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Determination of geometric objects of the type [2, 2, 1] with a linear homogeneous transformation formula *

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Introduction. The purpose of the present paper is to determine all differential geometric objects of the first class, with two components, in a two-dimensional space (i.e. objects of the type [2, 2, 1]—cf. [2], p. 15) which have a linear homogeneous transformation formula (1). Geometric objects with two components (of an arbitrary class and in a space of an arbitrary dimension) have been determined and classified by J. E. Pensov [8]; however, for the determination of these objects Pensov applied the theory of Lie groups, and thus tacitly assumed that the functions occurring in the transformation formula of the object are analytic. Therefore he has not obtained all objects of the type investigated. In the present paper we shall determine all linear homogeneous objects of the type [2, 2, 1], without any suppositions whatever about the functions occurring in the transformation formula. In the sequel we shall investigate the equivalence (cf. [2], [3]) of the objects obtained in the case where the functions occurring in the transformation formula are measurable.

§ 1. We seek purely differential geometric objects of the first class with two components in a two-dimensional space with the linear homogeneous transformation formula:

$$(1)\qquad \begin{aligned} \omega_1' &= f_{11}\omega_1 + f_{12}\omega_2\,, \\ \omega_2' &= f_{21}\omega_1 + f_{22}\omega_2\,. \end{aligned}$$

Having adopted the matrix notation

$$(2)\qquad \Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}, \quad \Omega' = \begin{Vmatrix} \omega_1' \\ \omega_2' \end{Vmatrix}, \quad F = \begin{Vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{Vmatrix},$$

* The results of this paper have been announced (without proofs) in our note [7].

(1) Linear differential geometric objects with two components of the type J (cf. [2], p. 47) in a two-dimensional space were determined by means of differentiation (and thus under suitable differentiability conditions) by A. Woźniacki in 1952. However, he has never published this result.

we can write formula (1) shortly in the form

$$\Omega' = F\cdot\Omega . \tag{3}$$

The functions f_{ij} in formula (1) as well as the function F in (3), depend by definition on the derivatives

$$A_\lambda^{\lambda'} \overset{\mathrm{df}}{=} \frac{\partial\bar{\xi}^{\lambda'}}{\partial\xi^\lambda}, \qquad \lambda = 1, 2;\ \lambda' = 1', 2',$$

of the new variables with respect to the old ones. Writing shortly

$$A = \begin{Vmatrix} A_1^{1'} & A_2^{1'} \\ A_1^{2'} & A_2^{2'} \end{Vmatrix},$$

we can write (3) more precisely as

$$\Omega' = F(A)\cdot\Omega . \tag{4}$$

From the group property of the transformation formula (4) it follows that the function $F(A)$ must satisfy for all regular matrices A, B the functional equation

$$F(A\cdot B) = F(A)\cdot F(B) , \tag{5}$$

(which corresponds to a system of four equations for the functions f_{ij}) and must be a regular matrix for every regular matrix A.

The general solution of equation (5) has been given in our paper [5] (cf. also [6]). The non-singular matrix-function $F(A)$, fulfilling equation (5) for all regular matrices A and B, must have one of the following forms:

$$F(A) = C\cdot\begin{Vmatrix} \varphi(J) & 0 \\ 0 & \varphi(J) \end{Vmatrix}\cdot A\cdot C^{-1}, \tag{6}$$

$$F(A) = C\cdot\begin{Vmatrix} \varphi_1(J) & 0 \\ 0 & \varphi_2(J) \end{Vmatrix}\cdot C^{-1}, \tag{7}$$

$$F(A) = C\cdot\begin{Vmatrix} \varphi(J) & 0 \\ 0 & \varphi(J) \end{Vmatrix}\cdot\begin{Vmatrix} 1 & a(J) \\ 0 & 1 \end{Vmatrix}\cdot C^{-1}, \tag{8}$$

$$F(A) = C\cdot\begin{Vmatrix} k(J) & -s(J) \\ s(J) & k(J) \end{Vmatrix}\cdot C^{-1}. \tag{9}$$

In formulae (6)-(9) $J = \det A$, C is an arbitrary regular matrix, $\varphi(x)$, $\varphi_1(x), \varphi_2(x)$ are arbitrary functions satisfying the functional equation

$$\varphi(xy) = \varphi(x)\varphi(y) , \qquad xy \neq 0 , \tag{10}$$

$a(x)$ is an arbitrary function satisfying the functional equation

$$a(xy) = a(x) + a(y) , \qquad xy \neq 0 , \tag{11}$$

and $k(x)$ and $s(x)$ are arbitrary functions satisfying the system of functional equations

$$\begin{aligned} k(xy) &= k(x)k(y) - s(x)s(y)\,, \\ s(xy) &= k(x)s(y) + s(x)k(y)\,, \end{aligned} \qquad xy \neq 0\,, \tag{12}$$

and the condition

$$s(x) \not\equiv 0 \tag{13}$$

(cf. [1], [6]).

Hence we immediately get the following

THEOREM 1. *Every differential geometric object of the first class with two components in a two-dimensional space with a linear homogeneous transformation formula must be of form* (4) (*with shortened notation* (2)), *where* $F(A)$ *is one of the matrices* (6)-(9).

§ 2. In the sequel we shall always assume that the functions f_{ij} occurring in the transformation formula (1) are measurable. Therefore we must insert in formulae (6)-(9), in the place of the functions $\varphi, \varphi_1, \varphi_2$, a, k, s, the measurable solutions of the corresponding equations (10)-(12). These solutions are given by the following formulae (cf. [1], [6]):

For equation (10)

$$\varphi(x) = |x|^d \quad \text{or} \quad \varphi(x) = |x|^d \operatorname{sgn} x\,, \qquad d = \text{const}; \tag{14}$$

for equation (11)

$$a(x) = c\ln|x|\,, \qquad c = \text{const}; \tag{15}$$

for equation (12)

$$k(x) = |x|^d \cos(c\ln|x|)\,, \qquad s(x) = |x|^d \sin(c\ln|x|)\,, \tag{16}$$

or

$$k(x) = (\operatorname{sgn} x)|x|^d \cos(c\ln|x|)\,, \qquad s(x) = (\operatorname{sgn} x)|x|^d \sin(c\ln|x|)\,, \tag{17}$$

$c, d = \text{const}$. It follows from (13) that in formulae (16) and (17) $c \neq 0$.

