ANNALES
POLONICI MATHEMATICI
XXVIII (1973)

On some structures defined by algebraic equations

by U. C. VOHRA and K. D. SINGH (Lucknow, India)

Abstract. Structures on a differentiable manifold obtained by introducing a vector-valued linear function J satisfying some algebraic relations have been extensively studied by a number of mathematicians under various topics, such as complex and almost complex spaces, almost product spaces, contact and almost contact spaces, and f-structure spaces. Recently Duggal [1] defined a new structure called a GF-structure on a differentiable manifold, which is more general than almost complex, almost product and almost tangent structures. In the present paper we define some tensors in the GF-structure space and study the properties of these tensors. In Section 4 we define some stuctures on the GF-structure and obtain an inclusion relation between them.

1. Introduction. We consider a differentiable manifold M of class C^∞. Let there exist on M a vector-valued linear function J of class C^∞ such that

$$J^2X = a^2X \tag{1.1}$$

for an arbitrary vector field X, and a is complex number. Such a structure is defined to be a GF-structure [1]. It is well known that M is endowed with an almost complex, almost product, and almost tangent structures according as $a = \pm i$ or $a = \pm 1$ or $a = 0$, respectively. If the GF-structure is endowed with a Hermitian structure, i.e. the metric tensor $\langle\ ,\ \rangle$ satisfying

$$\langle JX, JY\rangle = -a^2\langle X, Y\rangle \tag{1.2)a}$$

or

$$\langle JX, Y\rangle = -\langle X, JY\rangle, \tag{1.2)b}$$

then it is defined to be an H-structure [1] subordinate to the GF-structure.

DEFINITION. A bilinear function φ is said to be *pure in two slots* if $\varphi(JX, JY) - a^2\varphi(X, Y) = 0$. It is said to be *hybrid in its slots* if $\varphi(JX, JY) + a^2\varphi(X, Y) = 0$.

Let us consider on M, equipped with an H-structure, a tensor F of type $(0, 2)$ such that

$$F(X, Y) \stackrel{\mathrm{df}}{=} \langle JX, Y\rangle = -\langle X, JY\rangle = -F(Y, X). \tag{1.3}$$

It is easy to verify the following relations:

(1.4)a $$F(JX, Y) = a^2\langle X, Y\rangle = -F(X, JY),$$

(1.4)b $$F(JX, JY) = -a^2\langle JX, Y\rangle = a^2\langle X, JY\rangle = -a^2F(X, Y),$$

and

(1.5) $$(\nabla_X F)(Y, Z) = \langle(\nabla_X J)Y, Z\rangle,$$

where ∇_X is a connexion satisfying the following conditions:

(1.6) $$\begin{cases} \text{(I)} & \nabla_{(X+Y)}Z = \nabla_X Z + \nabla_Y Z, \\ \text{(II)} & \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \\ \text{(III)} & \nabla_{fX} Y = f\nabla_X Y, \\ \text{(IV)} & \nabla_X(fY) = f\nabla_X Y + (Xf)Y, \\ \text{(V)} & X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \end{cases}$$

where X, Y, Z are arbitrary vector fields, f is a real-valued function, and $\langle\ ,\ \rangle$ denotes metric tensor defined by (1.2). Hence the connexion ∇_X defined by (1.6) is a connexion with torsion, the torsion tensor $T(X, Y)$ being given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

In view of (1.4)b, we have

PROPOSITION 1.1. *The 2-form F defined by* (1.3) *is hybrid.*

2. Nijenhuis tensor. The *Nijenhuis tensor* with respect to J is a vector-valued bi-linear function N given by [1]

(2.1) $$N(X, Y) = J^2[X, Y] - J[JX, Y] - J[X, JY] + [JX, JY] + \\ + J^2T(X, Y) - JT(JX, Y) - JT(X, JY) + T(JX, JY),$$

where T is the torsion tensor with respect to the connexion ∇_X.

