1, \\ \varkappa(\beta_1\sigma_1+\beta_2\sigma_2) &= s+\varrho\sigma, \\ 2\varkappa(\beta_1\sigma_1^2+\beta_2\sigma_2^2) - q(\beta_1\sigma_1+\beta_2\sigma_2)^2 &= -s^2-\sigma^2-2\sigma_1\sigma_2-4\varrho\sigma s, \\ \beta_1+\beta_2 &= 2. \end{aligned} \tag{38'}$$

Let us point out that adopting the notation

$$\hat{\sigma}_1 = \sigma_1 e^{\frac{\pi}{3}i}, \quad \hat{\sigma}_2 = \sigma_2 e^{\frac{\pi}{3}i}, \quad \hat{\sigma} = \sigma e^{\frac{\pi}{3}i}, \quad \hat{s} = \hat{\sigma}_1+\hat{\sigma}_2 \tag{39}$$

we obtain from (38′) system (38) with unknowns $\hat{\sigma}_k$ $(k = 1, 2, 3)$, β_1, β_2, ϱ. Because of (39) and (32) the solutions of (38′) yield in every case a reverse value of the coefficient $a_{4(2)}$ in comparison with the solution of (38).

Let us therefore consider system (38).

Let us at first assume that $\varkappa \neq 0$, i.e. $a = (\sqrt{17}-1)/8$. In this case system (38) will assume the following form:

$$\begin{aligned} \sigma_1\sigma_2\sigma &= 1, \\ s &= -\varrho\sigma, \\ q(\beta_1\sigma_1+\beta_2\sigma_2)^2 &= \sigma^2+2\bar{\sigma}-3\varrho^2\sigma^2, \\ \beta_1+\beta_2 &= 2. \end{aligned} \tag{40}$$

From the first two equations we have $s = -\varrho\sigma$ and $s = -\varrho\bar{\sigma}^2$ and hence

$$\sigma^3 = 1, \quad \text{that is} \quad \sigma = 1, \quad \sigma = e^{\frac{2\pi}{3}i}, \quad \sigma = e^{\frac{4\pi}{3}i}.$$

1° Assuming $\sigma = 1$, we obtain the following from (40):

$$\begin{aligned} \sigma_1\sigma_2 &= 1, \\ \sigma_1+\sigma_2 &= -\varrho, \\ q(\beta_1\sigma_1+\beta_2\sigma_2)^2 &= 3(1-\varrho^2), \\ \beta_1+\beta_2 &= 2. \end{aligned} \tag{40'}$$

From the first equation (40′) we have $\sigma_1 = e^{i\varphi}$ and $\sigma_2 = e^{-i\varphi}$, from the second

$$\cos\varphi = -\tfrac{1}{2}\varrho, \tag{41}$$

while from the third equation we obtain:

$$\begin{aligned} \beta_1 e^{i\varphi}+\beta_2 e^{-i\varphi} &= \pm\sqrt{3(1-\varrho^2)/q}, \\ \beta_1 e^{-i\varphi}+\beta_2 e^{i\varphi} &= \pm\sqrt{3(1-\varrho^2)/q}. \end{aligned} \tag{42}$$

Adding the members of equation (42), we have

$$\cos\varphi = \pm\frac{1}{2}\sqrt{\frac{3(1-\varrho^2)}{q}}. \tag{42'}$$

From (41) and (42′), taking (36) into consideration, we have at once

$$\varrho = \sqrt{\frac{3}{3+q}} \quad \text{and} \quad \cos\varphi = -\frac{1}{2}\sqrt{\frac{3}{3+q}}.$$

Subtracting both sides of the equation from (42) and taking into consideration the fact that $\sigma_1 \neq \sigma_2$, we have $\beta_1 = \beta_2 = 1$. Substituting $\beta_1 = \beta_2 = 1$, $\sigma_1 = e^{i\varphi}$, $\sigma_2 = e^{-i\varphi}$, $\cos\varphi = -\dfrac{1}{2}\sqrt{\dfrac{3}{3+q}}$ to (32) we have:

$$a_{4(2)} = \frac{2\sqrt{3}}{9}a(2-3a^2)\sqrt{2-3a^2} \quad \text{for } a = \frac{\sqrt{17}-1}{8}. \tag{43}$$

2° Let $\sigma = e^{\frac{2\pi}{3}i}$; set (40) then has the form:

$$(40'') \qquad \begin{aligned} \sigma_1\sigma_2 &= e^{-\frac{2}{3}\pi i},\\ \sigma_1+\sigma_2 &= -\varrho e^{\frac{2\pi}{3}i},\\ q(\beta_1\sigma_1+\beta_2\sigma_2)^2 &= 3e^{\frac{4\pi}{3}i}(1-\varrho^2),\\ \beta_1+\beta_2 &= 2. \end{aligned}$$

Adopting a new notation, $\hat{\sigma}_1 = e^{\frac{\pi}{3}i}\sigma_1$, $\hat{\sigma}_2 = e^{\frac{\pi}{3}i}\sigma_2$, $\hat{\varrho} = -\varrho$, we easily notice that set (40″) is reduced to (40′). Hence follow immediately the solutions of (40″) of the form:

$$\beta_1 = \beta_2 = 1, \quad \sigma_1 = e^{i\left(\varphi-\frac{\pi}{3}\right)}, \quad \sigma_2 = e^{-i\left(\varphi+\frac{\pi}{3}\right)}, \quad \cos\varphi = \frac{1}{2}\sqrt{\frac{3}{3+q}}.$$

Substituting the above solution into formula (32), we get the value of $a_{4(2)}$ equal to the value of $a_{4(2)}$ from formula (43), case 1°, which means that this case can be omitted([2]).