If an object Ω is transformed according to formula (4) and if in a particular coordinate system we have $\omega_1 \overset{*}{=} 0$ and $\omega_2 \overset{*}{=} 0$ (²), then $\omega_1' = 0$ and $\omega_2' = 0$ in every coordinate system. The object Ω is then a couple of scalars and thus an object of class zero. Since in the present paper we aim at determining and classifying the objects of the strictly first class, in the sequel we shall always assume that

$$\omega_1^2 + \omega_2^2 > 0\,. \tag{18}$$

Relation (18) is invariant under transformation of the coordinate system.

(²) The sign $\overset{*}{=}$ means that the equality holds in a particular coordinate system (i. e. the relation need not be invariant). The notation is due to J. A. Schouten (cf. [10], p. 2).

We shall give now the definition of the equivalence (similarity) of geometric objects (cf. [2], [3]):

DEFINITION 1. Geometric objects Ω and Σ are called *equivalent* (or *similar*) if there exists an invertible function H such that the relation

$$\Omega = H(\Sigma)$$

holds in every coordinate system (is invariant under transformation of the coordinate system).

The relation of the equivalence of objects is reflexive, symmetric and transitive.

Now we shall prove the following

LEMMA 1. *Every object Σ with the transformation formula*

$$\Sigma' = C\cdot F(A)\cdot C^{-1}\cdot \Sigma, \tag{19}$$

where C is a regular matrix, is equivalent to an object Ω with transformation formula (4).

Proof. Let us put $H(\Sigma) \overset{\mathrm{df}}{=} C^{-1}\cdot\Sigma$. The function $H(\Sigma)$ is invertible, since the matrix C is regular. Moreover, we have by (19)

$$H(\Sigma') = C^{-1}\cdot\Sigma' = C^{-1}\cdot C\cdot F(A)\cdot C^{-1}\cdot\Sigma = F(A)\cdot C^{-1}\cdot\Sigma = F(A)\cdot H(\Sigma),$$

and consequently the object $\Omega \overset{\mathrm{df}}{=} H(\Sigma)$ is transformed according to formula (4), which was to be proved.

Given n objects $\Omega_1, \ldots, \Omega_n$ (each with an arbitrary number of components) we may unify them in one new object $\Omega = (\Omega_1, \ldots, \Omega_n)$ (cf. [2], p. 13).

LEMMA 2. *If an object Ω_i is equivalent to an object Σ_i (for $i = 1, \ldots, n$), then the object $\Omega = (\Omega_1, \ldots, \Omega_n)$ is equivalent to the object $\Sigma = (\Sigma_1, \ldots, \Sigma_n)$.*

LEMMA 3. *Object $\Omega = (\Omega_1, \ldots, \Omega_n)$ is equivalent to the object $\Omega^* = (\Omega_{\mu_1}, \ldots, \Omega_{\mu_n})$, where $\mu_1, \ldots, \mu_n$ is an arbitrary permutation of the sequence $1, \ldots, n$.*

We omit the simple proofs of Lemmas 2 and 3.

LEMMA 4. *Object Ω with transformation formula* (4) *with the function $F(A)$ of form* (6) *is not equivalent to an object with transformation formula* (4) *with the function $F(A)$ of form* (7), (8) *or* (9).

Proof. For an indirect proof let us suppose that the objects

$$\Omega' = C\cdot\begin{Vmatrix}\varphi(J) & 0\\ 0 & \varphi(J)\end{Vmatrix}\cdot A\cdot C^{-1}\cdot\Omega, \tag{20}$$

and

$$\Sigma' = G(J)\cdot\Sigma,$$

where $G(J)$ denotes one of the matrices (7)-(9), are equivalent. According to Lemma 1 object Ω is equivalent to an object $\bar{\Omega} = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\bar{\Omega}' = \begin{Vmatrix} \varphi(J) & 0 \\ 0 & \varphi(J) \end{Vmatrix} \cdot A \cdot \bar{\Omega}. \tag{21}$$

Since the relation of the equivalence of objects is transitive, the objects $\bar{\Omega}$ and Σ are also equivalent. Consequently there exists an invertible function $H(\Sigma)$ such that the relation

$$\bar{\Omega} = H(\Sigma) \tag{22}$$

is invariant under transformation of the coordinate system. Since we can always choose a transformation of the coordinate system in such a manner that the derivatives $A_\lambda^{\lambda'}$ assume given values (with the only restriction $\det(A_\lambda^{\lambda'}) \neq 0$), let us put $A = \begin{Vmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{Vmatrix}$. Then we have $J = 1$, $\varphi(J) = 1$, and—as easily follows from (10)-(13)—$G(J) = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$ in all cases (7)-(9). Consequently $\Sigma' = \Sigma$ and by (22) $\bar{\Omega}' = \bar{\Omega}$. But according to (21) $\bar{\Omega}' = A \cdot \bar{\Omega}$, i.e.

$$\omega_1' = 2\omega_1, \quad \omega_2' = \tfrac{1}{2}\omega_2.$$

Hence it follows that $\omega_1 = 0$ and $\omega_2 = 0$, which contradicts relation (18). Thus objects Ω and Σ cannot be equivalent.

Now we shall recall the definitions of some particular geometric objects with one or two components in a two-dimensional space.

DEFINITION 2. Object with two components $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = A \cdot \Omega$$

is called a *contravariant vector*.

Object $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = (A^{-1})^T \cdot \Omega \ (^3)$$

is called a *covariant vector*.

Object $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = \begin{Vmatrix} \operatorname{sgn} J & 0 \\ 0 & \operatorname{sgn} J \end{Vmatrix} \cdot A \cdot \Omega$$

is called a *contravariant G-vector*.

(3) X^T denotes the transposed matrix of the matrix X.

Object $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = \begin{Vmatrix} \operatorname{sgn} J & 0 \\ 0 & \operatorname{sgn} J \end{Vmatrix} \cdot (A^{-1})^T \cdot \Omega$$

is called a *covariant G-vector*.

Object $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = \begin{Vmatrix} |J|^p & 0 \\ 0 & |J|^p \end{Vmatrix} \cdot A \cdot \Omega, \quad p \neq 0,$$

is called a *contravariant vector-W-density* (*Weyl-density*) *of weight* $-p$.

Object $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = \begin{Vmatrix} |J|^p & 0 \\ 0 & |J|^p \end{Vmatrix} \cdot (A^{-1})^T \cdot \Omega, \quad p \neq 0,$$

is called a *covariant vector-W-density of weight* $-p$.

Object $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = \begin{Vmatrix} |J|^p \operatorname{sgn} J & 0 \\ 0 & |J|^p \operatorname{sgn} J \end{Vmatrix} \cdot A \cdot \Omega, \quad p \neq 0,$$

is called a *contravariant vector-G-density* (*ordinary density*) *of weight* $-p$.

