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Continuability and estimates of solutions of $(a(t)\psi(x)x')' + c(t)f(x) = 0$

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Abstract. This paper discusses some basic properties of solutions of the non-linear equation $(a(t)\psi(x)x')' + c(t)f(x) = 0$. A necessary and sufficient condition is given for non-continuability of solutions. An estimate is given on minimal value assumed by a solution on a given interval, and also a proof of a restricted version of Leighton's variational theorem.

1. Introductory remarks. In non-linear mechanics it is quite common for the equations of motion of a system to be of the form:

$$\begin{aligned} \big(a(t)p(t)\big)' &= c(t)f(x) \\ \psi(x)x' &= -p(t) \end{aligned} \qquad \left(' \equiv \frac{d}{dt}\right).$$

For example a slight modification of the non-linear motion studied by Poincaré ([7], Chapter 7) leads to such systems of equations. Assuming in this case that the mass is varying with time and the strength of the field is also time dependent, leads to the following expression for the angular momentum (in the usual polar coordinate system r, θ):

$$p_\theta = a(t)r^2\theta'.$$

Introducing a constraint $r = \psi(\theta)$ we have

$$p_\theta = a(t)\big(\psi(\theta)\big)^2\theta',$$

$$p'_\theta = -c(t)\varphi(r, \theta) = -c(t)\varphi\big(\psi(\theta), \theta\big) = -c(t)f(\theta).$$

This system is equivalent to the single equation

$$\Big(a(t)\big(\psi(\theta)\big)^2\theta'\Big)' + c(t)f(\theta) = 0,$$

which is of the form

$$(1) \qquad \big(a(t)\psi(x)x'\big)' + c(t)f(x) = 0.$$

Here p is the angular momentum of a particle with mass $a(t)$ varying in time. $c(t)\varphi(r, \theta)$ is the moment about the origin of the forces acting on the particle. This moment is assumed to depend on position and also on time. We comment that there is no loss of generality in assuming $a(t) \geqslant 0$ when studying equation (1) ($c(t)$ can be negative).

2. Non-continuability of solutions of equation (1). We shall extend the result of [2] and consequently generalize older results such as [3] by proving that the basic arguments of [2] apply to equations of the type:

$$\big(a(t)\psi(x)x'\big)' + c(t)f(x) = 0. \tag{1}$$

In the remainder of this paper we assume the conditions:

$$a(t) \geqslant 0, \quad a(t) \in C^1[t_0, T] \quad (T = +\infty \text{ not excluded}), \quad c(t) \in C[t_0, T],$$

$$f(\xi) \in C(\xi_0, +\infty), \quad \psi(\xi) \in C^1(\xi_0, +\infty) \quad \text{for some } \xi_0 \in R, \text{ and that} \tag{$*$}$$

$$\xi\psi(\xi) \geqslant 0, \quad \xi f(\xi) > 0 \quad \text{whenever } \xi \neq 0.$$

We pose the initial value problem for equation (1), and seek a local solution $\hat{x}(t)$ of (1) satisfying conditions $\hat{x}(t_0) = x_0 > \xi_0$, $\hat{x}(t_0) = v_0$. The proof of the existence of local solutions is fairly routine and is thus omitted. Under the above conditions on $\psi(x), f(x)$ and if $a(t)$ and $c(t)$ have the same sign on $[t_0, t_2]$ it is easy to duplicate the argument of [3] to show that all solutions of (1) are continuable on to $[t_0, t_1]$. Therefore we shall assume that $a(t_0)c(t_0) < 0$.

We shall rewrite equation (1) as a pair of equations:

$$\begin{aligned} x' &= [a(t)]^{-1}\big[\psi\big(x(t)\big)\big]^{-1}\big(\sqrt{a(t)}\cdot p\big), \\ (\sqrt{a}\cdot p)' &= c(t)f\big(x(t)\big), \end{aligned} \tag{3}$$

where $p = \sqrt{\big(a(t)\big)}\,\psi\big(x(t)\big)x'(t)$ (where $\sqrt{\ }$ denotes the positive square root). This form will be useful in subsequent arguments, particularly is the proof of Theorem 1.

We shall now prove that under a certain integrability hypothesis on $f(x)$, $\psi(x)$, given any $t_1 > t_0$, x_0 and v_0 can be chosen so that the solutions of (1) satisfying $x(t_0) = x_0$, $v(t_0) = v_0$ is not continuable on to $[t_0, t_1]$.

We denote $F(x) = \int\limits_{x_0}^{x} \psi(\xi)f(\xi)\,d\xi$.

THEOREM 1. (Sufficiency condition for non-continuability.) *Assume conditions* $(*)$ *and* $a(t_0)\cdot c(t_0) < 0$. *If*

$$\int\limits_{x_0}^{\infty} \frac{\psi(x)}{[1+F(x)]^{1/2}}\,dx < \infty,$$

then, given any closed interval $[t_0, t_1]$ *and given any* x_0 *such that* $x_0 \geqslant \xi_0 > 0$, *there exists a number* $K_1 > 0$ *such that for any* $v_0 > K_1$ *a solution* $\hat{x}(t)$ *of* (1) *satisfying* $\hat{x}(t_0) = x_0$, $\hat{x}'(t_0) = v_0$ *is not continuable on* $[t_0, t_1]$.