Assuming in turn that $\varkappa \neq 0$, we can write (38) in the following form:

$$(44) \qquad \begin{aligned} \sigma_1\sigma_2\sigma &= 1,\\ \beta_1\sigma_1+\beta_2\sigma_2 &= \frac{1}{\varkappa}s+\frac{\varrho}{\varkappa}\sigma,\\ \beta_1\sigma_1^2+\beta_2\sigma_2^2 &= \frac{q-\varkappa^2}{2\varkappa^3}s^2+\frac{\varrho(q-2\varkappa^2)}{\varkappa^3}\sigma s+\frac{q\varrho^2-\varkappa^2}{2\varkappa^3}\sigma^2-\frac{1}{\varkappa}\bar{\sigma},\\ \beta_1+\beta_2 &= 2. \end{aligned}$$

From (44) we easily get

$$(44') \qquad \begin{vmatrix} 1 & 1 & 2\\ \sigma_1 & \sigma_2 & \frac{1}{\varkappa}s+\frac{\varrho}{\varkappa}\sigma\\ \sigma_1 & \sigma_2 & \frac{1}{\varkappa}\bar{s}+\frac{\varrho}{\varkappa}\bar{\sigma} \end{vmatrix} = 0,$$

$$(44'') \qquad \begin{vmatrix} 1 & 1 & 2\\ \sigma_1 & \sigma_2 & \frac{1}{\varkappa}s+\frac{\varrho}{\varkappa}\sigma\\ \sigma_1^2 & \sigma_2^2 & \frac{q-\varkappa^2}{2\varkappa^3}s^2+\frac{\varrho(\dot{q}-2\varkappa^2)}{\varkappa^3}\sigma\cdot s+\frac{qs^2-\varkappa^2}{2\varkappa^3}\sigma^2-\frac{1}{\varkappa}\bar{\sigma} \end{vmatrix} = 0.$$

([2]) Analogously we can show that for 3°, i.e. for $\sigma = e^{\frac{4}{3}\pi i}$, we get the value of $a_{4(2)}$ equal to the value of $a_{4(2)}$ from (43).

From equations (44′) and (44″), after suitable transformations have been made and the condition $\sigma_1 \neq \sigma_2$ has been considered, we get

$$(45)\qquad (\sigma^3+1)\varrho-2(\varkappa-1)s\sigma^2 = 0,$$

$$(46)\qquad (q\varrho^2-\varkappa^2)\sigma^3+2(q-3\varkappa^2)\varrho s\sigma^2+(q-3\varkappa^2)s^2\sigma+2\varkappa^2(2\varkappa-1) = 0.$$

Equation (45) gives

$$(45')\qquad s = \frac{\varrho}{2(\varkappa-1)}\frac{\sigma^3+1}{\sigma^2};$$

hence, on inserting (45′) into (46), we have

$$(46')\qquad \varrho^2\left(\frac{q-3\varkappa^3}{\varkappa-1}(\sigma^3+1)\sigma^3+\frac{q-3\varkappa^2}{4(\varkappa-1)^2}(\sigma^3+1)^2+q\sigma^6\right) = \varkappa^2\sigma^6-2\varkappa^2(2\varkappa-1)\sigma^3$$

and consequently, also the equation

$$(46'')\qquad \varrho^2\left(\frac{q-3\varkappa^2}{\varkappa-1}(\sigma^3+1)+\frac{q-3\varkappa^2}{4(\varkappa-1)^2}(\sigma^3+1)^2+q\right) = \varkappa^2-2\varkappa^2(2\varkappa-1)\sigma^3.$$

From (46′) and (46″), after ϱ has been ejected and the whole ordered we obtain

$$(47)\qquad (\sigma^6-1)\{(q-3\varkappa^2)\sigma^6+2(2\varkappa-1)[4\varkappa(\varkappa-1)(q-3\varkappa)+q-3\varkappa^2]\sigma^3+ +(q-3\varkappa^2)\} = 0.$$

The following equations result from (47):

$$(48)\qquad \sigma^3-1 = 0,$$

$$(49)\qquad \sigma^3+1 = 0,$$

$$(50)\qquad (q-3\varkappa^2)\sigma^6+2(2\varkappa-1)[4\varkappa(\varkappa-1)(q-3\varkappa)+q-3\varkappa^2]\sigma^3+q-3\varkappa^2 = 0.$$

Let us consider these equations in turn.

In case of (48) we get for σ:

$$\sigma = 1,\quad \sigma = e^{\frac{2\pi}{3}i},\quad \sigma = e^{-\frac{2\pi}{3}i}.$$

Assuming that $\sigma = 1$ from (44), in view of (45′), we obtain:

$$(51)\qquad \begin{aligned} \sigma_1\sigma_2 &= 1,\\ \beta_1\sigma_1+\beta_2\sigma_2 &= \frac{\varrho}{\varkappa-1},\\ \beta_1\sigma_1^2+\beta_2\sigma_2^2 &= \frac{q-4\varkappa+3}{2\varkappa(\varkappa-1)^2}\varrho^2-\frac{3}{2\varkappa},\\ \beta_1+\beta_2 &= 2.\end{aligned}$$

From the first equation (51) we have: $\sigma_1 = e^{i\varphi}$, $\sigma_2 = e^{-i\varphi}$, from the second

$$\cos\varphi = \frac{\varrho}{2(\varkappa-1)}. \tag{52}$$

From the third, in turn,

$$\begin{aligned} \beta_1 e^{2i\varphi} + \beta_2 e^{-2i\varphi} &= \frac{q-4\varkappa+3}{2\varkappa(\varkappa-1)}\varrho^2 - \frac{3}{2\varkappa}, \\ \beta_1 e^{-2i\varphi} + \beta_2 e^{2i\varphi} &= \frac{q-4\varkappa+3}{2\varkappa(\varkappa-1)}\varrho^2 - \frac{3}{2\varkappa}. \end{aligned} \tag{53}$$

Adding both sides of equation (53), we get

$$\cos 2\varphi = \frac{q-4\varkappa+3}{2\varkappa(\varkappa-1)}\varrho^2 - \frac{3}{2\varkappa}. \tag{53'}$$

From (52) and (53′) we finally have

$$\varrho^2 = \frac{(\varkappa-1)^2(3-4\varkappa)}{q-6\varkappa+3} \quad \text{and} \quad \cos^2\varphi = \frac{1}{4}\,\frac{3-4\varkappa}{q-6\varkappa+3}. \tag{54}$$

Considering condition (36) and relation (33), the first of the formulae implies the inequality

$$240a^4 + 432a^3 + 163a^2 - 128a - 50 \leqslant 0,$$

which, for $a \in (0,1)$, implies the condition $0 < a \leqslant x_0$, where $x_0 \approx 0.5087$ is the only root in the interval $(0,1)$ of the equation:

$$240x^4 + 432x^3 + 163x^2 - 128x - 50 = 0. \tag{54'}$$

From the second formula (54), in view of (36), (52), (54′) and the inequality $\varkappa - 1 < 0$ for $a \in (0,1)$ we have:

$$\cos\varphi = -\frac{1}{2}\sqrt{\frac{3-4\varkappa}{q-6\varkappa+3}} \quad \text{for } a \in (0, x_0]. \tag{55}$$

Subtracting the members of equations (53), by (36), (52) and $\sigma_1 \neq \sigma_2$, we get: $\beta_1 = \beta_2 = 1$.