We can easily verify the following relations:

(2.2)a $$N(X, Y) = -N(Y, X),$$

(2.2)b $$N(JX, Y) = N(X, JY) = -JN(X, Y)$$

and

(2.2)c $$N(JX, JY) = -JN(X, JY) = -JN(JX, Y) = a^2N(X, Y).$$

The above relations provide the proof of the following:

PROPOSITION 2.1. *The Nijenhuis tensor N in a GF-structure is pure in X and Y.*

THEOREM 2.1. *Let M be equipped with an H-structure. If we put*

$$N_1(X, Y) = J^2[X, Y]+[JX, JY]-J[X, JY]-J[JX, Y] \tag{2.3}$$

and

$$N_2(X, Y) = J^2T(X, Y)+T(JX, JY)-JT(X, JY)-JT(JX, Y), \tag{2.4}$$

then

$$N(X, Y) = N_1(X, Y)+N_2(X, Y).$$

Moreover, N_1 and N_2 are skew-symmetric and pure in X and Y.

Proof. By the definition of N_1, N_2 and N, we get

$$N(X, Y) = N_1(X, Y)+N_2(X, Y).$$

Moreover, N_1 and N_2 are obviously skew-symmetric, while

$$\begin{aligned} N_1(JX, JY) &= J^2[JX, JY]+[J^2X, J^2Y]-J[JX, J^2Y]-J[J^2X, JY] \\ &= a^2N_1(X, Y) \end{aligned}$$

proves that $N_1(X, Y)$ is pure in X and Y. Similarly it can be verified that $N_2(X, Y)$ is pure in X and Y.

THEOREM 2.2. *If the torsion tensor $T(X, Y)$ satisfies*

$$T(JX, Y) = JT(X, Y), \tag{2.5}$$

then it is pure in its slots and

$$N(X, Y) = N_1(X, Y).$$

Proof. In view of (2.5) and (1.1) we have

$$T(JX, JY) = JT(X, JY) = -JT(JY, X) = a^2T(X, Y).$$

Hence $T(X, Y)$ is pure in X, Y. Using (2.5) in (2.1) we get

$$N(X, Y) = N_1(X, Y).$$

THEOREM 2.3. *If the torsion tensor $T(X, Y)$ satisfies*

$$T(JX, Y) = -JT(X, Y), \tag{2.6}$$

then it is pure in its slots and

$$N(X, Y) = N_1(X, Y)+4a^2T(X, Y).$$

Proof. As in Theorem 2.2 we get

$$T(JX, JY) = a^2T(X, Y),$$

which proves that $T(X, Y)$ is pure in X, Y. Also using (2.6) in (2.1) we have

$$N(X, Y) = N_1(X, Y)+4a^2T(X, Y).$$

COROLLARY 2.1. *In an almost tangent space, if the torsion tensor satisfies* (2.6), *then*

$$N(X, Y) = N_1(X, Y).$$

THEOREM 2.4. *The tensor fields* N_1 *and* N_2 *defined by* (2.3) *and* (2.4), *respectively, satisfy the following relations*:

$$N_1(JX, Y) = N_1(X, JY) = -JN_1(X, Y)$$

and

$$N_2(JX, Y) = N_2(X, JY) = -JN_2(X, Y).$$

Proof. From (2.3) we have

$$N_1(JX, Y) = J^2[JX, Y]+[J^2X, JY]-J[J^2X, Y]-J[JX, JY].$$

Using (1.1) in the above equation we get

$$N_1(JX, Y) = -JN_1(X, Y).$$

Similarly

$$N_1(X, JY) = -JN_1(X, Y)$$

and

$$N_2(X, JY) = N_2(JX, Y) = -JN_2(X, Y).$$

THEOREM 2.5. *Let us put*

$$'N(X, Y, Z) = -a^2\langle N(X, Y), Z\rangle = -\langle N(JX, JY), Z\rangle; \tag{2.7}$$

then

$$'N(X, Y, Z) = -'N(Y, X, Z) \tag{2.8}$$

and

$$'N(JX, Y, Z) = 'N(X, JY, Z) = 'N(X, Y, JZ). \tag{2.9}$$

Proof. Relation (2.8) follows directly from (2.2)a while (2.9) follows from (2.7) and (2.2)b.