We remark that an identical theorem can be stated for the case when $\psi(\xi)$ and $f(\xi)$ are defined on $(-\infty, \xi_0)$, with hypothesis

$$\int_{x_0}^{-\infty} \frac{\psi(x)}{[1+F(x)]^{1/2}}\, dx > -\infty,$$

and the signs of initial conditions reversed.

We comment that the original technique of the proof of this theorem was developed by Burton and Grimmer in [2].

We shall need two lemmas.

LEMMA 1. *Assume that a solution $\hat{x}(t)$ of* (1) *is non-continuable onto an interval $[t_0, \tilde{t}]$. Then there exists $t_1 \in (t_0, \tilde{t}]$ such that the solution is continuable onto $[t_0, t_1)$ and is not continuable onto $[t_0, t_2)$ for any $t_2 > t_1$.*

The proof is easy, and is omitted.

LEMMA 2. *Assume $a(t)\cdot c(t) < 0$ on some interval $[t_0, t_1]$ and that a solution $\hat{x}(t)$ of equation* (1) *satisfying hypothesis* (*) *is non-continuable onto $[t_0, t_1]$, while it is continuable onto $[t_0, t]$ for any $t \in [t_0, t_1)$. Then there exists an open interval $(t_2, t_1) \subset (t_0, t_1)$ such that $\hat{x}(t)\hat{x}'(t) > 0$ for all $t \in (t_2, t_1)$.*

Proof. Without any loss of generality we assume $a(t) > 0$ on $[t_0\ t_1]$. Assume, contrary to the assertion, that $\hat{x}(t)\cdot\hat{x}'(t)$ fails to be positive on every interval $(\bar{t}, t_1) \subset (t_0, t_1)$. The case $\hat{x}(t)\cdot\hat{x}'(t) < 0$ for all $t \in (\bar{t}, t_1)$ is easily disposed of, and we can consider the other possiblility, which turns out to be that $x'(t)$ vanishes on every interval $(\bar{t}, t_1)$, $t_0 < \bar{t} < t_1$. Let us first demonstrate that the infinite set of points (in $(\bar{t}, t_1)$) on which $\hat{x}'(t)$ vanishes has only one possible limit point, namely t_1. Otherwise if $\hat{x}'(t_i) = 0$ for some sequence of points $t_i \in (t_0, t_1)$ with $\lim_{i\to\infty} t_i = \tau < t_1$, then by continuity $\hat{x}'(\tau) = 0$ and $\hat{x}''(\tau) = 0$. Since $\hat{x}(t)$ is a solution of (1) it follows that $c(\tau)\cdot\big(f(\hat{x})(\tau)\big) = 0$, which is possible only if $f\big(\hat{x}(t)\big) = 0$, i. e. if $\hat{x}(\tau) = 0$, since $\hat{x}\cdot f(\hat{x}) > 0$ if $\hat{x} \neq 0$. $x \equiv 0$ satisfies the initial value problem for equation (1) with $\hat{x}(\tau) = 0$, $\hat{x}'(\tau) = 0$. Since the solution $\hat{x}(t) \equiv 0$, $t \geqslant \tau$, can be continued past $t = t_1$, we have a contradiction. Hence t_1 is the only possible limit point of zeros of $\hat{x}(t)$, and such zeros form a countable set of points on (t_0, t_1). Denote it by $\{t_j\}$, $j = 1, 2, \ldots$

Since $\hat{x}(t)$ satisfies equation (1):

$$\big(a(t)\psi(\hat{x})\big)'\hat{x}' + a(t)\psi(\hat{x})\hat{x}'' + c(t)f(\hat{x}) = 0,$$

we have

$$\hat{x}(t_j)\big[a(t_j)\psi\big(\hat{x}(t_j)\big)\hat{x}''(t_j) + c(t_j)f\big(\hat{x}(t_j)\big)\big] = 0.$$

Since $a(t_j) > 0$, and

$$\hat{x}(t_j)\psi\big(\hat{x}(t_j)\big) > 0 \qquad \hat{x}(t_j)f\big(\hat{x}(t_j)\big) > 0 \qquad \text{if } x(t_j) \neq 0,$$

we have either $x(t_j) = 0$ (which leads to a contradiction after repeating our previous argument using the uniqueness of solutions), or else $x''(t_j) \neq 0$ and is of constant sign at all points t_j, $j = 1, 2, \ldots$ But this is impossible since a C^1 function can not have a non-zero derivative of constant sign at more than one consecutive zero of the function. This proves the lemma.

Proof of Theorem 1. We have

$$\frac{d}{dt}(ap^2) = 2(\sqrt{a}p)(\sqrt{a}p)' = 2\big(a\psi(\hat{x})\hat{x}'\big)\big(a\psi(\hat{x})\hat{x}'\big)' \tag{4}$$
$$= -2a(t)c(t)\psi\big(\hat{x}(t)\big)f\big(\hat{x}(t)\big)\hat{x}'(t).$$

Hence for a given $t > t_0$ (and $t \in [t_0, t_1]$ and sufficiently close to t_0), we have

$$(ap^2)_t = (ap^2)_{t_0} - 2\int_{t_0}^{t} \big|a(\xi)c(\xi)\psi\big(\hat{x}(\xi)\big)f\big(\hat{x}(\xi)\big)\hat{x}'(\xi)\big|d\xi$$
$$\geqslant (ap^2)_{t_0} + 2m\int_{t_0}^{t} \psi\big(\hat{x}(\xi)\big)f\big(\hat{x}(\xi)\big)\hat{x}'(\xi)d\xi$$
$$= (ap^2)_{t_0} + 2m\int_{x_0}^{x(t)} \psi(\hat{x})f(\hat{x})dx$$
$$= (ap^2)_{t_0} + 2mF\big(\hat{x}(t)\big),$$

where $m > 0$ denotes $m = \min\big(-a(t)c(t)\big)$, $t \in [t_0, t_1]$. Note: $x' > 0$, $\psi(x)f(x) > 0$, on a sufficiently small interval $[t_0, \tilde{t}] \subset [t_0, t_1]$.