Inserting

$$\beta_1 = \beta_2 = 1, \quad \sigma_1 = e^{i\varphi}, \quad \sigma_2 = e^{-i\varphi}, \quad \cos\varphi = -\frac{1}{2}\sqrt{\frac{3-4\varkappa}{q-6\varkappa+3}}$$

into (32) we obtain

$$a_{4(2)}^{(1)} = \frac{2\sqrt{3}}{9} a(2-3a^2)\sqrt{2-3a^2} \quad \text{for } 0 < a \leqslant x_0, \tag{56}$$

where $x_0 \approx 0.5087$ is the root of equation (54′).

The values $\sigma = e^{\pm\frac{2\pi}{3}i}$ lead, similarly as in 2° and 3°, to the value $a_{4(2)}$ given by formula (56). This case can therefore be ignored. Moreover, it should be noticed that formula (43) for $a_{4(2)}$ obtained by $\varkappa = 0$ is a special case of (56).

In the case (49) we have:

$$\sigma = -1, \quad \sigma = e^{\frac{\pi}{3}i}, \quad \sigma = e^{-\frac{\pi}{3}i}.$$

Assuming that $\sigma = -1$, we obtain

$$s = 0 \tag{57}$$

from (45′). This and (44) imply the set of equations

$$\begin{aligned} \sigma_1\sigma_2 &= -1, \\ \beta_1\sigma_1+\beta_2\sigma_2 &= -\frac{\varrho}{\varkappa}, \\ \beta_1\sigma_1^2+\beta_2\sigma_2^2 &= \frac{q\varrho^2+\varkappa^2}{2\varkappa^3}, \\ \beta_1+\beta_2 &= 2. \end{aligned} \tag{58}$$

It follows from the first equation (58) and from (57) that

$$\sigma_1 = 1, \quad \sigma_2 = -1. \tag{59}$$

From the third and the fourth equation of (58) we have

$$\varrho^2 = \frac{\varkappa^2(4\varkappa-1)}{q}. \tag{60}$$

From the third and the fourth equation of (58) and (60) we get

$$\beta_1 = 1-\frac{|\varkappa|}{2\varkappa}\sqrt{\frac{4\varkappa-1}{q}}, \quad \beta_2 = 1+\frac{|\varkappa|}{2\varkappa}\sqrt{\frac{4\varkappa-1}{q}}. \tag{61}$$

Let us point out that (36) and (60) imply the following conditions:

$$\frac{-4\varkappa^3+\varkappa^2+q}{q} \leqslant 0, \quad \text{that is} \quad \begin{cases} -4\varkappa^3+\varkappa^2+q \leqslant 0, \\ q \neq 0, \end{cases}$$

$$\text{or} \quad \begin{cases} -4\varkappa^3+\varkappa^2+q \geqslant 0, \\ q \neq 0. \end{cases}$$

Hence, and in view of (33), the following inequalities are obtained:

$$(16a^2-11a+1)(400a^5+592a^4+a^3-211a^2-82a+20) \leqslant 0, \quad 16a^2-11a+1 \neq 0. \tag{62}$$

Inequalities (62), for $a \in (0,1)$ hold in the intervals

$$\frac{11-\sqrt{57}}{32} < a \leqslant x_1, \quad \frac{11+\sqrt{57}}{32} < a \leqslant x_2, \tag{63}$$

where $x_1 \approx 0.1737$ and $x_2 \approx 0.5932$ are roots of the equation

$$400x^5+592x^4+x^3-211x^2-82x+20 = 0. \tag{63'}$$

Another condition follows from (61) and (4):

$$\frac{1}{2}\sqrt{\frac{4\varkappa-1}{q}} < 1, \tag{64}$$

Hence, by (33), we get the inequality:

$$(85a^2-68a+10)(16a^2-11a+1) > 0. \tag{65}$$

It can be verified quite easily that inequality (65) for a determined by inequalities (63) does not hold. It follows that the case $\sigma = -1$ does not yield the extremal value of coefficient a_4. The same result can be obtained by considering the cases $\sigma = e^{\pm\frac{2\pi}{3}i}$. In order to do this, it is enough, similarly as in 2° and 3°, to reduce the set (44) to (58).

Finnally, let us consider the last of equations (48)–(50), i.e. equation (50). Subtracting both sides of equations (46′) and (46″) and omitting the cases where $\sigma^3 = \pm 1$ considered above, we have

$$\varrho^2 = \frac{\varkappa(\varkappa-1)}{q-3\varkappa}; \tag{68}$$

hence, in view of (36) and having utilized (33), we have the inequality

$$(4a-1)(48a^3+24a^2+61a-13) \leqslant 0,$$

from which for $a \in (0,1)$ follows the inequality

$$x_3 \leqslant a < 0.25, \tag{68'}$$

where $x_3 \approx 0.1928$ is the root of the equation:

$$48x^3+24x^2+61x-13 = 0. \tag{68''}$$

Combining our further discussion to the interval for a determined by (68′), let us examine, according to what has already been stated, equation (50). As we are looking for those solutions of this equation which satisfy the codition $|\sigma| = 1$, the existence of such solutions is determined by the non-negative value of the discriminant of equation (50). Hence follows an inequality of the form:

$$\varkappa(\varkappa-1)[2\varkappa(2\varkappa-1)(q-3\varkappa)+q-3\varkappa^2][2(2\varkappa-1)(\varkappa-1)(q-3\varkappa)+q-3\varkappa] \leqslant 0,$$

and so, because of the inequalities $\varkappa < 0$ and $\varkappa-1 < 0$ for $a \in [x_3, 0.25)$ we have

$$[2\varkappa(2\varkappa-1)(q-3\varkappa)+q-3\varkappa^2][2(2\varkappa-1)(\varkappa-1)(q-3\varkappa)+q-3\varkappa^2] \leqslant 0.$$

