

On concomitants of mixed tensors

by J. ACZÉL (Debrecen) and M. HOSSZÚ (Miskolc)

*To Professor S. Gołab on his 60-th birthday
 with expressions of cordial friendship*

Introduction. S. Gołab has mentioned in his paper [3] (p. 114) the problem of determining all concomitants

$$(1) \quad \Omega = \Psi(a_i^k)$$

of mixed tensors which are scalars (resp. densities). As he mentioned there, he has solved this problem under the supposition of continuous derivability of the function Ψ for tensors in 2-dimensional spaces (unpublished), while in these spaces all solutions, resp. all continuous solutions, are determined in [1]; in this paper we give a complete solution in spaces of an arbitrary number of dimensions without imposing any regularity conditions upon the function Ψ in (1). (Similar questions have been investigated by W. Graeb [6].)

Since mixed tensors are transformed under a transformation

$$\bar{\xi}^\kappa = \varphi^\kappa(\xi^k)$$

$$(k, \kappa = 1, 2, \dots, n; \text{Det } A_k^\kappa \neq 0, A_k^\kappa \triangleq \frac{\partial \bar{\xi}^\kappa}{\partial \xi^k}, A_\kappa^k \triangleq \frac{\partial \xi^k}{\partial \bar{\xi}^\kappa})$$

of coordinates by the formula

$$\bar{a}_\lambda^\kappa = a_i^k A_k^\kappa A_\lambda^i$$

(one has to sum with respect to indices figuring twice, once as subscript and once as supercript), while densities are transformed by the formula

$$\bar{\Delta} = \Delta |\text{Det } A_k^\kappa|^p \varepsilon$$

(where $\varepsilon = \text{sign Det } A_k^\kappa$, resp. $\varepsilon = 1$, according to whether it is an ordinary or a Weyl density; $\text{Det } A_k^\kappa$ is the determinant built from A_k^κ) and scalars are transformed by the formula

$$\bar{\Sigma} = \Sigma,$$

so the functions (1) having densities, resp. scalars, as function values must fulfil the functional equations

$$(2) \quad \Psi(a_i^k A_k^\alpha A_\lambda^1) = \Psi(a_i^k) |\text{Det } A_m^\alpha|^\varepsilon \quad (\varepsilon = 1 \text{ or } \varepsilon = \text{sign Det } A_m^\alpha),$$

resp.

$$(3) \quad \Psi(a_i^k A_k^\alpha A_\lambda^1) = \Psi(a_i^k).$$

In this paper we consider these functional equations as valid for a given tensor a_i^k and for arbitrary A_k^α with $\text{Det } A_k^\alpha \neq 0$ and so we get all density, resp. scalar concomitants, of a given mixed tensor (1) while, if (2) and (3) were supposed to be fulfilled also for arbitrary a_i^k , then our results would be only that there are no further general formulae ordering densities, resp. scalars, to all mixed tensors, leaving the question open whether certain particular tensors can have some more density, resp. scalar, concomitants or not. (For this remark cf. § 3 of this paper.)

In § 1 we determine the scalar concomitants of mixed tensors in 2-dimensional spaces by an elementary method; in § 2 we show that the complete solution of this problem in n -dimensional spaces is a simple consequence of some elementary facts of matrix algebra (we shall need nothing more than the normal forms of matrices). Finally, in § 3 we show that mixed tensors have no concomitants which are densities of weight different from 0, we examine the case of biscalar concomitants and we give a general remark on the determination of all density concomitants of geometric objects if all scalar concomitants are known.