Object $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = \begin{Vmatrix} |J|^p \operatorname{sgn} J & 0 \\ 0 & |J|^p \operatorname{sgn} J \end{Vmatrix} \cdot (A^{-1})^T \cdot \Omega, \quad p \neq 0,$$

is called a *covariant vector-G-density of weight* $-p$.

Object with one component ω with the transformation formula

$$\omega' = |J|^p \omega, \quad p \neq 0,$$

is called a *W-density* (*Weyl-density*) *of weight* $-p$.

Object ω with the transformation formula

$$\omega' = |J|^p (\operatorname{sgn} J)\, \omega, \quad p \neq 0,$$

is called a *G-density* (*ordinary density*) *of weight* $-p$.

Object ω with the transformation formula

$$\omega' = (\operatorname{sgn} J)\, \omega$$

is called a *biscalar* (4).

(4) More exactly, a biscalar is an object with one component which can assume only two different values. The name has been introduced by S. Gołąb. J. A. Schouten [9] calls such an object *W-scalar*. Physicists sometimes call it *pseudoscalar*.

Object ω with the transformation formula

$$\omega' = \omega$$

is called *scalar*.

The last four of the above mentioned objects (W-density, G-density, biscalar, scalar) are called *objects of type* J (cf. [2], p. 47).

The scalar is an object of class zero, all the remaining objects defined above are of the first class. All G-densities are equivalent to the G-density of weight -1, and similarly all W-densities are equivalent to the W-density of weight -1. The function establishing the equivalence is in both cases

$$H(\sigma) = |\sigma|^{1/p} \operatorname{sgn} \sigma .$$

On the other hand, no G-density is equivalent to any W-density. For, if a G-density ω were equivalent to a W-density σ, then there would have to exist an invertible function H such that the relation

$$\omega = H(\sigma) \tag{23}$$

should hold in every coordinate system. But after a transformation of the coordinate system such that $J = -1$ we have $\sigma' = \sigma$ and $\omega' = -\omega$, and hence $-\omega = H(\sigma)$. Hence it follows by (23) that $\omega = 0$. Thus ω is a scalar and not a density, as has been assumed.

§ 3. On account of Lemma 1 and formulae (14), object Ω with transformation formula (4) with a measurable function $F(A)$ of form (6) is equivalent to a contravariant vector-W-density, to a contravariant vector-G-density, to a contravariant vector, or to a contravariant G-vector. Now we shall prove

LEMMA 5. *Contravariant vector-W-densities of different weights are not equivalent; neither is a contravariant vector equivalent to any vector-W-density. Similarly, contravariant vector-G-densities of different weights are not equivalent; neither is a contravariant G-vector equivalent to any vector-G-density. Moreover, no contravariant vector-G-density or contravariant G-vector is equivalent to any contravariant vector-W-density or to a contravariant vector.*

Proof. Let $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ be a contravariant vector-W-density of weight $-p$, or a contravariant vector (for $p = 0$):

$$\Omega' = \begin{Vmatrix} |J|^p & 0 \\ 0 & |J|^p \end{Vmatrix} \cdot A \cdot \Omega , \tag{24}$$

and let $\Sigma = \begin{Vmatrix} \sigma_1 \\ \sigma_2 \end{Vmatrix}$ be a contravariant vector-W-density of weight $-q$:

$$\Sigma' = \begin{Vmatrix} |J|^q & 0 \\ 0 & |J|^q \end{Vmatrix} \cdot A \cdot \Sigma , \tag{25}$$

and let us assume that

(26) $$p \neq q\,.$$

Let us suppose that objects (24) and (25) are equivalent, i.e. that there exists an invertible function H such that in every coordinate system we have

(27) $$\Omega = H(\Sigma)\,.$$

Thus $\Omega' = H(\Sigma')$. Taking into account (24), (25) and (27) we get hence

(28) $$\begin{Vmatrix} |J|^p & 0 \\ 0 & |J|^p \end{Vmatrix} \cdot A \cdot H(\Sigma) = H\left(\begin{Vmatrix} |J|^q & 0 \\ 0 & |J|^q \end{Vmatrix} \cdot A \cdot \Sigma\right).$$

The function H should satisfy equation (28) for all matrices A with $J \neq 0$ and for all $\Sigma = \begin{Vmatrix} \sigma_1 \\ \sigma_2 \end{Vmatrix}$ such that $\sigma_1^2 + \sigma_2^2 > 0$.

Now let us put in (28) $A = \begin{Vmatrix} 1/\sigma_1 & 0 \\ \sigma_2 & -\sigma_1 \end{Vmatrix}$, $\sigma_1 \neq 0$. Then $J = 1$ and $A \cdot \Sigma = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}$. Writing $H_0 = H\left(\begin{Vmatrix} 1 \\ 0 \end{Vmatrix}\right)$, we obtain the formula

(29) $$H(\Sigma) = A^{-1} \cdot H_0 = \begin{Vmatrix} \sigma_1 & 0 \\ \sigma_2 & 1/\sigma_1 \end{Vmatrix} \cdot H_0\,,$$

valid for all $\sigma_1 \neq 0$ and σ_2.

Next let us choose in (28) $A = \begin{Vmatrix} \varrho & 0 \\ 0 & \varrho \end{Vmatrix}$, $\varrho > 0$. Taking into account (29) we obtain

$$\begin{Vmatrix} \varrho^{2p+1}\sigma_1 & 0 \\ \varrho^{2p+1}\sigma_2 & \varrho^{2p+1}/\sigma_1 \end{Vmatrix} \cdot H_0 = \begin{Vmatrix} \varrho^{2q+1}\sigma_1 & 0 \\ \varrho^{2q+1}\sigma_2 & 1/\varrho^{2q+1}\sigma_1 \end{Vmatrix} \cdot H_0\,,$$

whence it follows, according to (26), that H_0 must have the form $\begin{Vmatrix} 0 \\ h \end{Vmatrix}$, $h = \text{const}$. Thus we get from (29)

$$H(\Sigma) = \begin{Vmatrix} 0 \\ h/\sigma_1 \end{Vmatrix}.$$

But such a function cannot be invertible. Consequently the conjecture that objects (24) and (25) are equivalent has been false.

The proof for contravariant vector-G-densities is quite analogical. The proof that vector-G-densities and G-vectors are not equivalent to vector-W-densities and vectors is quite similar to the proof of the analogical property for G-densities and W-densities with one component (cf. § 2).

Since covariant vector-densities and vectors have the transformation formulae of form (4) with $F(A)$ of form (6), they must be equivalent to contravariant vector-densities or vectors. More precisely this equivalence ([5]) is established by the following

LEMMA 6. *Every covariant vector-W-density (vector-G-density) of weight $-p$ is equivalent to a vector-G-density (vector-W-density) of weight $-p+1$, or (if $p=1$) to a contravariant G-vector (vector). In particular, a covariant vector (G-vector) is equivalent to a contravariant vector-G-density (vector-W-density) of weight* 1.