From the above theorem we get the following:

COROLLARY 2.2. *Tensor* $'N(X, Y, Z)$ *defined by* (2.7) *satisfies the following relations*:

$$'N(JX, JY, Z)='N(X, JY, JZ)='N(JX, Y, JZ)=a^2\,'N(X, Y, Z). \tag{2.10}$$

THEOREM 2.6. *If we put*

$$'N_1(X, Y, Z) = -a^2\langle N_1(X, Y), Z\rangle = -\langle N_1(JX, JY), Z\rangle \tag{2.11}$$

and

$$'N_2(X, Y, Z) = -a^2\langle N_2(X, Y), Z\rangle = -\langle N_2(JX, JY), Z\rangle, \tag{2.12}$$

then

$$'N(X, Y, Z) = 'N_1(X, Y, Z) + 'N_2(X, Y, Z), \tag{2.13}$$

$$'N_1(X, Y, Z) = -'N_1(Y, X, Z), \tag{2.14}$$

$$'N_2(X, Y, Z) = -'N_2(Y, X, Z), \tag{2.15}$$

$$'N_1(JX, Y, Z) = 'N_1(X, JY, Z) = 'N_1(X, Y, JZ), \tag{2.16}$$

$$'N_2(JX, Y, Z) = 'N_2(X, JY, Z) = 'N_2(X, Y, JZ). \tag{2.17}$$

Proof. (2.13) follows from the definition of $'N_1$, $'N_2$ and N, (2.14) and (2.15) follow from the skew-symmetry of $N_1(X, Y)$ and $N_2(X, Y)$, while (2.16) and (2.17) follow from (1.2)b and Theorem 2.4.

THEOREM 2.7. *Let us put*

$$M(X, Y) = (\nabla_X J)JY - (\nabla_{JY} J)X; \tag{2.18}$$

then

$$N(X, Y) = M(X, Y) - M(Y, X). \tag{2.19}$$

Proof. We have

$$\begin{aligned} M(X, Y) - M(Y, X) &= (\nabla_X J)JY - (\nabla_{JY} J)X - (\nabla_Y J)JX + (\nabla_{JX} J)Y \\ &= a^2 \nabla_X Y - J\nabla_X JY - \nabla_{JY} JX + J\nabla_{JY} X - \\ &\quad - a^2 \nabla_Y X + J\nabla_Y JX + \nabla_{JX} JY - J\nabla_{JX} Y \\ &= a^2(\nabla_X Y - \nabla_Y X) + (\nabla_{JX} JY - \nabla_{JY} JX) - \\ &\quad - J(\nabla_{JX} Y - \nabla_Y JX) - J(\nabla_X JY - \nabla_{JY} X) \\ &= N(X, Y). \end{aligned}$$

PROPOSITION 2.2. *Tensor* $M(X, Y)$ *defined by* (2.18) *satisfies the following relations*:

$$M(JX, Y) = -M(JY, X) \tag{2.20a}$$

and

$$M(JX, JY) = -a^2 M(Y, X). \tag{2.20b}$$

Proof.