§ 1. In the case of a 2-dimensional space $n = 2$ we write in (3)

$$(4) \quad a_1^1 = x, \quad a_2^1 = y, \quad a_1^2 = u, \quad a_2^2 = v$$

and

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$$

in place of

$$\begin{vmatrix} \partial \bar{\xi}^\alpha \\ \partial \xi^k \end{vmatrix}$$

and so (3) becomes

$$(5) \quad \Psi(x, y, u, v) = \Psi \left[\frac{1}{J} (x\alpha\delta - y\alpha\gamma + u\beta\delta - v\beta\gamma), \frac{1}{J} (-x\alpha\beta + y\alpha^2 - u\beta^2 + v\alpha\beta), \frac{1}{J} (x\gamma\delta - y\gamma^2 + u\delta^2 - v\gamma\delta), \frac{1}{J} (-x\beta\gamma + y\alpha\gamma - u\beta\delta + v\alpha\delta) \right],$$

(1) Of course, if Ψ is defined for a mixed tensor a_i^k , it must also be defined for all values $\bar{a}_\lambda^\alpha = a_i^k A_k^\alpha A_\lambda^1$ which can be reached from the given a_i^k by transformation of coordinates, i.e. in a whole transitivity set (A. Nijenhuis [9], p. 186). So what we find (e.g. in Theorems 1, 3 and 5) are just the *relative concomitants* defined by M. Kucharszewski in his paper [7].

where

$$(6) \quad J \stackrel{\text{df}}{=} \alpha\delta - \beta\gamma \neq 0.$$

To solve (5) we put in this equation

$$\alpha = 0, \quad \beta = 1, \quad \gamma = u, \quad \delta = v$$

and write

$$(7) \quad \Theta(s, t) \stackrel{\text{df}}{=} \Psi(0, 1, -t, s)$$

to get

$$(8) \quad \Psi(x, y, u, v) = \Theta(x + v, xv - yu) \quad (u \neq 0)$$

valid for any x, y, v and for any $u \neq 0$ because of (6) and of

$$J = \alpha\delta - \beta\gamma = -u.$$

Now we suppose $y \neq 0$ and define

$$\Theta'(s, t) \stackrel{\text{df}}{=} \Psi(s, -t, 1, 0).$$

Thus by putting in (5):

$$\alpha = x, \quad \beta = y, \quad \gamma = 1, \quad \delta = 0$$

we get

$$(9) \quad \Psi(x, y, u, v) = \Theta'(x + v, xv - yu) \quad (y \neq 0)$$

valid for any x, u, v , and for any $y \neq 0$ (since in this case $J = -y$). If we take in (8) and (9) $uy \neq 0$, then we see that

$$\Theta(s, t) \equiv \Theta'(s, t).$$

So

$$(10) \quad \Psi(x, y, u, v) = \Theta(x + v, xv - yu) \quad (u^2 + y^2 \neq 0)$$

is proved for all x, y, u, v with the exception of $u = y = 0$. If we write in (5) $u = y = 0$ and choose

$$\alpha = 1, \quad \beta = 1, \quad \gamma = x, \quad \delta = v,$$

then from (5) and (7)

$$(11) \quad \Psi(x, 0, 0, v) = \Theta(x + v, xv) \quad (x \neq v)$$

follows $J = v - x$.

Finally, for $u = y = 0, v = x$, (5) becomes

$$\Psi(x, 0, 0, x) = \Psi(x, 0, 0, x);$$

so in this case (5) imposes no restriction upon the function Ψ

$$(12) \quad \Psi(x, 0, 0, x) = \Lambda(x).$$

Summing up (10), (11) and (12), we have

$$(13) \quad \Psi(x, y, u, v) = \begin{cases} \Theta(x+v, xv-yu) & \text{for } (x, y, u, v) \neq (x, 0, 0, x), \\ \Lambda(x) & \text{for } (x, y, u, v) = (x, 0, 0, x), \end{cases}$$

(Λ and Θ being arbitrary functions of one, resp. two, variables) and, conversely, one checks immediately that also any function of the form (13) satisfies (5).

So we have

THEOREM 1. (13) is the general solution of the functional equation (5).