Considering (33), the latter inequality can, be written in the form

$$(2384a^5+176a^4-1951a^3+91a^2+454a-74)(528a^4+360a^3-377a^2- -200a+58) \geqslant 0;$$

hence, utilizing (68′) at the same time, we find that

$$x_3 \leqslant a \leqslant x_4; \tag{69}$$

here $x_3 = 0.1928$ is a root of equation (68″) and $x_4 \approx 0.2288$ is a root of the equation

$$528x^4+360x^3-377x^2-200x+58 = 0. \tag{69′}$$

Let us now determine σ_1 and σ_2 in the case under consideration. It follows from the first set of equations (44), (45′) and (68) that σ_1 and σ_2 are roots of the equation

$$x^2+\frac{1}{2}\sqrt{\frac{\varkappa}{(\varkappa-1)(q-3\varkappa)}}\cdot\frac{\sigma^3+1}{\sigma^2}x+\bar{\sigma} = 0, \tag{70}$$

where σ is a number determined by equation (50). Because of the fact that only numbers of the form $\xi_1 = re^{i\varphi}$ and $\xi_2 = \frac{1}{r}e^{i\theta}$ are roots of equation (70), roots for which $|\xi_1| = |\xi_2| = 1$ can exist only if they satisfy the following equality:

$$|\xi_1|^2+|\xi_2|^2 = 2,$$

that is

(70′)

$$\frac{\varkappa}{8(\varkappa-1)(q-3\varkappa)}\cdot\frac{(\sigma^3+1)^2}{\sigma^2}+\frac{1}{2}\sqrt{\left(4-\frac{\varkappa}{4(\varkappa-1)(q-3\varkappa)}\cdot\frac{(\sigma^3+1)^2}{\sigma^3}\right)^2} = 2.$$

Equality (70′) holds when

$$\frac{\varkappa}{4(\varkappa-1)(q-3\varkappa)}\frac{(\sigma^3+1)^2}{\sigma^3} \leqslant 4. \tag{70″}$$

Taking (50) into consideration, we have

$$(71)\qquad \frac{(\sigma^3+1)^2}{\sigma^3} = -\frac{4(\varkappa-1)[(q-3\varkappa^2)(4\varkappa-1)+2q(\varkappa-1)(2\varkappa-1)]}{q-3\varkappa^2};$$

hence and in view of (33) inequality (70″) for $x_3 \leqslant a \leqslant x_4$ has the form

$$8256a^7-432a^6-4012a^5-1663a^4+7358a^3-5933a^2+652a+94 \geqslant 0.$$

It can be shown in an elementary way that in the interval under consideration such an inequality holds.

The next step in the examination of equation (50) will be to determine the values of β_1 and β_2. From the second and the forth equations of the set (44) as well as from (45′) and (68) we have

$$(73)\qquad \begin{aligned} \beta_1+\beta_2 &= 2, \\ \beta_1\sigma_1+\beta_2\sigma_2 &= \frac{1}{\varkappa}\sqrt{\frac{\varkappa(\varkappa-1)}{q-3\varkappa}}\left(\frac{1}{2(\varkappa-1)}\frac{\sigma^3+1}{\sigma^2}+\sigma\right), \end{aligned}$$

and hence

$$\beta_1 = \frac{2\sigma_2-\dfrac{1}{\varkappa}\sqrt{\dfrac{\varkappa(\varkappa-1)}{q-3\varkappa}}\left(\dfrac{1}{2(\varkappa-1)}\cdot\dfrac{\sigma^3+1}{\sigma^2}+\sigma\right)}{\sigma_2-\sigma_1},$$

$$(73')\qquad \beta_2 = \frac{\dfrac{1}{\varkappa}\sqrt{\dfrac{\varkappa(\varkappa-1)}{q-3\varkappa}}\left(\dfrac{1}{2(\varkappa-1)}\cdot\dfrac{\sigma^3+1}{\sigma^2}+\sigma\right)-2\sigma_1}{\sigma_2-\sigma_1}.$$

It follows at once from (73′), by (44), that

$$(73'')\qquad \beta_1\beta_2 = \frac{4-\dfrac{\varkappa-1}{\varkappa(q-3\varkappa)}-\dfrac{2\varkappa-1}{4\varkappa(\varkappa-1)(q-3\varkappa)}\cdot\dfrac{(\sigma^3+1)}{\sigma^3}}{4-\dfrac{\varkappa}{4(q-3\varkappa)(\varkappa-1)}\cdot\dfrac{(\sigma^3+1)^2}{\sigma^3}}$$

$$= A = -\frac{(4a^2+a+1)}{9(4a^2+a-1)}\times$$

$$\times\frac{59072a^7+30832a^6-145860a^5+34895a^4+56410a^3-20595a^2-2572a+778}{8256a^7-432a^6-4012a^5-1663a^4+7358a^3-5933a^2+652a+94}.$$

Let it be pointed out that for condition (4) to be satisfied in view of the first equation (73), (73′) and (73″) it suffices that the following inequality be fulfilled:

$$(74)\qquad 4-\frac{\varkappa-1}{\varkappa(q-3\varkappa)}-\frac{2\varkappa-1}{4\varkappa(\varkappa-1)(q-3\varkappa)}\cdot\frac{(\sigma^3+1)^2}{\sigma^3} > 0.$$

Inequality (74), by (33) and (71), implies for $x_3 \leqslant a \leqslant x_4$ the inequality

$$(75)\quad 59072a^7+30832a^6-145860a^5+34895a^4+56410a^3-20595a^2-\\ -2572a+778>0,$$

which is fulfilled in the whole interval under investigation. Finally, numbers β_1 and β_2, by (73'') and (4), are determined by the equation

$$(76)\qquad x^2-2x+A=0$$

for $a \in [x_3, x_4]$, $x_3 \approx 0.1928$ and $x_1 \approx 0.2228$ being defined by equations (68'') and (69') respectively.