Observe that since $x_0 > 0, v_0 > 0$, $\hat{x}(t)$ is a monotone increasing function of t on a sufficiently small interval $[t_0, \tilde{t}]$, $\tilde{t} \in [t_0, t_1]$. This fact is sufficient for the purpose of this proof. Hence

$$\sqrt{a}p(t) \geqslant \big[(ap^2)_{t=t_0} + 2mF\big(x(t)\big)\big]^{1/2}, \qquad \forall t \in [t_0, \tilde{t}].$$

Write $c_0^2 = (ap^2)_{t=t_0}$.

Then for any $\tilde{t}$ for which $x(t)$ is monotone increasing and continuable onto $[t_0, \tilde{t}]$ we have:

$$(2m)^{-1/2}\int_{\hat{x}(t_0)}^{\hat{x}(\tilde{t})} \frac{\psi(x)\,dx}{[c_0^2/2m + F(x)]^{1/2}} \geqslant \int_{t_0}^{\tilde{t}} [a(\xi)]^{-1/2}d\xi. \tag{i}$$

Having assumed that the solution can be continued to $\tilde{t}$, we have

$$(2m)^{-1/2}\int_{x_0}^{\hat{x}(\tilde{t})} \frac{\psi(x)\,dx}{[c_0^2/2m + F(x)]^{1/2}} \geqslant \delta > 0, \tag{ii}$$

where

$$\delta = \int_{t_0}^{\tilde{t}} [a(t)]^{-1/2}dt.$$

Since

$$\int_{x_0}^{\infty} \frac{\psi(x)}{[1+F(x)]^{1/2}}\, dx < \infty,$$

it follows that

$$\int_{x_0}^{\infty} \frac{\psi(x)\, dx}{[c_0^2/2m+F(x)]^{1/2}} < \infty,$$

and the value of this integral depends continuously on the parameter c_0^2. Moreover,

$$\lim_{c_0 \to \infty} \int_{x_0}^{T} \frac{\psi(x)\, dx}{[c_0^2/2m+F(x)]^{1/2}} = 0 \quad \text{for any } T > x_0.$$

Hence c_0^2 can be chosen sufficiently large so that

$$\text{(iii)} \qquad \int_{x_0}^{\hat{x}(\tilde{t})} \frac{\psi(x)\, dx}{[c_0^2/2m+F(x)]^{1/2}} < (2m)^{1/2}\delta,$$

which contradicts inequality (ii), proving that with such choice of c_0 the solution $\hat{x}(t)$ is not continuable onto $[t_0, \tilde{t}] \subset [t_0, t_1]$. Since

$$\begin{aligned} c_0^2 &= a(t_0)p^2(t_0) = a^2(t_0)\psi^2\big(\hat{x}(t_0)\big)\big(\hat{x}'(t_0)\big)^2 \\ &= a^2(t_0)\psi^2(x_0)v_0^2, \end{aligned}$$

the remainder of the proof follows.

Choosing any $x_0 > 0$ we can now choose v_0 to satisfy condition (iii) (or vice versa). Hence the corresponding solution $\hat{x}(t, t_0, x_0, v_0)$ is not continuable onto the interval $[t_0, t_1] \supseteq [t_0, \tilde{t}]$. This completes the proof.

We comment that the continuity of $f(x)$ was not used in the proof and only local integrability was required. (Continuity was needed to assert the existence of solution.) We also comment that the result of [2] is obtained by putting $\psi(x) \equiv 1$, $a(t) \equiv 1$.

THEOREM 2. **(A necessary condition for non-continuability.)**

$$\text{(iv)} \qquad \int_{x_0}^{\infty} \frac{\psi(x)\, dx}{[1+F(x)]^{1/2}} = +\infty$$

implies that all solutions of (1) *are continuable onto* $[t_0, \infty)$.