If we seek the continuous solutions of (5), then the function Θ in (13) must be continuous and, moreover,

$$\begin{aligned} \Lambda(x) &= \Theta(2x, x^2), \\ \Psi(x, y, u, v) &= \Theta(x+v, xv-yu) \end{aligned}$$

for any x, y, u, v .

Taking (4) into consideration, one sees that

$$x+v = a_1^1 + a_2^2 = \text{Tr} a_i^k$$

is the trace of the matrix a_i^k and

$$xv-yu = a_1^1 a_2^2 - a_1^2 a_2^1 = \text{Det} a_i^k$$

is its norm (determinant).

So we have

THEOREM 2. In 2-dimensional spaces every continuous scalar concomitant of a mixed tensor is a continuous function of the trace and of the norm of the matrix built from the components of the tensor.

§ 2. In order to solve (3) in full generality we write again this functional equation applying the matrix notation and thus obtaining an equation for functions defined for matrices of order n . In fact, by writing

$$(14) \quad \|a_i^k\| = X$$

and

$$\|A_k^*\| = A$$

and so

$$\|A_*^k\| = A^{-1},$$

(1) and (3) become

$$\Omega = \Psi(X)$$

and

$$(15) \quad \Psi(AXA^{-1}) = \Psi(X) \quad (\text{Det} A \neq 0).$$

Equation (15) shows that the matrix-scalar function $\Psi(X)$ is invariant with respect to similarity ⁽²⁾. Now, it is well known that every matrix X is similar to a matrix of the so-called second normal form (see e.g. [2], Ch. VI, § 6), i.e. to a generalized diagonal matrix with blocks of the form

$$\left\| \begin{array}{cccc} 0 & 0 & \theta & -a_{im_i} \\ 1 & 0 & 0 & -a_{i,m_i-1} \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{array} \right\|$$

where the a_{ik} 's ($k = 1, 2, \dots, m_i$) are the coefficients of the i th elementary divisor of X . The elementary divisors of X are of the form

$$f_1(\lambda)^{n_1}, f_2(\lambda)^{n_2}, \dots, f_u(\lambda)^{n_u} \quad (n_i = m_i \text{ or } n_i = \frac{m_i}{2}),$$

where the $f_i(\lambda)$'s are polynomials of first or second degree, not necessarily all different. The second normal form is unique up to the change of the blocks. So we may prescribe that the blocks should follow in lexicographic order of magnitude of the coefficients and exponents (n_i) of the $f_i(\lambda)$'s to make this form unique.

Now we choose in (15) for A the very matrix transforming X into the second normal form thus ordered. So we get

$$(16) \quad \Psi(X) = \Phi(a_{11}, \dots, a_{1m_1}; \dots; a_{u1}, \dots, a_{um_u}).$$

We also effect some changes in (16). As is well known,

$$f_1(\lambda)^{n_1} f_2(\lambda)^{n_2} \dots f_u(\lambda)^{n_u} = \text{Det}(\lambda E - X)$$

(E being the unit matrix), and so the coefficients of the elementary divisors determine the coefficients of the characteristic polynomial $\text{Det}(\lambda E - X)$ (and they determine, of course, also their exponents n_1, n_2, \dots, n_u). Conversely, if the exponents n_1, \dots, n_u of the elementary divisors and their lexicographical order are prescribed, then these and the coefficients of $\text{Det}(\lambda E - X)$ determine in a unique manner the elementary divisors in the prescribed order, as the factorization of $\text{Det}(\lambda E - X)$

⁽²⁾ The transcription $\bar{X} = AXA^{-1}$ of the transformation formula $\bar{a}_\lambda^i = a_\lambda^k A_k^i A_\lambda^l$ of mixed tensors gives an answer to the question posed and solved for $n = 2$ by S. Gołab and E. Siwek in their work [5], namely the determination of the transitivity sets for mixed tensors. In fact, one sees that the necessary and sufficient condition for two mixed tensors to belong to the same transitivity set is that their matrices be similar, i.e. that their elementary divisors be the same, in other words that the set of numbers $a_1, a_2, \dots, a_n, n_1, \dots, n_u$ (see our Theorem 3) coincide. If one considers the Theorem and formula (16) of Gołab and Siwek, one sees that they assert and prove exactly the special case $n = 2$ of this statement.

determines the different $f_j(\lambda)$'s and the sum of the exponents belonging to them and the n_i 's determine the exponents themselves of the elementary divisors in the order prescribed above.