Proof. The lemma follows immediately from the identity

$$(A^{-1})^T = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \frac{1}{J} & 0 \\ 0 & \frac{1}{J} \end{Vmatrix} \cdot A \cdot \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}^{-1},$$

and from Lemma 1.

§ 4. On account of Lemmas 1, 2, 3 and of formulae (14), an object with transformation formula (4) with a measurable function $F(A)$ given by (7) is equivalent to a pair of densities, to a pair consisting of a density and a biscalar, to a pair consisting of a density and a scalar, to a pair of biscalars, to a pair consisting of a biscalar and a scalar, or to a pair of scalars (this latter object is of class zero). Now we shall prove that an object Ω with transformation formula (4) with a measurable function $F(A)$ given by (8) is also equivalent to one of the above-mentioned objects.

According to Lemma 1 and formulae (14) and (15), we may confine ourselves to objects $\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}$ with the transformation formula

$$\Omega' = \begin{Vmatrix} |J|^d & 0 \\ 0 & |J|^d \end{Vmatrix} \cdot \begin{Vmatrix} 1 & c\ln|J| \\ 0 & 1 \end{Vmatrix} \cdot \Omega, \tag{30}$$

or

$$\Omega' = \begin{Vmatrix} |J|^d \operatorname{sgn} J & 0 \\ 0 & |J|^d \operatorname{sgn} J \end{Vmatrix} \cdot \begin{Vmatrix} 1 & c\ln|J| \\ 0 & 1 \end{Vmatrix} \cdot \Omega. \tag{31}$$

Moreover, we may assume that $c \neq 0$, for it is obvious that in the contrary case object (30) represents a pair of W-densities or scalars, and object (31) represents a pair of G-densities or biscalars.

In order to prove the equivalence announced we start from the following

([5]) The equivalence of vector-densities (in a space of an arbitrary dimension) has been recently investigated by S. Gołąb [4]. The following lemma is a particular case of his results.

LEMMA 7. *There exists a function $r(a)$ such that*

$$(32)\qquad \frac{d}{dt}\left[\sqrt{t}\,e^{r(a)t}\left(\sqrt{t^2+1}\right)^a\right] > 1$$

for all $t \in (0, \infty)$, $a \in (-\infty, \infty)$.

Proof. Let us put

$$(33)\qquad f(t;\, a, r) \overset{\mathrm{df}}{=} \sqrt{t}\,e^{rt}\sqrt{t^2+1}^{\,a}, \qquad t \in (0, \infty), \qquad a, r \in (-\infty, \infty).$$

We have

$$\frac{d}{dt} f(t;\, a, r) = \frac{1}{2\sqrt{t}} e^{rt}\left(\sqrt{t^2+1}\right)^a + r\sqrt{t}\,e^{rt}\left(\sqrt{t^2+1}\right)^a + at\sqrt{t}\,e^{rt}\left(\sqrt{t^2+1}\right)^{a-2},$$

i.e.

$$(34)\qquad \frac{d}{dt} f(t;\, a, r) = \sqrt{t}\,e^{rt}\left(\sqrt{t^2+1}\right)^a\left(\frac{1}{2t} + r + \frac{at}{t^2+1}\right).$$

Since the function

$$(35)\qquad g(t;\, a) \overset{\mathrm{df}}{=} \sqrt{t}\left(\sqrt{t^2+1}\right)^a\left(\frac{1}{2t} + \frac{at}{t^2+1}\right)$$

increases boundlessly as $t \to 0$, one can find a $\delta > 0$ (in general depending on a) such that

$$(36)\qquad g(t;\, a) > 1 \quad \text{for} \quad t \in (0, \delta).$$

Hence we have for $t \in (0, \delta)$ and $r > 0$

$$(37)\qquad \frac{d}{dt} f(t;\, a, r) = g(t;\, a)e^{rt} + r\sqrt{t}\,e^{rt}\left(\sqrt{t^2+1}\right)^a > g(t;\, a)e^{rt} > 1.$$

The function $f(t;\, a, r)$ is continuous with respect to t and positive for $t \in \langle\delta, \infty)$, moreover for $r > 0$ we have $\lim\limits_{t\to\infty} f(t;\, a, r) = \infty$. Consequently $f(t;\, a, r)$ is in $\langle\delta, \infty)$ bounded from below by a positive constant:

$$(38)\qquad f(t;\, a, r) \geqslant \beta > 0 \quad \text{for} \quad t \in \langle\delta, \infty),$$

where β depends on a, but it can be made independent of r if we confine ourselves to $r \geqslant 1$. Similarly the function $\dfrac{1}{2t} + \dfrac{at}{t^2+1}$ is continuous for $t \in \langle\delta, \infty)$, and tends to zero as $t \to \infty$. Consequently it is bounded from below for $t \in \langle\delta, \infty)$ (the bounding constant can depend on a). Thus one can choose $r(a)$ in such a manner that

$$(39)\qquad \frac{1}{2t} + r(a) + \frac{at}{t^2+1} > \frac{1}{\beta} \quad \text{for} \quad t \in \langle\delta, \infty), \qquad a \in (-\infty, \infty).$$

From (37), (34), (38) and (39) follows relation (32), which was to be proved.

LEMMA 8. *If $d \neq 0$, then the object with transformation formula (30) is equivalent to a pair of W-densities of weight -1, or to a pair consisting of a W-density of weight -1 and a scalar. Similarly the object with the transformation formula (31) is equivalent to a pair of G-densities of weight -1, or to a pair consisting of a G-density of weight -1 and a scalar.*

Proof. Let us put

$$(40)\qquad \begin{aligned} \omega_1 &= \begin{cases} \left(\sqrt{\sigma_1^2+\sigma_2^2}\right)^d \left[-c\sqrt{\left|\dfrac{\sigma_1}{\sigma_2}\right|}\, e^{r(d)|\sigma_1/\sigma_2|}\operatorname{sgn}\sigma_1 + \right. & \\ \qquad\qquad \left. + c\left(\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)\left|\dfrac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right|^d \operatorname{sgn}\sigma_2\right] & \text{for} \quad \sigma_2 \neq 0, \\ |\sigma_1|^d \operatorname{sgn}\sigma_1 \quad \text{for} \quad \sigma_1 \neq 0, \quad \sigma_2 = 0, & \\ 0 \quad \text{for} \quad \sigma_1 = \sigma_2 = 0, & \end{cases} \\ \omega_2 &= \begin{cases} |\sigma_2|^d \operatorname{sgn}\sigma_2 & \text{for} \quad \sigma_2 \neq 0, \\ 0 & \text{for} \quad \sigma_2 = 0, \end{cases} \end{aligned}$$

where $r(d)$ is so chosen that for $a = d$ relation (32) hold. We shall prove that relations (40) are invertible, i.e. that σ_1 and σ_2 are uniquely determined by ω_1 and ω_2. From the second relation of (40) we obtain

$$(41)\qquad \sigma_2 = \begin{cases} |\omega_2|^{1/d} \operatorname{sgn}\omega_2 & \text{for} \quad \omega_2 \neq 0, \\ 0 & \text{for} \quad \omega_2 = 0. \end{cases}$$