In the case under consideration, i.e. equation (50), it only remains now to determine the value of $a_{4(2)}$ from (32). Let us point out that using the set of equations (44), (45') and (68), we can write formula (32) in the form

$$a_{4(2)}^{(2)} = p\sqrt{\frac{\varkappa(\varkappa-1)}{q-3\varkappa}}\,\frac{q}{24\varkappa^2(\varkappa-1)^2(q-3\varkappa)}(2\varkappa-1+\bar{\sigma}^3)^3\sigma^3-\\ -\frac{1}{4(\varkappa-1)^2(q-3\varkappa)}(2\varkappa-1+\bar{\sigma}^3)^2(1+\sigma^3)+\frac{2}{\varkappa-1}(2\varkappa-1+\bar{\sigma})^3+\\ +\frac{1}{8(\varkappa-1)^2(q-3\varkappa)}(2\varkappa-1+\bar{\sigma}^3)\frac{(1+\sigma^3)^2}{\sigma^3}-\\ -\frac{1}{2\varkappa(\varkappa-1)}(2\varkappa-1+\bar{\sigma}^3)-\left(\frac{1}{\varkappa-1}\cdot\frac{1+\sigma^3}{\sigma^3}\right);$$

hence, after rearrangement,

$$a_{4(2)}^{(2)} = \frac{p\sqrt{\dfrac{\varkappa(\varkappa-1)}{q-3\varkappa}}}{24\varkappa^2(\varkappa-1)^2(q-3\varkappa)}\{3q(32\varkappa^4-64\varkappa^3+40\varkappa^2-8\varkappa+1)-\\ -3\varkappa^2(96\varkappa^3-184\varkappa^2+104\varkappa-13)+(2\varkappa-1)[q(2\varkappa-1)^2-12\varkappa^3+9\varkappa^2]\sigma^3+\\ +3(2\varkappa-1)[q(2\varkappa-1)^2-12\varkappa^3+9\varkappa^2]\bar{\sigma}^3+(q-3\varkappa^2)\bar{\sigma}^6\},$$

which means that in view of (12), we get

$$(76')\quad a_{4(2)}^{(2)} = \frac{p\sqrt{\dfrac{\varkappa(\varkappa-1)}{q-3\varkappa}}}{24\varkappa^2(\varkappa-1)^2(q-3\varkappa)}\{6q(32\varkappa^4-64\varkappa^3+40\varkappa^2-8\varkappa+1)-\\ -6\varkappa^2(96\varkappa^3-184\varkappa^2+104\varkappa-13)+4(2\varkappa-1)[q(2x-1)^2-12\varkappa^3+9\varkappa^2]\times\\ \times(\sigma^3+\bar{\sigma}^3)+(q-3\varkappa^2)(\sigma^6+\bar{\sigma}^6)\};$$

from (76′) and (50) we now have

$$a_{4(2)}^{(2)} = \frac{-4p\varkappa(\varkappa-1)(4q-12\varkappa+3)}{3(q-3\varkappa^2)}\sqrt{\frac{\varkappa(\varkappa-1)}{q-3\varkappa}},$$

and hence, having used (33), we finally arrive at the formula

$$(77)\quad a_{4(2)}^{(2)} = \frac{2a(1-a-4a^2)(8a+7)(7a^2-8a+2)}{16a^2+64a+1}\sqrt{\frac{(1-a-4a^2)(8a+7)}{5(1-4a)(a^2+a+1)}},$$
$$x_3 \leqslant a < x_4,$$

where x_3 is the root of equation (68″), x_4 is the root of equation (69′) and numbers σ_1, σ_2 are given by equation (70), while numbers β_1, β_2 are given by equation (76).

The above reasoning leads to the following corollary. If $g \in \mathfrak{G}_2^M$, then the maximum $\mathrm{re}\, a_4$ can only determine formula (56) for $0 < a \leqslant x_0$, where $x_0 \approx 0.5087$ is the root of (54), and formula (77) for $x_3 \leqslant a \leqslant x_4$, where $x_3 \approx 0.1928$ and $x_4 \approx 0.228$ are roots of equations (68″) and (69′) respectively.

4.3. Let us finally assume that $g \in \mathfrak{G}_3^M$, then the function g is determined by the equation

$$(78)\qquad \frac{g}{\prod_{k=1}^{3}(1-\sigma_k g)^{\beta_k}} = \frac{1}{M}\,\frac{z}{\prod_{k=1}^{3}(1-\sigma_k z)^{\beta_k}}$$

and its coefficient $a_{4(3)}$ is given by the formula

$$a_{4(3)} = p\big(\tfrac{1}{3}q(\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3)^3 - 2\varkappa(\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3)(\beta_1\sigma_1^2+ \\ +\beta_2\sigma_2^2+\beta_3\sigma_3^2) + (\beta_1\sigma_1^3+\beta_2\sigma_2^3+\beta_3\sigma_3^3)\big),$$

with we use of the notation from (33).

From Theorem 1 [1] it follows that the function $\mathscr{R}(z)$ given by formula (34) with coefficients

$$\mathscr{D}_1 = p\big(2\varkappa(\beta_1\sigma_1^2+\beta_2\sigma_2^2+\beta_3\sigma_3^2) - q(\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3)^2\big),$$
$$\mathscr{D}_2 = 4p\varkappa(\beta_1\sigma_1+\beta_2\sigma_2+\beta_3\sigma_3),$$
$$\mathscr{D}_3 = -3p,$$

has double roots at points $\bar{\sigma}_k$ $(k = 1, 2, 3)$. From the above, as well as from (4), we obtain the following set of equations:

$$(80)\qquad \begin{aligned} \sigma_1^2\sigma_2^2\sigma_3^2 &= 1,\\ \varkappa(\sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3) &= s_1,\\ -2\varkappa(\sigma_1^2\beta_1+\sigma_2^2\beta_2+\sigma_3^2\beta_3)+q(\sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3)^2 &= S_1^2+2\bar{s}_1\sigma_1\sigma_2\sigma_3,\\ \beta_1+\beta_2+\beta_3 &= 2, \end{aligned}$$

where

$$s_1 = \sigma_1 + \sigma_2 + \sigma_3. \tag{81}$$

Let us consider two sets in turn:

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 &= 1, \\ \varkappa(\sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3) &= s_1, \\ -2\varkappa(\sigma_1^2\beta_1+\sigma_2^2\beta_2+\sigma_3^2\beta_3)+q(\sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3)^2 &= s_1^2+2\bar{s}_1\sigma_1\sigma_2\sigma_3, \\ \beta_1+\beta_2+\beta_3 &= 2, \end{aligned} \tag{82}$$

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 &= -1, \\ \varkappa(\sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3) &= s_1, \\ -2\varkappa(\sigma_1^2\beta_1+\sigma_2^2\beta_2+\sigma_3^2\beta_3)+q(\sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3)^2 &= s_1^2+2\bar{s}_1\sigma_1\sigma_2\sigma_3, \\ \beta_1+\beta_2+\beta_3 &= 2. \end{aligned} \tag{82'}$$

Note that adopting the notation

$$\sigma_k e^{\frac{\pi}{3}i} = \hat{\sigma}_k \quad (k = 1, 2, 3), \qquad \hat{s}_1 = \hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3, \tag{82''}$$

we obtain from (82′) the set (82) with the unknowns $\hat{\sigma}_k$, β_k $(k = 1, 2, 3)$ In view of (82″) and (79) the solutions of (82′) lead in every case to the reverse value of the coefficient in comparison with the solutions of (82). With this in mind we can confine our discussion to set (82).