$$M(JX, Y) = (\nabla_{JX} J)JY - (\nabla_{JY} J)JX = -M(JY, X).$$

From the above relation we have

$$M(JX, JY) = -a^2 M(Y, X).$$

THEOREM 2.8. *Let us put*

$$'M(X, Y, Z) = -a^2 \langle M(X, Y,)Z\rangle = \langle M(JY, JX), Z\rangle; \tag{2.21}$$

then

$$'N(X, Y, Z) = 'M(X, Y, Z) - 'M(Y, X, Z). \tag{2.22}$$

Furthermore, $'M(X, Y, Z)$ is skew-symmetric in Y and Z if and only if

$$(\nabla_{JY}J)Z+(\nabla_{JZ}J)Y = 0.$$

Proof. Relation (2.22) follows from (2.19) and (2.21). Furthermore

$$\begin{aligned} M(X, Y, Z)+M(X,Z, Y) &= (\nabla_X F)(JY, Z)-(\nabla_{JY}F)(X,Z)+ \\ &\quad +(\nabla_X F)(JZ, Y)-(\nabla_{JZ}F)(X, Y) \\ &= \langle(\nabla_{JY}J)Z, X\rangle+\langle(\nabla_{JZ}J)Y, X\rangle, \end{aligned}$$

which proves the theorem.

Relations (2.20)a, (2.20)b and (2.22) provide the proof of the following:

COROLLARY 2.3. *We have*

$$'N(JX, Y, Z)= 2\,'M(X, JY, Z) \quad \text{and} \quad 'N(JX, JY, Z)= 2a^2\,'M(X, Y, Z).$$

THEOREM 2.9. *If the covariant derivative ∇_X satisfies*

$$(\nabla_{JY}J)Z+(\nabla_{JZ}J)Y = 0,$$

for arbitrary Y, Z, then

$$\begin{aligned} &'N(X, Y, Z)+'N(Y, Z, X)+'N(Z, X, Y) \\ &\qquad = 2\,'M(X, Y, Z)+2\,'M(Y, Z, X)+2\,'M(Z, X, Y). \end{aligned}$$

Proof. The relation follows immediately from Theorem 2.8. In view of (2.18) and (2.21) we get

THEOREM 2.10. *We have*

$$\begin{aligned} &'M(JX, Y, Z)+'M(JY, Z, X)+'M(JZ, X, Y) \\ &\qquad = -2a^2[\nabla_{JX}F(JY, Z)+\nabla_{JY}F(JZ, X)+\nabla_{JZ}F(JX, Y)]. \end{aligned}$$

3. Remarks.

1. If for an H-structure $(\nabla_X J)Y = 0$ is satisfied, then we define M to be a Kähler space in the broad sense. It follows immediately from (2.10) and (2.19) that for a Kähler space in the broad sense $M(X, Y)$ and $N(X, Y)$ both vanish. Consequently, $'M(X, Y, Z)$ and $'N(X, Y, Z)$ also vanish.

2. If for an H-structure $(\nabla_{JX}J)JY-a^2(\nabla_X J)Y = 0$ is satisfied, then we say that M is a quasi-Kählerian space in the broad sense. It can easily be deduced that in a quasi-Kählerian space in the broad sense $M(JX, Y) = -JM(X, Y)$.

4. It can easily be checked that the covariant derivative $\nabla_X F$ and exterior derivative dF of the 2-form F are given by

$$(\nabla_X F)(Y, Z) = \langle(\nabla_X J)Y, Z\rangle \tag{4.1}$$

and

$$dF(X, Y, Z) = \mathop{\mathfrak{S}}_{X,Y,Z} \nabla_X(F)(Y, Z) + \mathop{\mathfrak{S}}_{X,Y,Z} F\big(T(X, Y), Z\big), \tag{4.2}$$

where $\mathop{\mathfrak{S}}_{X,Y,Z}$ denotes the cyclic sum over X, Y, Z.