Now, if $\omega_2 = 0$, then we have by (40)

$$\omega_1 = \begin{cases} |\sigma_1|^d \operatorname{sgn}\sigma_1 & \text{for} \quad \sigma_1 \neq 0, \\ 0 & \text{for} \quad \sigma_1 = 0, \end{cases}$$

i.e.

$$\sigma_1 = \begin{cases} |\omega_1|^{1/d} \operatorname{sgn}\omega_1 & \text{for} \quad \omega_1 \neq 0, \\ 0 & \text{for} \quad \omega_1 = 0. \end{cases}$$

If, instead, $\omega_2 \neq 0$, then according to (41) also $\sigma_2 \neq 0$, and we obtain from (40)

$$(42)\qquad \omega_1 = -c|\sigma_2|^d \operatorname{sgn}\sigma_2 \left(\sqrt{\left|\frac{\sigma_1}{\sigma_2}\right|}\, e^{r(d)|\sigma_1/\sigma_2|}\left(\sqrt{\left(\frac{\sigma_1}{\sigma_2}\right)^2+1}\right)^d \operatorname{sgn}\frac{\sigma_1}{\sigma_2} - \right.$$
$$\left. - \ln\sqrt{\left(\frac{\sigma_1}{\sigma_2}\right)^2+1} - \ln|\sigma_2|\right).$$

Let us put

$$f(t) \stackrel{\text{df}}{=} \sqrt{|t|}\, e^{r(d)|t|}\left(\sqrt{t^2+1}\right)^d \operatorname{sgn} t - \ln\sqrt{t^2+1}.$$

Since

$$\frac{d}{dt}\ln\sqrt{t^2+1} = \frac{t}{t^2+1} \leqslant \frac{1}{2},$$

on account of Lemma 7 we have for $t > 0$

$$f'(t) > \tfrac{1}{2}.$$

Thus for $t \in \langle 0, \infty)$ the function $f(t)$ increases from zero to ∞. On the other hand, it is evident that for $t \in (-\infty, 0)$ the function $f(t)$ increases from $-\infty$ to zero. Thus the function $f(t)$ increases in $(-\infty, \infty)$ from $-\infty$ to $+\infty$, being, of course, continuous. Consequently, having determined σ_2 from relation (41), we can uniquely determine σ_1/σ_2 from (42), and then also σ_1.

Now, let σ_1 and σ_2 be W-densities of weight -1:

$$\sigma_1' = |J|\sigma_1, \quad \sigma_2' = |J|\sigma_2 \tag{43}$$

(or a scalar and a W-density if $\sigma_1 = 0$ or $\sigma_2 = 0$). We shall verify in what manner σ_1 and σ_2 are transformed with a change of the coordinate system. If $\sigma_2 \neq 0$ (this condition is invariant under transformations of the coordinate system, and thus implies the relation $\sigma_2' \neq 0$), then we obtain from (40) and (43)

$$\begin{aligned} \omega_1' &= |J|^d\omega_1 + |J|^d\omega_2 c\ln|J|, \\ \omega_2' &= |J|^d\omega_2, \end{aligned} \tag{44}$$

which, after putting $\Omega = \left\| \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\|$, can be written in form (30). And if $\sigma_2 = 0$, then we obtain from (40) and (43)

$$\omega_1' = |J|\omega_1, \quad \omega_2' = 0,$$

which again can be written in form (44), and thus, after putting $\Omega = \left\| \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\|$, also in form (30).

Consequently formulae (40) establish the equivalence of object (30) and the pair of objects (43); the latter represents a pair of W-densities of weight -1 (when $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$), or a pair consisting of a W-density of weight -1 and a scalar (when $\sigma_1\sigma_2 = 0$, $\sigma_1^2 + \sigma_2^2 > 0$. The case $\sigma_1 = 0$ and $\sigma_2 = 0$ would imply $\omega_1 = 0$ and $\omega_2 = 0$, which, according to (18), is impossible).

Now let σ_1 and σ_2 be G-densities of weight -1:

$$\sigma_1' = J\sigma_1, \quad \sigma_2' = J\sigma_2 \tag{45}$$

(or a scalar and a G-density, if $\sigma_1 = 0$ or $\sigma_2 = 0$). We shall verify in what manner ω_1 and ω_2 are now transformed with a change of the coordinate system. If $\sigma_2 \neq 0$, then we obtain from (40) and (45)

$$\begin{aligned} \omega_1' &= |J|^d(\operatorname{sgn}J)\,\omega_1 + |J|^d(\operatorname{sgn}J)\,\omega_2 c\ln|J|, \\ \omega_2' &= |J|^d(\operatorname{sgn}J)\,\omega_2, \end{aligned} \tag{46}$$

which, after putting $\Omega = \left\| \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\|$, can be written in form (31). And if $\sigma_2 = 0$, then we obtain from (40) and (45)

$$\omega_1' = |J|^d(\operatorname{sgn} J)\,\omega_1\,, \qquad \omega_2' = 0\,,$$

which again can be written in form (46), and thus, after putting $\Omega = \left\| \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\|$, also in form (31).

Consequently formulae (40) establish the equivalence of object (31) and the pair of objects (45); the latter represents a pair of G-densities of weight -1 (when $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$) or a pair consisting of a G-density of weight -1 and a scalar (when $\sigma_1\sigma_2 = 0$, $\sigma_1^2+\sigma_2^2 > 0$).

Thus the lemma has been completely proved.