Proof. It is sufficient to prove this theorem for initial conditions $\operatorname{sign}\big(x(t_0)\big) = \operatorname{sign}\big(x'(t_0)\big)$, since by Lemma 2 any non-continuable solution $\hat{x}(t)$ must have the property that $\hat{x}(t)$ and $\hat{x}'(t)$ assume the same sign for all t greater than some $\tau > t_0$ on the maximal interval $[t_0, t_1)$ of continuability of $\hat{x}(t)$. Without any loss of generality let us assume that $\hat{x}(t_0) > 0$,

$\hat{x}'(t_0) > 0$, and by way of contradiction let us assume that a solution satisfying condition (iv) is not continuable onto some interval $[t_0, t_1]$, while $\hat{x}(t)$ and $\hat{x}'(t)$ are positive on any open interval $[t_0, t)$ on which $\hat{x}(t)$ is continuable. We shall first consider the possibility $\limsup_{t\to t_1}(\hat{x}(t)) = +\infty$, while $\hat{x}(t)$ is defined on $[t_0, t_1)$, and show that this assumption leads to a contradiction. (The case $\liminf_{t\to t_1}(\hat{x}(t)) = -\infty$ is analogous.) As before we obtain the estimate:

$$a(t)p^2(t) \leqslant (ap^2)_{t=t_0} + 2M\cdot\left|\int_{t_0}^{t} \psi(x(\xi))f(x(\xi))x'(\xi)\,d\xi\right|,$$

$t\in[t_0, t_1)$, where $M = \max_{t\in[t_0,t_1]} |a(t)c(t)|$.

Note that without knowing the length of $[t_0, t_1]$ we cannot in general assume $a(t)c(t) < 0$ on $[t_0, t_1]$ even if $a(t_0)c(t_0) < 0$.

$$a(\tilde{t})p^2(\tilde{t}) \leqslant (ap^2)_{t=t_0} + 2M\cdot|F(x(\tilde{t})) - F(x(t_0))|,$$

$\tilde{t}\in[t_0, t_1)$. As before c_0^2 will denote $(ap^2)_{t=t_0}$.

Since $\limsup_{t\to t_1} \hat{x}(t) = +\infty$, we can choose a sequence of points $\tilde{t}_i$ converging to t_1 such that

$$a(\tilde{t}_i)p^2(\tilde{t}_i) \leqslant c_0^2 + 2M\big(F(\hat{x}(\tilde{t}_i)) - F(\hat{x}(t_0))\big),$$

while

$$\lim_{i\to\infty}\int_{t_0}^{t_i} \frac{\psi(\hat{x}(t))\hat{x}'(t)}{[1+F(\hat{x}(t))]^{1/2}}\,dt = +\infty.$$

Hence

$$\lim_{x\to\infty}\int_{x_0}^{\hat{x}(\tilde{t}_i)} \frac{\psi(x)\,dx}{[c_0^2 + 2M(F(x) - F(x_0))]^{1/2}} \leqslant \int_{t_0}^{t} [a(t)]^{-1/2}\,dt < \infty,$$

which is a contradiction. Lemma 2 easily takes care of the remainder of the proof.

Note. The possibility $\limsup |\hat{x}(t)| < \infty$, but $\hat{x}(t)$ cannot be continued to t_1, because $\lim_{t\to t_1}\hat{x}(t)$ does not exist, can be disposed of by the following argument. We assume as before that $\hat{x}(t)$ is defined on $[t_0, t_1)$. We examine the behavior along the trajectory $\hat{x}(t)$, $t\in[t_0, t_1)$, of the function:

$$a(t)p^2(t) = a^2(t)\psi^2(\hat{x}(t))(\hat{x}')^2(t) = c_0^2 - 2\int_{t_0}^{t} a(t)c(t)\psi(\hat{x}(t))f(\hat{x}(t))\,dt$$
$$\leqslant c_0^2 + 2\bar{M}\big(F(\hat{x}(t)) - F(\hat{x}(t_0))\big),$$

where $\bar{M} = (t_1 - t_0)\max_{t\in[t_0,t_1]} |a(t)c(t)|$

(see formula (4) and recall again that $\psi(x)f(x) \geqslant 0$).

Hence $\limsup_{t\to t_1} |p(t)| < \infty$. Choosing $\varepsilon > 0$ it follows that $x'(t) \in C^1[t_0, t_1)$ is bounded uniformly on the subset of $[t_0, t_1)$ on which $|x(t)| \geqslant \varepsilon$. It follows easily that zero is the only possible limit point for any sequence $\{x(t_i)\}$, $t_i \in [t_0, t_1)$, $\lim t_i = t_1$, such that $\lim_{t_i\to t_1} x'(t_i) = +\infty$. This possiblity is excluded by our hypothesis that $\lim_{t\to t_1} |x(t)|$ does not exist, completing the proof.

The understanding of the case when $a(t)$ and $c(t)$ are of the same sign is made easier by the following theorem, which indicates a basic behavior property of all solutions of [1].

THEOREM 3. *Let $\hat{x}(t)$ be a solution of* (1) *on interval* $[t_1, t_2]$, *such that* $\hat{x}'(t_1) = \hat{x}'(t_2) = 0$. *Let the signs of $a(t)$ and of $c(t)$ be the same on* $[t_1, t_2]$. *Then $\hat{x}(t)$ must vanish on the interval* $[t_1, t_2]$.