THEOREM 4.1. *Using the above formulae we get the following results*:

$$N(X, Y) = (\nabla_X J)JY + (\nabla_{JX} J)Y - (\nabla_{JY} J)X - (\nabla_Y J)JX, \tag{4.3}$$

$$\begin{gathered} 2a^2(\nabla_X F)(Y, Z) = a^2 dF(X, Y, Z) + dF(X, JY, JZ) - \\ - \langle N(Y, JZ), X\rangle - a^2 \mathop{\mathfrak{S}}_{X,Y,Z} F\big(T(X, Y), Z\big) - \mathop{\mathfrak{S}}_{X,JY,JZ} F\big(T(X, JY), JZ\big). \end{gathered} \tag{4.4}$$

$$\begin{aligned} 2(\nabla_{JX} F)(JY, Z) - 2a^2(\nabla_X F)(Y, Z) \\ = -a^2 dF(X, Y, Z) - dF(X, JY, JZ) + dF(Z, JX, JY) + \\ + dF(Y, JZ, JX) + a^2 \mathop{\mathfrak{S}}_{X,Y,Z} F\big(T(X, Y), Z\big) + \mathop{\mathfrak{S}}_{X,JY,JZ} F\big(T(X, JY), JZ\big) - \\ - \mathop{\mathfrak{S}}_{JX,JY,Z} F\big(T(Z, JX), JY\big) - \mathop{\mathfrak{S}}_{JX,Y,JZ} F\big(T(JZ, JX), Y\big), \end{aligned} \tag{4.5}$$

$$\begin{aligned} 2(\nabla_{JX} F)(JY, Z) + 2a^2(\nabla_X F)(Y, Z) \\ = \langle N(X, JY), Z\rangle - \langle N(X, Z), JY\rangle - \langle N(JY, Z), X\rangle. \end{aligned} \tag{4.6}$$

Proof. The proof of (4.3) follows from (2.1), (1.5) and the fact that

$$\nabla_X Y - \nabla_Y X = [X, Y] + T(X, Y),$$

while (4.4), (4.5) and (4.6) are consequences of (2.5) and the formula

$$(\nabla_X F)(JY, Z) = (\nabla_X F)(Y, JZ). \tag{4.7}$$

We shall call an H-structure space
a *BK-space* iff

$$\nabla_X(J) = 0,$$

a *B A K-space* iff

$$dF(X, Y, Z) = \mathop{\mathfrak{S}}_{X,Y,Z} F\big(T(X, Y), Z\big),$$

a *B N K-space* iff

$$\nabla_X(J)Y + \nabla_Y(J)X = 0,$$

a *B Q K-space* iff

$$(\nabla_{JX} J)JY - a^2(\nabla_X J)Y = 0,$$

and a *B H-space* iff

$$N(X, Y) = 0$$

for all X, Y, Z.

We study the inclusion relation between the special spaces defined above and prove

THEOREM 4.2.

$$BK \begin{array}{l} \subseteq BAK \\ \\ \subseteq BNK \end{array} \subseteq BQK \quad \textit{and} \quad BK \subseteq BH.$$

Proof. We prove that $BK \subseteq BAK$ follows from (4.2); $BAK \subseteq BQK$ follows from (4.2) and (4.5), while $BK \subseteq BH$ follows from (4.3). It is obvious that $BK \subseteq BNK$, while $BNK \subseteq BQK$ is a consequence of (4.7).

References

[1] K. L. Duggal, *On differentiable structures defined by algebraic equations*, I, Nijenhuis tensor, Tensor N.S. 22 (1971).

[2] A. Gray, *Some examples of almost Hermitian manifolds*, Illinois J. Math. 10 (1966), p. 353–366.

[3] N. J. Hicks, *Notes on Differential Geometry*, New York 1964.

[5] S. Koto, *Some theorems on almost Käheleriаn spaces*, J. Math. Soc. Japan 12 (1960), p. 422–433.

[6] K. D. Singh and R. C. Srivastava, *Almost Hermitian manifolds with torsion*, to be published in Matematicki Vesnik.

DEPARTMENT OF MATHEMATICS AND ASTRONOMY
LUCKNOW UNIVERSITY
Lucknow, India

Reçu par la Rédaction le 7. 1. 1972