Let us then consider set (82). Suppose first that $\varkappa = 0$, that is $a = (\sqrt{17}-1)/8$. Set (82) then has the form

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 &= 1, \\ \sigma_1+\sigma_2+\sigma_3 &= 0, \\ \sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3 &= 0, \\ \beta_1+\beta_2+\beta_3 &= 2. \end{aligned} \tag{83}$$

From the first two equations (83) it follows immediately that

$$\sigma_1 = 1, \quad \sigma_2 = e^{\frac{2}{3}\pi i}, \quad \sigma_3 = e^{-\frac{2}{3}\pi i} \tag{84}$$

and then from the last two equations we get

$$\beta_k = 2/3 \quad (k = 1, 2, 3). \tag{84'}$$

Inserting (84) and (84′) is to (79), we obtain

$$a_{4(3)} = \tfrac{2}{3}a(1-a^3) \quad \text{for } a = (\sqrt{17}-1)/8. \tag{85}$$

Let us assume now that $\varkappa = 0$; then system (82) can be written in the form

(86)
$$\begin{aligned}\sigma_1\sigma_2\sigma_3 &= 1,\\ \sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3 &= \frac{1}{\varkappa}s_1,\\ \sigma_1^2\beta_1+\sigma_2^2\beta_2+\sigma_3^2\beta_3 &= \frac{q-3\varkappa^2}{2\varkappa^3}s_1^2-\frac{1}{\varkappa}\bar{s}_1,\\ \beta_1+\beta_2+\beta_3 &= 2.\end{aligned}$$

The following equations follow from (86):

(87)
$$\begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \frac{1}{\varkappa}s_1 \\ \bar{\sigma}_1 & \bar{\sigma}_2 & \bar{\sigma}_3 & \frac{1}{\varkappa}\bar{s}_1 \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 & \frac{q-3\varkappa^2}{2\varkappa^3}-\frac{1}{\varkappa}\bar{s}_1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 0,$$

(88)
$$\begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \frac{1}{\varkappa}s_1 \\ \bar{\sigma}_1 & \bar{\sigma}_2 & \bar{\sigma}_3 & \frac{1}{\varkappa}\bar{s}_1 \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 & \frac{q-3\varkappa^2}{2\varkappa^3}-\frac{1}{\varkappa}\bar{s}_1 \\ \bar{\sigma}_1^2 & \bar{\sigma}_2^2 & \bar{\sigma}_3^2 & \frac{q-3\varkappa^2}{2\varkappa^3}-\frac{1}{\varkappa}s_1 \end{vmatrix} = 0.$$

Hence, after suitable transformations and taking the condition $\sigma_i \neq \sigma_j$ for $i \neq j$ $(i, j = 1, 2, 3)$, we obtain

(87′)
$$(q-3\varkappa^2)s_1^2+4\varkappa^2(\varkappa-1)\bar{s}_1 = 0,$$

(88′)
$$(q-3\varkappa^2)(s_1^3-\bar{s}_1^3) = 0.$$

It follows from equation (88′), in view of $q-3\varkappa^2 \neq 0$, that

(89)
$$s_1 = \bar{s}_1,$$

(89′)
$$s_1 = \bar{s}_1 e^{\pm\frac{2}{3}\pi i}.$$

Inserting in turn (89) and (89′) in to (87′), we have

$$s_1 = 0, \tag{90}$$

$$s_1 = \frac{4\varkappa^2(1-\varkappa)}{q-3\varkappa^2}, \tag{91}$$

$$s_1 = \frac{4\varkappa^2(1-\varkappa)}{q-3\varkappa^2} e^{\pm\frac{2}{3}\pi i}. \tag{92}$$

Let us now consider (86) for the cases (90)–(92). Let us first assume that $s_1 = 0$. Then

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 &= 1, \\ \sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3 &= 0, \\ \sigma_1^2\beta_1+\sigma_2^2\beta_2+\sigma_3^2\beta_3 &= 0, \\ \beta_1+\beta_2+\beta_3 &= 2. \end{aligned}$$

Hence, in view of $s_1 = \sigma_1+\sigma_2+\sigma_3 = 0$, we easily obtain

$$\sigma_1 = 1, \quad \sigma_2 = e^{\frac{2\pi}{3}i}, \quad \sigma_3 = e^{-\frac{2\pi}{3}i}, \quad \beta_1 = \beta_2 = \beta_3 = \tfrac{2}{3}. \tag{93}$$

Inserting solution (93) into (79), we get

$$a_{4(3)}^{(1)} = \tfrac{2}{3}a(1-a^3) \quad \text{for } 0 < a < 1. \tag{94}$$

Moreover, it is worth noting that the case $\varkappa = 0$ and formula (85) which follows from it is a special case of formula (94) obtained above.

Let us now assume that condition (91) holds. Set (86) then has the form

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 &= 1, \\ \sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3 &= \frac{4\varkappa(1-\varkappa)}{q-3\varkappa^2}, \\ \sigma_1^2\beta_1+\sigma_2^2\beta_2+\sigma_3^2\beta_3 &= \frac{4\varkappa(1-\varkappa)(2\varkappa^3+\varkappa^2-2q\varkappa+q)}{(q-3\varkappa^2)^2}, \\ \beta_1+\beta_2+\beta_3 &= 2. \end{aligned} \tag{95}$$

What is more, numbers σ_k $(k = 1, 2, 3)$, in view of $s_1 = \bar{s}_1$, satisfy the conditions

$$\begin{aligned} \sigma_1+\sigma_2+\sigma_3 &= s_1, \\ \sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3 &= s_1, \\ \sigma_1\cdot\sigma_2\cdot\sigma_3 &= 1. \end{aligned}$$

Hence we have

$$\sigma_1 = e^{i\varphi}, \quad \sigma_2 = e^{-i\varphi}, \quad \sigma_3 = 1, \tag{96}$$

and, by (91), the inequality

(97) $$-1 \leqslant \frac{4\varkappa^2(1-\varkappa)}{q-3\varkappa^2} \leqslant 3$$

must hold, i.e., in view of (33) and the inequality $q-3\varkappa^2 < 0$ for $a \in (0, 1)$, we have the inequalities

$$400a^5+592a^4 \pm a^3-211a^2-82a+20 \leqslant 0,$$
$$112a^5+112a^4-a^3-49a^2+58a+8 \geqslant 0.$$