Proof. Assume to the contrary that $\hat{x}(t) \neq 0$, for all $t \in [t_1, t_2]$, hence $f(\hat{x}(t)) \neq 0$ and $\psi(\hat{x}(t)) \neq 0$ on $[t_1, t_2]$. It follows that the function $F(\hat{x}) = \int_0^{\hat{x}} \psi(\xi) f(\xi)\, d\xi = F(\hat{x}(t)) > 0$ on $[t_1, t_2]$, and that $f(\hat{x})$ and $f(\hat{x})F(\hat{x})$ are of the same sign of $[t_1, t_2]$. Consider the function

$$\varphi(t) = \frac{a(t)\psi(\hat{x}(t))\hat{x}'(t)}{F(\hat{x}(t))}$$

which is continuously differentiable on $[t_1, t_2]$. Clearly $\varphi(t_1) = \varphi(t_2) = 0$. However,

$$\begin{aligned}\int_{t_1}^{t_2} \varphi'(t)\, dt &= \int_{t_1}^{t_2} \left[-\frac{c(t) f(\hat{x}(t))}{F(\hat{x}(t))} - \frac{a(t)\psi^2(\hat{x}(t)) f(\hat{x}(t))(x')^2}{F^2(\hat{x}(t))} \right] dt \\ &= -\int_{t_1}^{t_2} \left[\frac{c(t) f(\hat{x}(t)) F(\hat{x}(t)) + a(t)\psi^2(\hat{x}(t)) f(\hat{x}(t))(\hat{x}')^2}{F^2(\hat{x}(t))} \right. dt \\ &= \varphi(t_2) - \varphi(t_1)\end{aligned}$$

is a non-zero number, since the integrand is of constant sign, because $a(t)$ and $c(t)$ are of the same sign, and $f(\hat{x})$ and $f(\hat{x})F(\hat{x})$ are of the same sign on $[t_1, t_2]$. This contradiction proves the theorem.

3. An estimate and a generalization of Leighton's variational theorem for equation (1). In this part of the paper we can prove an estimate of the type introduced by the author in [5] for equation (2). Equation (1) is of course a special case of (2):

$$(2) \qquad \big(A(t)\varphi(x, t)x'\big)' + C(t)f(x) = 0, \qquad t \in [t_0, \infty].$$

It is clear that $A(t)$ can be absorbed in $\varphi(x, t)$ without any loss of generality. I have left it deliberately in this form to make easier the identification of the special case when $\varphi(x, t) \equiv \psi(x)$. We assume that

(i) $f(\xi) \in C^1(-\infty, +\infty)$, $f(0) = 0$, $f(\xi) \neq 0$ if $\xi \neq 0$,

(ii) $A(t) \in C^1[t_0, \infty)$, $A(t) > 0$, $t \in [t_0, \infty)$, $C(t) \in C[t_0, \infty)$,

(iii) $\varphi(x, t) \in C^1[(-\infty, +\infty) \times C^1[t_0, \infty)]$, $\varphi(x, t) \neq 0$ if $x \neq 0$

$$\big(\forall t \in (t_0, \infty)\big).$$

Let us write $h(x, x', t) = \varphi(x, t)x'$. $J_{\alpha,\beta}\big(u, G(u)\big)$ will denote the following functional:

$$J_{\alpha,\beta}\big(u, G(u)\big) = \int_\alpha^\beta \big[A(t)h^2\big(u(t), u'(t), t\big) - C(t)G\big(u(t)\big)\big]dt,$$

where

$$u(t) \in C^1[\alpha, \beta], \quad g(u) \in C^1(-\infty, +\infty),$$
$$\varphi\big(u(t), t\big) \neq 0 \quad \forall t \in [\alpha, \beta].$$

THEOREM 4. *Let $[\alpha, \beta]$ be any closed subinterval of $[t_0, \infty)$. Let $u(t)$ be any function of class $C^2[\alpha, \beta]$, such that $\varphi\big(u(t), t\big) \neq 0$ for all $t \in [\alpha, \beta]$. Assume that there exists a function $G(\xi) \in C^1(-\infty, +\infty)$, such that $G\big(u(t)\big) > 0$ for all $t \in (\alpha, \beta)$, $G\big(u(\alpha)\big) = G\big(u(\beta)\big) = 0$, and $J_{\alpha,\beta}\big(u, G(u)\big) < 0$. Let $g\big(u(t), t\big) = \dfrac{dG(u)}{du}\big[\varphi\big(u(t), t\big)\big]^{-1}$. Write $\hat{G}(t) = G\big(u(t)\big)$, $\hat{g}(t) = g\big(u(t)\big)$. If there exists a number m such that $m = \max\limits_{t\in[\alpha,\beta]} \dfrac{\hat{g}^2(t)}{4\hat{G}(t)}$, then any solution $v(t)$ of (2), $t \in [\alpha, \beta]$, such that $v(t) \neq 0$ on $[\alpha, \beta]$, will satisfy the inequality $m > f'\big(v(t)\big)\big[\varphi\big(v(t), t\big)\big]^{-1}$ on some open subinterval $(\gamma, \delta) \subset [\alpha, \beta]$, where $f'(v)$ stands for $df(v)/dv$. (Clearly with $\varphi\big(x(t), t\big) \neq 0$ on (γ, δ).)*