LEMMA 9. *If $d = 0$, then object with transformation formula* (30) *is equivalent to the object consisting of a W-density of weight -1 and a scalar, or of a pair of scalars. Similarly the object with transformation formula* (31) *is equivalent to the object consisting of a G-density of weight -1 and a biscalar, or of a biscalar and a scalar.*

Proof. Let $\Omega = \left\| \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\|$ be an object with the transformation formula (30). Since $d = 0$, formula (30) can be written as

$$\omega_1' = \omega_1 + c\ln|J|\,\omega_2\,, \qquad \omega_2' = \omega_2\,. \tag{47}$$

If $\omega_2 = 0$, then object (47) consists of two independent objects

$$\omega_1' = \omega_1\,, \qquad \omega_2' = 0\,,$$

i.e. it is a pair of scalars. If, on the other hand, $\omega_2 \neq 0$, then the object with components

$$\sigma_1 = \exp\frac{\omega_1}{c\omega_2}\,, \qquad \sigma_2 = \omega_2\,, \tag{48}$$

is equivalent to the object Ω, for we have from (48)

$$\omega_1 = c\sigma_2\ln\sigma_1\,, \qquad \omega_2 = \sigma_2\,,$$

and thus formulae (48) are invertible. Further we get from (48) and (47) with a change of the coordinate system

$$\sigma_1' = \exp\frac{\omega_1'}{c\omega_2'} = \exp\left(\frac{\omega_1}{c\omega_2} + \ln|J|\right) = |J|\,\sigma_1\,,$$
$$\sigma_2' = \sigma_2\,,$$

which means that σ_1 is a W-density of weight -1 and σ_2 is a scalar.

Now let $\omega = \left\| \begin{matrix} \omega_1 \\ \omega_2 \end{matrix} \right\|$ be an object with the transformation formula (31). Since $d = 0$, formula (31) can be written as

$$\omega_1' = (\operatorname{sgn} J)\,\omega_1 + (\operatorname{sgn} J)(c \ln|J|)\,\omega_2\,, \qquad \omega_2' = (\operatorname{sgn} J)\,\omega_2\,. \tag{49}$$

If $\omega_2 = 0$, then Ω consists of two objects with one component

$$\omega_1' = (\operatorname{sgn} J)\,\omega_1\,, \qquad \omega_2' = 0\,,$$

i.e. it consists of a biscalar and a scalar (according to (18) necessarily $\omega_1 \neq 0$). If, on the other hand, $\omega_2 \neq 0$, then the object with components

$$\sigma_1 = \left(\exp \frac{\omega_1}{c\omega_2}\right)(\operatorname{sgn} \omega_2)\,, \qquad \sigma_2 = \omega_2\,, \tag{50}$$

is equivalent to the object Ω, and, moreover, we have by (49) and (50) with a change of the coordinate system

$$\begin{aligned} \sigma_1' &= \left(\exp \frac{\omega_1'}{c\omega_2'}\right)(\operatorname{sgn} \omega_2') = \left[\exp\left(\frac{\omega_1}{c\omega_2} + \ln|J|\right)\right](\operatorname{sgn} \omega_2)(\operatorname{sgn} J) \\ &= J\left(\exp \frac{\omega_1}{c\omega_2}\right)(\operatorname{sgn} \omega_2) = J\sigma_1\,, \\ \sigma_2' &= (\operatorname{sgn} J)\,\sigma_2\,, \end{aligned}$$

which means that σ_1 is a G-density of weight -1 and σ_2 is a biscalar. This completes the proof of the lemma.

§ 5. Now we shall deal with the objects with transformation formula (4) with a measurable function $F(A)$ given by (9). According to Lemma 1 and formulae (16) and (17) we may confine ourselves to the objects with the transformation formula

$$\Omega' = \left\| \begin{matrix} |J|^d \cos(c \ln|J|) & -|J|^d \sin(c \ln|J|) \\ |J|^d \sin(c \ln|J|) & |J|^d \cos(c \ln|J|) \end{matrix} \right\| \cdot \Omega\,, \tag{51}$$

or

$$\Omega' = \left\| \begin{matrix} |J|^d (\operatorname{sgn} J) \cos(c \ln|J|) & -|J|^d (\operatorname{sgn} J) \sin(c \ln|J|) \\ |J|^d (\operatorname{sgn} J) \sin(c \ln|J|) & |J|^d (\operatorname{sgn} J) \cos(c \ln|J|) \end{matrix} \right\| \cdot \Omega\,, \tag{52}$$

where, according to (13), $c \neq 0$.

LEMMA 10. *If $d \neq 0$, then the object with transformation formula* (51) *is equivalent to a pair of W-densities of weight -1, or to a pair consisting of a W-density of weight -1 and a scalar. Similarly the object with transformation formula* (52) *is equivalent to a pair of G-densities of weight -1, or to a pair consisting of a G-density of weight -1 and a scalar.*

Proof. Let us put

$$(53)\qquad \begin{aligned}\omega_1 &= \left(\sqrt{\sigma_1^2+\sigma_2^2}\right)^d\left[\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\cos\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)-\frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\sin\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)\right],\\ \omega_2 &= \left(\sqrt{\sigma_1^2+\sigma_2^2}\right)^d\left[\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\sin\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)+\frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\cos\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)\right].\end{aligned}$$

We shall show that relations (53) are invertible, i.e. that to every couple of numbers ω_1, ω_2 fulfilling condition (18) there corresponds exactly one couple σ_1, σ_2, $\sigma_1^2+\sigma_2^2>0$, such that relations (53) hold. For this purpose we put

$$(54)\qquad \sigma_1 = \varrho\cos a\,, \qquad \sigma_2 = \varrho\sin a\,.$$

Relations (54) establish a one-to-one correspondence between the couples σ_1, σ_2 $(\sigma_1^2+\sigma_2^2>0)$ and the couples ϱ, a from the region $\varrho>0$, $0\leqslant a<2\pi$. Formulae (53) will now have the form

$$(55)\qquad \begin{aligned}\omega_1 &= \varrho^d\cos(a+c\ln\varrho)\,,\\ \omega_2 &= \varrho^d\sin(a+c\ln\varrho)\,.\end{aligned}$$

In order to prove the invertibility of relations (53) it is enough to show that to every couple ω_1, ω_2 fulfilling (18) there corresponds exactly one couple ϱ, a, $\varrho>0$, $0\leqslant a<2\pi$, such that relations (55) hold. Taking the squares of both sides of (55) and adding the corresponding sides, we obtain

$$\varrho = (\omega_1^2+\omega_2^2)^{1/2d}.$$

Hence and from (55) we can determine $a+c\ln\varrho$, and thus also a, up to a multiplicity of 2π. Taking into account the condition $a\in\langle 0, 2\pi)$ we can determine a uniquely. This proves the invertibility of relations (53).