It can easily be shown that the second of these inequalities holds for any $a \in (0, 1)$, while the first holds for

(97′) $$x_1 \leqslant a \leqslant x_2,$$

where $x_1 \approx 0.1737$ and $x_2 \approx 0.5992$ are roots of equation (63′). Inserting (96) into (95) we get

$$\beta_1 e^{i\varphi}+\beta_2 e^{-i\varphi}+\beta_3 = \frac{4\varkappa(1-\varkappa)}{q-3\varkappa^2},$$
$$\beta_1 e^{2i\varphi}+\beta_2 e^{-2i\varphi}+\beta_3 = \frac{4\varkappa(1-\varkappa)(2\varkappa^3+\varkappa^2-2q\varkappa+q)}{(q-3\varkappa^2)^2},$$
$$\beta_1+\beta_2+\beta_3 = 2;$$

hence

(98) $$\beta_1 = \beta_2 = \frac{2(q-\varkappa^2-2\varkappa)}{3q-13\varkappa^2+4\varkappa^3}, \quad \beta_3 = \frac{2(q-11\varkappa^2+4\varkappa^3+4\varkappa)}{3q-13\varkappa^2+4\varkappa^3}$$

and at the same time, by (4), the inequalities

(98′) $$\frac{q-\varkappa^2-2\varkappa}{3q-13\varkappa^2+4\varkappa^3} > 0,$$

(98″) $$\frac{q-11\varkappa^2+4\varkappa^3+4\varkappa}{3q-13\varkappa^2+4\varkappa^3} > 0,$$

should be fulfilled. Since

$$3q-13\varkappa^2-4\varkappa^3 = \frac{9(a-1)(112a^5+112a^4-a^3-49a^2+58a+8)}{16(a^2+a+1)^3} < 0$$

for $a \in (0, 1)$, inequalities (98′) and (98″) are equvivalent with the following inequalities:

(99′) $$48a^3+24a^2+61a-13 > 0,$$

(99″) $$240a^4+432a^3+163a^2-128a-50 < 0$$

which are simultaneously satisfied in the interval

$$x_3 < a < x_0, \tag{100}$$

where $x_3 \approx 0.1928$ and $x_0 \approx 0.5087$ are roots of equations (68″) and (54′) respectively. Finally, having assumed (91), we have obtained the solution of (86) of the form

$$\sigma_1 = e^{i\varphi}, \quad \sigma_2 = e^{-i\varphi}, \quad \sigma_3 = 1, \quad \cos\varphi = \frac{-4\varkappa^3+7\varkappa^2-q}{2(q-3\varkappa^2)},$$

$$\beta_1 = \beta_2 = \frac{2(q-\varkappa^2-2\varkappa)}{3q-13\varkappa^2+4\varkappa^3}, \quad \beta_3 = \frac{2(q-11\varkappa^2+4\varkappa^3+4\varkappa)}{3q-13\varkappa^2+4\varkappa^3}$$

while, in view of (100), $a \in (x_3, x_0)$. Inserting the above solution into (79) we have

$$a_{4(3)}^{(2)} = 2p\left(\frac{16(\varkappa-1)^3\varkappa^3}{3(q-3\varkappa^2)^2}+1\right),$$

that is, by (33),

$$a_{4(3)}^{(2)} = \frac{2}{3}a\,(1-a^3)\left(\frac{(8a+7)^3(4a^2+a-1)^3}{(a-1)(a^2+a+1)^2(16a^2+64a+1)^2}+1\right), \tag{101}$$
$$x_3 < a < x_0.$$

Let us now consider the case (92). Inserting it in to system (86), we obtain

$$\begin{aligned} \sigma_1\sigma_2\sigma_3 &= 1, \\ \sigma_1\beta_1+\sigma_2\beta_2+\sigma_3\beta_3 &= \frac{4\varkappa(1-\varkappa)}{q-3\varkappa^2}e^{\pm\frac{2}{3}\pi i}; \\ \sigma_1^2\beta_1+\sigma_2^2\beta_2+\sigma_3^2\beta_3 &= \frac{4\varkappa(1-\varkappa)(2\varkappa^3+\varkappa^2-2q\varkappa+q)}{(q-3\varkappa^2)^2}e^{\pm\frac{4}{3}\pi i}, \\ \beta_1+\beta_2+\beta_3 &= 2. \end{aligned} \tag{102}$$

Adopting the new notation $\hat{\sigma}_k = \sigma_k e^{\pm\frac{2}{4}\pi i}$ $(k = 1, 2, 3)$ we can easily notice that system (102) is reduced to (95), and its solution, in view of (79), leads to $a_{4(3)}$ defined by formula (101), which means that it can be ignored. The above solutions lead to the following corollary.

If $g \in \mathfrak{G}_3^M$, then the maximum $\mathrm{re}\, a_4$ can only define formula (94) for $0 < a < 1$ and formula (101) for $x_3 < a < x_0$, where x_3 and x_0 are roots of equations (68″) and (54′) respectively.

The results of 4.1, 4.2, 4.3 given by formulae (31), (31′), (56), (77), (94) and (101) lead to the following corollaries.

If the function g is extremal with respect to functional (30), then $g \in \bigcup_{k=1}^{3} \mathfrak{G}_k^M$ and the only possible values of this functional are:

$$(103) \quad a_{4(1)}^{(1)}, \quad a_{4(2)}^{(1)}, \quad a_{4(3)}^{(1)} \qquad \text{when } 0 < a < x_3,$$

$$(103') \quad a_{4(1)}^{(1)}, \quad a_{4(2)}^{(1)}, \quad a_{4(3)}^{(1)}, \quad a_{4(2)}^{(2)}, \quad a_{4(3)}^{(2)} \qquad \text{when } x_3 < a < x_4,$$

$$(103'') \quad a_{4(1)}^{(1)}, \quad a_{4(2)}^{(1)}, \quad a_{4(3)}^{(1)}, \quad a_{4(3)}^{(2)} \qquad \text{when } x_4 < a < \frac{4-\sqrt{2}}{7},$$

$$(103''') \quad a_{4(1)}^{(2)}, \quad a_{4(2)}^{(1)}, \quad a_{4(3)}^{(1)}, \quad a_{4(3)}^{(2)} \qquad \text{when } \frac{4-\sqrt{2}}{7} < a < x_0,$$

$$(103^{\mathrm{IV}}) \quad a_{4(1)}^{(2)}, \quad a_{4(3)}^{(1)} \qquad \text{when } x_0 < a < \frac{4+\sqrt{2}}{7},$$

$$(103^{\mathrm{V}}) \quad a_{4(1)}^{(1)}, \quad a_{4(3)}^{(1)} \qquad \text{when } \frac{4+\sqrt{2}}{7} < a < 1.$$