Proof. Let us assume that there exists a solution $v(t)$ of (2) which is not equal to zero on $[\alpha, \beta]$. (Otherwise the conclusion of the theorem is trivially correct.) We examine the non-negative function

$$\int_\alpha^\beta A(t)\left[h\big(u(t), u'(t), t\big) - \frac{g\big(u(t), t\big)h(v, v', t)}{2f\big(v(t)\big)}\right]^2 dt$$
$$= \int_\alpha^\beta \left\{\left(\frac{A(t)h(v, v', t)G\big(u(t)\big)}{f\big(v(t)\big)}\right)' + A(t)h^2\big(u(t), u'(t), t\big) - \right.$$
$$- A(t)\left(\frac{h\big(u(t), u'(t), t\big)g\big(u(t), t\big)h\big(v(t), v'(t), t\big)}{f\big(v(t)\big)} + \right.$$
$$\left.\left. + \frac{g^2\big(u(t), t\big)h^2\big(v(t)v'(t), t\big)}{4f^2\big(v(t)\big)}\right)\right\} dt$$

$$= \int_{\alpha}^{\beta} \left\{ \frac{[A(t)h(v(t), v'(t), t)]'}{f(v(t))} G(u(t)) + A(t)h^2(u(t), u'(t), t) - \right.$$

$$- A(t)\left[\frac{h(v(t)v'(t), t)\, G(u(t)) f'(v(t)) v'(t)}{f^2(v(t))} + \right.$$

$$\left.\left. + \frac{1}{4} \frac{g^2(u(t), t) h^2(v(t), v'(t), t)}{f^2(v(t))} \right] \right\} dt$$

$$= J_{\alpha,\beta}(u, G(u)) + \int_{\alpha}^{\beta} \left\{ \frac{1}{4} \frac{g^2((u(t), t), t) h^2(v(t), v'(t), t)}{f^2(v(t))} - \right.$$

$$\left. - \frac{G(u(t)) h(v(t), v'(t), t) f'(v(t)) v'(t)}{f^2(v(t))} \right\} A(t)\, dt.$$

Since $J_{\alpha,\beta}(u, G(u)) < 0$, the last integral in this chain of equalities is positive. Hence on some subinterval of $[\alpha, \beta]$ we have the strict inequality:

$$\tfrac{1}{4} g^2(u(t), t) h^2(v(t), v'(t), t) > G(u(t))\, h(v(t), v'(t), t) f'(v(t)) v'(t),$$
$$\forall t \in (\gamma, \delta) \subset [\alpha, \beta].$$

It follows that $h(v(t), v'(t), t) \neq 0$ for all $t \in (\gamma, \delta)$, and

$$\frac{1}{4} g^2(u(t), t) > \frac{G(u(t)) f'(v(t)) v'(t)}{h(v(t), v'(t), t)} = \frac{G(u(t)) f'(v(t))}{\varphi(v(t), t)}, \quad t \in (\gamma, \delta).$$

Since (γ, δ) is an open subinterval of $[\alpha, \beta]$, $G(u(t)) > 0$ for all $t \in (\gamma, \delta)$, and

$$\frac{g^2(u(t), t)}{4G(u(t))} > \frac{f'(v(t))}{\varphi(v(t), t)}, \quad \forall t \in (\gamma, \delta).$$

Hence if $m = \max\limits_{t \in [\alpha,\beta]} \dfrac{g^2(u(t), t)}{4G(u(t), t)}$ exists, then $m > \dfrac{f'(v(t))}{\varphi(x(t), t)}$, $\forall t \in (\gamma, \delta)$, completing the proof.

COROLLARY. *If $f(v) = v^a$, $\varphi(v(t), t) = \psi(t) v^b$, where $\psi(t) > 0$ on $[\alpha, \beta]$, and $a > b+1$, then $v(t) < (mk/a)^{1/(a-b-1)}$, $t \in (\gamma, \delta)$, where $k = \min\limits_{t \in [\alpha,\beta]} \psi(t)$.*

4. Example of application. Consider the equation $(tx^2x')' + Ctx^K = 0$, $t > 0$, where $C \geqslant 1$, $K > 3$.

We wish to show that on any interval $[\alpha = n\pi + \pi/4, \beta = n\pi + \pi/2]$, $n = 0, 1, 2, \ldots$, any solution of this equation will attain values smaller than $(4/K)^{1/(K-3)}$.

Proof. Choose $u = \sin t$, $G(u) = [(u-\sqrt{2}/2)^2(1-u)]^2$, then

$$J_{\alpha,\beta}(u, G(u)) = \int_\alpha^\beta t\{2(u-\sqrt{2}/2)^5(1-u)^5(-2u+1-\sqrt{2}/2)^2 - \\ - C(u-\sqrt{2}/2)^2(1-u)^2\}\,dt < 0,$$

$$g(u) = u^{-2}\frac{dG(u)}{du} = 2u^{-2}(-2u+1-\sqrt{2}/2)(u-\sqrt{2}/2)(1-u),$$

$$\frac{g^2(u)}{4G(u)} = \frac{(-2u+1-\sqrt{2}/2)^2}{u^4} = \frac{[(1-\sqrt{2}/2)-2\sin t]^2}{\sin^2 t} < 4,$$

and $x < (4/K)^{1/(K-3)}$ as required.

In particular, if $K = 4$, we observe that $x < 1$ on $[n\pi+\pi/4, n\pi+ +\pi/2]$, $n = 0, 1, 2, \ldots$ This also implies that any solution $\hat{x}(t)$, $\hat{x}(0) > 0$ must attain values smaller in absolute value then one on the interval $[\pi/4, \pi/2]$.