Now let σ_1 and σ_2 be W-densities of weight -1

$$(56)\qquad \sigma_1' = |J|\sigma_1\,, \qquad \sigma_2' = |J|\sigma_2$$

(or a scalar and a W-density if $\sigma_1 = 0$ or $\sigma_2 = 0$). We shall verify in what manner ω_1 and ω_2 are transformed with a change of the coordinate system. We have from (53) and (56)

$$\begin{aligned}\omega_1' &= |J|^d\left(\sqrt{\sigma_1^2+\sigma_2^2}\right)^d\Bigg[\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\cos\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}+c\ln|J|\right)-\\ &\qquad\qquad -\frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\sin c\ln\left(\sqrt{\sigma_1^2+\sigma_2^2}+c\ln|J|\right)\Bigg]\\ &= |J|^d\left(\sqrt{\sigma_1^2+\sigma_2^2}\right)^d\left[\left(\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\cos\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)-\right.\right.\end{aligned}$$

$$-\frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\sin\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)\Big)\cos(c\ln|J|)-$$

$$-\left(\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}\sin\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)+\frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\cos\left(c\ln\sqrt{\sigma_1^2+\sigma_2^2}\right)\right)\sin(c\ln|J|)\Big]$$

$$=|J|^d\omega_1\cos(c\ln|J|)-|J|^d\omega_2\sin(c\ln|J|)\,,$$

and similarly

$$\omega_2' = |J|^d\omega_1\sin(c\ln|J|)+|J|^d\omega_2\cos(c\ln|J|)\,,$$

which, after putting $\Omega = \begin{Vmatrix}\omega_1\\ \omega_2\end{Vmatrix}$, can be written in form (51). Consequently object (51) is equivalent to a pair of W-densities of weight -1 (when $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$), or to a pair consisting of a W-density of weight -1 and a scalar (when $\sigma_1 = 0$ or $\sigma_2 = 0$; from condition (18) and the form of formulae (53) it follows that we must have $\sigma_1^2+\sigma_2^2 > 0$).

Now let σ_1 and σ_2 be G-densities of weight -1

$$\sigma_1' = J\sigma_1\,,\qquad \sigma_2' = J\sigma_2 \tag{57}$$

(or a scalar and a G-density if $\sigma_1 = 0$ or $\sigma_2 = 0$). It follows from (53) and (57) that with a change of the coordinate system ω_1 and ω_2 are transformed according to the formulae

$$\omega_1' = |J|^d(\operatorname{sgn}J)\,\omega_1\cos(c\ln|J|)-|J|^d(\operatorname{sgn}J)\,\omega_2\sin(c\ln|J|)\,,$$
$$\omega_2' = |J|^d(\operatorname{sgn}J)\,\omega_1\sin(c\ln|J|)+|J|^d(\operatorname{sgn}J)\,\omega_2\cos(c\ln|J|)\,,$$

which, after putting $\Omega = \begin{Vmatrix}\omega_1\\ \omega_2\end{Vmatrix}$, can be written in form (52). Consequently object (52) is equivalent to a pair of G-densities of weight -1, or to a pair consisting of a G-density of weight -1 and a scalar. This completes the proof of the lemma.

In the case where $d = 0$ formulae (51) and (52) have the form

$$\Omega' = \begin{Vmatrix}\cos(c\ln|J|) & -\sin(c\ln|J|)\\ \sin(c\ln|J|) & \cos(c\ln|J|)\end{Vmatrix}\cdot\Omega \tag{58}$$

and

$$\Omega' = \begin{Vmatrix}(\operatorname{sgn}J)\cos(c\ln|J|) & -(\operatorname{sgn}J)\sin(c\ln|J|)\\ (\operatorname{sgn}J)\sin(c\ln|J|) & (\operatorname{sgn}J)\cos(c\ln|J|)\end{Vmatrix}\cdot\Omega \tag{59}$$

respectively. We shall prove the following

LEMMA 11. *Neither of the objects* (58) *and* (59) *is equivalent to any object consisting of two objects of type J with one component. Neither are the objects* (58) *and* (59) *equivalent to each other.*

Proof. Let Ω be an object with transformation formula (58) or (59). For an indirect proof let us suppose that there exists an invertible function

H such that $H(\Omega) = \begin{Vmatrix} h_1(\Omega) \\ h_2(\Omega) \end{Vmatrix}$ is an object consisting of a pair of objects of type J. Thus we have

$$h_1' = \varphi_1(J) h_1, \qquad h_2' = \varphi_2(J) h_2,$$

where $\varphi_1(x)$ and $\varphi_2(x)$ denote some of the functions

$$\varphi(x) = |x|^d, \quad \varphi(x) = |x|^d \operatorname{sgn} x, \quad \varphi(x) = \operatorname{sgn} x, \quad \varphi(x) \equiv 1$$

($d \neq 0$). Now we effect a transformation of the coordinate system such that $J = e^{2k\pi/c}$. Then we have according to (58) or (59) $\Omega' = \Omega$ and thus also $h_1' = h_1$ and $h_2' = h_2$. Hence it follows that $h_i = 0$, $\varphi_i(x) \equiv 1$, or $\varphi_i(x) = \operatorname{sgn} x$ ($i = 1, 2$), and this means that H is a pair of scalars, or a pair of biscalars, or a pair consisting of a scalar and a biscalar. Consequently $H(\Omega)$ can assume at most two different values, while the set of values assumed by Ω is (on account of (18)) infinite. This contradicts the supposition that the function $H(\Omega)$ is invertible.

Similarly, objects (58) and (59) cannot be equivalent to each other, because under a transformation of the coordinate system with $J = -1$ the components of object (58) remain unchanged, while the components of object (59) change the sign. This completes the proof of the lemma.

Formulae (58) and (59) give two families of objects, since they contain the parameter c. The equivalence of objects of these families is established by the following

LEMMA 12. *Every object with transformation formula* (58) (*or* (59)) *with a parameter* ($-c$) *is equivalent to an object with transformation formula* (58) (*or* (59)) *with the parameter* c. *On the other hand, two objects with transformation formula* (58) (*or* (59)) *with parameters* c_1 *and* c_2, *where* $|c_1| \neq |c_2|$, *are not equivalent.*

Proof. Let

$$F_1(J) = \begin{Vmatrix} \cos(c_1 \ln|J|) & -\sin(c_1 \ln|J|) \\ \sin(c_1 \ln|J|) & \cos(c_1 \ln|J|) \end{Vmatrix}$$

and

$$F_2(J) = \begin{Vmatrix} \cos(c_2 \ln|J|) & -\sin(c_2 \ln|J|) \\ \sin(c_2 \ln|J|) & \cos(c_2 \ln|J|) \end{Vmatrix}$$

and let $c_1 = -c_2$. Further let Ω_1 be an object with the transformation formula