After elementary but nevertheless arduous calculations the following inequalities can be obtained:

$$(104) \quad a_{4(3)}^{(1)} < a_{4(2)}^{(1)} < a_{4(1)}^{(1)} \qquad \text{for } 0 < a < x_3,$$

$$(105) \quad a_{4(3)}^{(1)} < a_{4(1)}^{(1)} < a_{4(3)}^{(2)} < a_{4(2)}^{(2)} \qquad \text{for } x_3 < a < x_4,$$

$$(105') \quad a_{4(2)}^{(1)} < a_{4(2)}^{(2)} \qquad \text{for } x_3 < a < x_4,$$

$$(106) \quad a_{4(1)}^{(1)} < a_{4(3)}^{(2)} < a_{4(2)}^{(1)} \qquad \text{for } x_4 < a < \frac{4-\sqrt{2}}{7},$$

$$(106') \quad a_{4(3)}^{(1)} < a_{4(3)}^{(2)} \qquad \text{for } x_4 < a < \frac{4-\sqrt{2}}{7},$$

$$(106'') \quad a_{4(1)}^{(2)} < a_{4(3)}^{(1)} < a_{4(3)}^{(2)} < a_{4(2)}^{(1)} \qquad \text{for } \frac{4-\sqrt{2}}{7} < a < \frac{\sqrt{17}-1}{8},$$

$$(106''') \quad a_{4(1)}^{(2)} < a_{4(3)}^{(2)} < a_{4(3)}^{(1)} < a_{4(2)}^{(1)} \qquad \text{for } \frac{\sqrt{17}-1}{8} < a < x_6,$$

$$(106^{\mathrm{IV}}) \quad a_{4(1)}^{(2)} < a_{4(3)}^{(2)} < a_{4(3)}^{(1)} \qquad \text{for } x_6 < a < x_0,$$

$$(107) \quad a_{4(1)}^{(1)} < a_{4(3)}^{(1)} \qquad \text{for } x_0 < a < \frac{4+\sqrt{2}}{7},$$

$$(108) \quad a_{4(1)}^{(1)} < a_{4(3)}^{(1)} \qquad \text{for } \frac{4+\sqrt{2}'}{7} < a < 1.$$

Moreover, it can be verified that

$$
\begin{array}{lll}
 & a^{(1)}_{4(3)} = a^{(1)}_{4(2)} = a^{(1)}_{4(1)} & \text{for } a = x_3, \\
(109) & a^{(2)}_{4(2)} = a^{(1)}_{4(2)} & \text{for } a = x_4, \\
 & a^{(1)}_{4(2)} = a^{(1)}_{4(3)} & \text{for } a = x_6.
\end{array}
$$

From the above inequalities follows

THEOREM 3. *For any quasi-starlike function g of the form* (1) *the following inequalities are true*:

$$
(110) \qquad |a_4| \leqslant \begin{cases} 2a(1-a)(7a^2-8a+2) & \textit{for } 0 < a \leqslant x_3, \\[2mm] \dfrac{2a(1-a-4a^2)^{\frac{3}{2}}(8a+7)^{\frac{3}{2}}(7a^2-8a+2)}{\sqrt{5}\,(16a^2+64a+1)(1-4a)^{\frac{1}{2}}(a^2+a+1)^{\frac{1}{2}}} & \textit{for } x_3 \leqslant a \leqslant x_4, \\[2mm] a(2-3a^2)^{\frac{3}{2}} & \textit{for } x_4 \leqslant a \leqslant x_6, \\[2mm] a(1-a^3) & \textit{for } x_6 \leqslant a \leqslant 1, \end{cases}
$$

where the notation $a_1 = a = 1/M$ are adopted, and numbers x_3, x_4, x_6 are roots of the equations

$$
\begin{aligned}
48x^3+24x^2+61x-13 &= 0, \\
528x^4+360x^3-377x^2-200x+58 &= 0, \\
30x^6-54x^4-6x^3+36x^2-5 &= 0
\end{aligned}
$$

respectively. Moreover, the values x_3, x_4, x_6 are approximately equivalent to 0.1928, 0.2228, 0,4685, *respectively, x_3 and x_6 are the only roots of the interval* (0, 1), *and x_4 of the interval* (0, 0.25).

Functions g for which the equality sign in (110) holds are determined by the equations

$$
(111) \qquad \frac{g}{(1-g)^2} = \frac{1}{M}\,\frac{z}{(1-z)^2},
$$

$$
(112) \qquad \frac{g}{(1-\sigma_1 g)^{\beta_1}(1-\sigma_2 g)^{\beta_2}} = \frac{1}{M}\,\frac{z}{(1-\sigma_1 z)^{\beta_1}(1-\sigma_2 z)^{\beta_2}},
$$

where numbers σ_1 and σ_2 are roots of the equations

$$
x^2+\frac{1}{1-a}\sqrt{\frac{(a^2+a+1)(4a^2+a-1)}{5(8a+7)(4a-1)}}\;\frac{\sigma^3+1}{\sigma}\,x+\bar{\sigma} = 0,
$$

σ is determined by equation (50) and numbers β_1, β_2 are roots of the equation

$$
x^2-2x+A = 0,
$$

A being determined by formula (73''),

$$\frac{g}{1-2g\cos\varphi+g^2} = \frac{1}{M}\,\frac{z}{1-2z\cos\varphi+z^2}, \tag{113}$$

where

$$\cos\varphi = -\frac{\sqrt{2-3a^2}}{2\sqrt{3}\,(1-a)},$$

$$\frac{g}{1-g^3} = \frac{1}{M}\,\frac{z}{1-z^3}, \tag{114}$$

respectively. Formulae (111)–(114) determine functions g with an accuracy up to an arbitrary rotation angle.

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