A generalization of a classical result of Leighton. The so-called "Leighton's variational theorem" (see reference [6] for the original proof) appears to be difficult to formulate as in [6], in the most general case of equation (1). We offer below a restricted version of such a generalization. This result also generalizes the author's former result in [5]. Again we consider equation (1), subject to hypotheses (i), (ii) and (iii) as stated in Section 3, and in addition to the hypothesis

(iv) $$\varphi(x) \neq 0 \quad \text{if } x \neq 0.$$

We shall denote by $I_{\alpha,\beta}(G(u), u)$ the following functional:

$$I_{\alpha,\beta}(G(u), u) = \int_\alpha^\beta \{\varphi(u(t))[a(t)(u(t))(u'(t))^2 - c(t)G(u(t))]\}\,dt,$$

where $u(t)$, $G(u(t))$ are functions satisfying the hypotheses:

$$u(t) \in C^1[\alpha, \beta], \qquad G(u) \in C^1(-\infty, +\infty).$$

THEOREM 5. *Suppose that there exists a function $u(t)$ defined on $[\alpha, \beta]$, and $u(t) \in C^1[\alpha, \beta]$, while $\varphi(u(t)) > 0$ for all $t \in [\alpha, \beta]$, and a function $G(\xi) \in C^1(-\infty, +\infty)$ such that the continuous function*

$$g(u(t)) = \frac{\big(G(u(t))\varphi(u(t))\big)'}{\varphi(u(t))}$$

satisfies the inequality $g^2(u(t)) \leqslant 4G(u(t))\varphi(u(t))$ for all $t \in [\alpha, \beta]$, while $G(u(\alpha)) = G(u(\beta)) = 0$. Suppose that there exists $m > 0$ such that $\dfrac{f'(\xi)}{\varphi(\xi)} \geqslant 1$ (*) for all ξ satisfying $|\xi| > m$. Then $I_{\alpha,\beta}(G(u(t)), u(t)) < 0$ implies *that every solution $\hat{x}(t)$ of equation* (1) *will have the property that* $|\hat{x}(t)| \leqslant m$ *for some* $t \in [\alpha, \beta]$. *In particular,*

$$\textit{if } \frac{f'(\xi)}{\varphi(\xi)} \geqslant 1 \quad \textit{for all } \xi, \quad \textit{then } I_{\alpha,\beta}\big(G(u), u\big) < 0$$

implies that every solution of (1) *will have a zero on* $[\alpha, \beta]$.

Proof. Assume to the contrary that there exists a solution of (1) $\hat{x}(t)$, such that $|\hat{x}(t)| > m$ for all $t \in [\alpha, \beta]$. Our hypothesis implies that $f\big(\hat{x}(t)\big) \neq 0$ for all $t \in [\alpha, \beta]$. Consider the integral

$$K_{\alpha,\beta}\Big(G\big(u(t)\big), u(t)\Big) = \int_{\alpha}^{\beta} \Bigg\{\bigg(\frac{a(t)\varphi\big(\hat{x}(t)\big)\varphi\big(u(t)\big)G\big(u(t)\big)\hat{x}'(t)}{f\big(\hat{x}(t)\big)}\bigg)' + \\ + a(t)\bigg[\varphi\big(u(t)\big)u'(t) - \frac{g\big(u(t)\big)\varphi\big(\hat{x}(t)\big)\hat{x}'(t)}{2f\big(\hat{x}(t)\big)}\bigg]^2\Bigg\}dt;$$

$K_{\alpha,\beta}\big(G(u), u\big)$ is non-negative since $G\big(u(\alpha)\big) = G\big(u(\beta)\big) = 0$ and the first term of the integrand contributes nothing to the value of the integral, while $a(t) \geqslant 0$ on the interval $[\alpha, \beta]$. We have

$$\begin{aligned} K_{\alpha,\beta}\big(u, G(u)\big) &= \int_{\alpha}^{\beta} -c(t)\varphi\big(u(t)\big)G\big(u(t)\big) + a(t)\big(\hat{x}(t)\big)\hat{x}'(t)\varphi\big(u(t)\big)\frac{g\big(u(t)\big)u'(t)}{f\big(\hat{x}(t)\big)} - \\ &\quad - \frac{g(u)f'\big(\hat{x}(t)\big)\hat{x}'(t)}{f^2\big(\hat{x}(t)\big)} + a(t)\bigg[\varphi^2\big(u(t)\big)\big(u'(t)\big)^2 - \\ &\quad - \frac{\varphi\big(u(t)\big)\varphi\big(\hat{x}(t)\big)g\big(u(t)\big)u'(t)\hat{x}'(t)}{f\big(\hat{x}(t)\big)} + \frac{g^2\big(u(t)\big)\varphi^2\big(\hat{x}(t)\big)\big(\hat{x}'(t)\big)^2}{4f^2\big(\hat{x}(t)\big)}\bigg]dt \\ &= I_{\alpha,\beta}\big(G(u), u\big) + \int_{\alpha}^{\beta}\bigg\{a(t)\,\frac{g^2(u)\varphi^2(\hat{x})\big(\hat{x}'(t)\big)^2}{4f^2\big(\hat{x}(t)\big)} - \\ &\quad - a(t)\,\frac{\varphi(\hat{x})\varphi(u)G(u)f'\big(\hat{x}(t)\big)\big(\hat{x}'(t)\big)^2}{f^2\big(\hat{x}(t)\big)}\bigg\}dt \\ &= I_{\alpha,\beta}\big(G(u), u\big) + \int_{\alpha}^{\beta} \frac{a(t)\varphi^2(\hat{x})\big(\hat{x}'(t)\big)^2}{4f^2(\hat{x})}\bigg[g^2\big(u(t)\big) - \\ &\quad - \frac{4\varphi(u)G(u)f'\big(\hat{x}(t)\big)}{\varphi\big(\hat{x}(t)\big)}\bigg]dt. \end{aligned}$$