(60) $$\Omega_1' = F_1(J) \cdot \Omega_1 .$$

But, as one can easily verify,

$$F_1(J) = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \cdot F_2(J) \cdot \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}^{-1} .$$

Thus, by Lemma 1, the object Ω_1 is equivalent to an object Ω_2 with the transformation formula

$$\Omega_2' = F_2(J)\cdot\Omega_2 . \tag{61}$$

On the other hand, let $|c_1| \neq |c_2|$, say $|c_1| > |c_2|$, and let us suppose that the objects Ω_1 and Ω_2 (with transformation formulae (60) and (61) respectively) are equivalent:

$$\Omega_1 = H(\Omega_2) . \tag{62}$$

We effect a change of the coordinate system such that $J = \exp(2\pi/c_1)$. Then $F_1(J) = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$ and Ω_1 remains unchanged, while

$$F_2(J) = \begin{Vmatrix} \cos 2\pi\dfrac{c_2}{c_1} & -\sin 2\pi\dfrac{c_2}{c_1} \\ \sin 2\pi\dfrac{c_2}{c_1} & \cos 2\pi\dfrac{c_2}{c_1} \end{Vmatrix}$$

and consequently Ω_2 is changed (by the assumption $c \neq 0$ (cf. p. 84) we have $0 < |c_2/c_1| < 1$). Thus the function H in (62) cannot be invertible and the objects Ω_1 and Ω_2 cannot be equivalent.

The proof for objects with transformation formula (59) is analogous.

§ 6. As we have seen, in many cases objects with transformation formula (4) have turned out to be equivalent to an object consisting of a pair of objects of type J with one component. Here arises the problem of the equivalence of the latter objects. The following lemma solves this problem:

LEMMA 13. *Every object consisting of two objects of type J with one component is equivalent to a pair of W-densities of weight -1, to a pair of G-densities of weight -1, to a pair of biscalars, or to a pair of scalars. These four objects are not equivalent to one another.*

Proof. Taking into account Lemmas 2 and 3 and the fact that every object of type J with one component is equivalent to a W-density of weight -1, to a G-density of weight -1, to a biscalar, or to a scalar, in order to prove the first part of the lemma it is enough to show that every "mixed" object (i.e. an object consisting of two different ones of the above-mentioned objects with one component) is equivalent to one of the objects occurring in the assertion of the lemma.

Let σ_1 and σ_2 be a W-density and a G-density of weight -1 respectively:

$$\sigma_1' = |J|\sigma_1 , \qquad \sigma_2' = J\sigma_2 .$$

Then the object

$$\omega_1 = \sqrt{|\sigma_1\sigma_2|}\operatorname{sgn}\sigma_1\sigma_2 , \qquad \omega_2 = \sigma_2$$

is equivalent to the object $\Sigma = \begin{Vmatrix} \sigma_1 \\ \sigma_2 \end{Vmatrix}$ and represents a pair of G-densities of weight -1.

If σ_1 and σ_2 are a W-density of weight -1 and a biscalar respectively:

$$\sigma_1' = |J|\sigma_1\,, \qquad \sigma_2' = (\operatorname{sgn} J)\sigma_2\,,$$

then the object

$$\omega_1 = |\sigma_1|\sqrt{|\sigma_2|}\operatorname{sgn}\sigma_2\,, \qquad \omega_2 = \sigma_1\sigma_2$$

is equivalent to the object $\Sigma = \begin{Vmatrix} \sigma_1 \\ \sigma_2 \end{Vmatrix}$ and represents a pair of G-densities of weight -1. And if σ_1 and σ_2 are a G-density of weight -1 and a biscalar respectively:

$$\sigma_1' = J\sigma_1\,, \qquad \sigma_2' = (\operatorname{sgn} J)\sigma_2\,,$$

then the object

$$\omega_1 = \sigma_1\,, \qquad \omega_2 = |\sigma_1|\sigma_2$$

is equivalent to the object $\Sigma = \begin{Vmatrix} \sigma_1 \\ \sigma_2 \end{Vmatrix}$ and represents a pair of G-densities of weight -1.

If σ_1 is a W-density or a G-density of weight -1, or a biscalar, and σ_2 is a scalar different from zero, then the object

$$\omega_1 = \sigma_1\,, \qquad \omega_2 = \sigma_1\sigma_2$$

is equivalent to the object $\Sigma = \begin{Vmatrix} \sigma_1 \\ \sigma_2 \end{Vmatrix}$ and represents a pair of W-densities of weight -1, a pair of G-densities of weight -1, or a pair of biscalars, respectively. If, on the other hand, $\sigma_2 = 0$ (a separate treatment of this case is allowed, since the properties $\sigma_2 = 0$ and $\sigma_2 \neq 0$ are invariant under transformation of the coordinate system), then the object

$$\omega_1 = \omega_2 = \sigma_1$$

is equivalent to the object $\Sigma = \begin{Vmatrix} \sigma_1 \\ \sigma_2 \end{Vmatrix}$ and also represents a pair of W-densities of weight -1, a pair of G-densities of weight -1, or a pair of biscalars.

Now we proceed to the proof of the second part of the lemma. Let Σ be an object consisting of a pair of W-densities, or of a pair of scalars, and let Ω be an object consisting of a pair of G-densities, or of a pair of biscalars. If Σ and Ω were equivalent:

$$\Omega = H(\Sigma)\,,$$

then after a change of the coordinate system such that $J = -1$ we should have $\Sigma' = \Sigma$, and thus also $\Omega' = \Omega$; but the components of the object Ω after such a transformation change the sign.

Similarly an object Σ consisting of a pair of scalars or of a pair of biscalars cannot be equivalent to an object Ω consisting of a pair of densities, because Σ can assume at most two different values, while Ω assumes infinitely many. This completes the proof of the lemma.

Thus finally we have obtained the following result:

THEOREM 2. *Every differential geometric object of the first classwith two components in a two-dimensional space, with linear homogeneous transformation formula* (4) *is equivalent to one and only one of the following objects*:

1. *Contravariant vector-W-density of some definite weight* $-p$.
2. *Contravariant vector-G-density of some definite weight* $-p$.
3. *Contravariant vector.*
4. *Contravariant G-vector.*
5. *Pair of W-densities of weight* -1.
6. *Pair of G-densities of weight* -1.
7. *Pair of biscalars.*
8. *Object with transformation formula* (58) *with a definite parameter* $c > 0$.
9. *Object with transformation formula* (59) *with a definite parameter* $c > 0$.

Remark 1. Object consisting of a pair of scalars is not an object of the first class, but of class zero.

Remark 2. As far as we know, objects (58) and (59) have not been known till now, apart from the unpublished work of Woźniacki. In particular, they do not appear in [8].

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