Since $I_{\alpha,\beta}\big(G(u)u\big) < 0$, the integrand in the above expression must attain positive values in some subinterval of $[\alpha, \beta]$. Hence for some $\tilde{t} \in [\alpha, \beta]$

$$g^2\big(u(\tilde{t})\big) > 4\varphi\big(u(\tilde{t})\big)G\big(u(\tilde{t})\big)\frac{f'\big(\hat{x}(\tilde{t})\big)}{\varphi\big(\hat{x}(\tilde{t})\big)}.$$

(Observe that $\varphi(\hat{x}(\tilde{t})) \neq 0$.) However, since by hypothesis $g^2(u) \leqslant 4\varphi(u)G(u)$, and since by our assumption $|\hat{x}(\tilde{t})| > m$, and $f'(\hat{x}(t))/\varphi(\hat{x}(t)) \geqslant 1$, this inequality cannot be true. This contradiction completes the proof.

We observe that in the linear case $f(x) \equiv x$, $\varphi(x) \equiv 1$, we have $\frac{f'(x)}{\varphi(x)} \equiv 1$ and we obtain the classical result of Leighton (see [6]) by putting $G(u) = u^2$, $u(\alpha) = u(\beta) = 0$.

An example of application. Every solution $\hat{x}(t)$ of the equation

$$\big((1+\cos^2(x))x'\big)' + t^2 x^\mu = 0$$

(where μ is an odd positive integer, $t \geqslant 0$) will assume values smaller in absolute value than $(2/\mu)^{1/\mu-1}$ on the interval $[0, \pi]$. To prove it we simply check that all conditions of Theorem 5 are satisfied when

$$|x| > (2/\mu)^{1/\mu-1}, \quad \frac{f'(x)}{\varphi(x)} = \frac{\mu x^{\mu-1}}{1+\cos^2 x} \geqslant \frac{\mu(2/\mu)}{2} = 1.$$

We set $u(t) = \sin t$, $G(u) = u^2$. Then we check the inequality

$$\begin{aligned} 4\varphi(u)G(u) &= 4\big(1+\cos^2(\sin t)\big)\sin^2 t > \frac{[1+\cos^2(\sin t)\sin^2 t]'}{1+\sin^2 t} \\ &= \left(\frac{-2\cos(\sin t)\cos t\sin^2 t + 2\cos^2(\sin t)\sin t\cos t}{1+\sin^2 t}\right)^2 \\ &= \frac{4\sin^2 t\cos^2(\sin t)\cos t}{(1+\sin^2 t)^2}[\sin t - \cos^2(\sin t)]^2, \end{aligned}$$

i. e. we need to check that for all $t \in [0, \pi]$

$$1 > \frac{\sin^2 t\cos^2(\sin t)\cos t}{(1+\sin^2 t)\big(1+\cos^2(\sin t)\big)}[\sin t - \cos^2 t\sin t]^2.$$

An elementary argument confirms this inequality, thus completing the proof.

Remark. Some weakening of the hypothesis is possible in Theorem 5, if the initial conditions are given at $t_0 = a$. If $\hat{x}(a) > 0$, then condition $(*)$ may be restated as a one-sided inequality $\frac{f'(\xi)}{\varphi(\xi)} \geqslant 1$ for all $\xi > m$. Similarly, if $\hat{x}(a) < 0$, we shall only require $\frac{f'(\xi)}{\varphi(\xi)} \geqslant 1$ for all $\xi < -m$. No changes are required in the subsequent argument.

In closing I would like to mention the papers of Tippett and Ford who studied a somewhat similar differential equation in [4] and [9].

However, their investigation concerned primarily the existence of solutions of the two point boundary value problem. The author would like to thank the referee for a very careful reading and correcting of the original manuscript.

References

[1] P. B. Bailey, L. F. Shampine and P. E. Waltman, *Non-linear two point boundary value problems*, Academic Press, New York 1968.

[2] T. Burton and R. Grimmer, *On continuability of solutions of second order differential equations*, Proc. Amer. Math. Soc. 29, 2 (1971), p. 277–283.

[3] C. V. Coffman and D. F. Ulrich, *On the continuation of certain non-linear differential equations*, Monatsch. Math. 71 (1967), p. 385–392.

[4] W. T. Ford, *On the first boundary value problem for* $[h(x, s', t)] = f(x, x', t)$, Proc. Amer. Math. Soc. (to appear).

[5] V. Komkov, *A generalization of Leighton's variational theorem*, J. Appl. Anal. 1 (1972), p. 377–383.

[6] W. Leighton, *Comparison theorems for linear differential equations of second order*, Proc. Amer. Math. Soc. 13, 4 (1962), p. 603–610.

[7] H. Poincaré, *Oeuvres*, vol. 1, Paris 1928.

[8] C. A. Swanson, *Comparison and oscillation theory of linear differential equation*, Academic Press, New York 1968.

[9] J. M. Tippett, *The first boundary value problem for* $x'' = f(x, x', t)$, M. S. Thesis, Texas Tech. 1971.

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