Thus we can change (16) into

$$(17) \quad \Psi(X) = \Theta(a_1, a_2, \dots, a_n; n_1, n_2, \dots, n_u),$$

a_1, a_2, \dots, a_n being the coefficients of the characteristic equation

$$(18) \quad \text{Det}(\lambda E - X) = 0.$$

Evidently, (17) always satisfies equation (15) and we have

THEOREM 3. *The general solution of equation (15) is (17), where a_1, a_2, \dots, a_n are the coefficients of (18) and n_1, \dots, n_u are the exponents of the elementary divisors $f_i(\lambda)^{n_i}$ ($i = 1, 2, \dots, u$) belonging to X and ordered in the lexicographic order of magnitude of the coefficients of the f_i 's and of the exponents n_i of the elementary divisors.*

If Ψ is continuous, then Θ in (17) cannot depend upon the discrete quantities n_1, \dots, n_u (since matrices with the same eigenvalues, i.e. with the same coefficients of their characteristic equation, can always be transformed into each other by a continuous alteration of their elements). On the other hand, Ψ being continuous and a_1, a_2, \dots, a_n being continuous functions of the elements of X , also the dependence of $\Theta(a_1, a_2, \dots, a_n)$ upon them must be continuous; thus we have

THEOREM 4. *The continuous scalar concomitants of mixed tensors are all continuous functions of the coefficients in the characteristic equation of the matrix built from the components of this tensor.*

Since—apart from the sign—the coefficient of λ^{n-1} in (18) is $\text{Tr} X$ and that of λ^0 is $\text{Det} X$, in case $n = 2$ Theorem 4 reduces to Theorem 2 and also the reduction of Theorem 3 to Theorem 1 can easily be made.

Remark. The proof of Theorems 3 and 4 becomes easier in complex spaces, since there the Jordan normal form of matrices (cf. [2], loc. cit.) can be used.

§ 3. Now we consider concomitants which are non-trivial densities. If we write again (2) applying the matrix notation, we get

$$(19) \quad \Psi(AXA^{-1}) = \Psi(X) |\text{Det} A|^p \varepsilon \quad (\varepsilon = 1 \text{ or } \varepsilon = \text{sign Det } A, p \neq 0).$$

We show that *matrix-functional equations of this form have only the trivial solution $\Psi(X) \equiv 0$.*

To prove this, it is enough to substitute into (19) a diagonal matrix

$$D = \begin{vmatrix} d & 0 & \dots & 0 \\ 0 & d & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d \end{vmatrix} \quad (d > 1)$$

in place of A . As $DXD^{-1} = X$ we have for $A = D$

$$\Psi(X) = \Psi(X)|\text{Det } D|^p;$$

thus — in view of $p \neq 0$ — necessarily $\Psi(X) \equiv 0$, q.e.d. Thus we have proved

THEOREM 5. *Mixed tensors have no non-trivial density concomitants of weight different from 0.*

The question arises what can be said of density concomitants of weight 0. If in (19) $p = 0$ and $\varepsilon = 1$, then one has (15), i.e. the problem of scalar concomitants solved in §§ 2-3. If $p = 0$ and $\varepsilon = \text{sign Det } A$, then one is led to the question of *biscalar concomitants of mixed tensors* and (19) becomes

$$(20) \quad \Psi(AXA^{-1}) = \Psi(X)\text{sign Det } A .$$

In order to solve this equation we substitute for A the matrix transforming X into the second normal form treated in § 3. We get

$$(21) \quad \Psi(X) = \Phi[L(X)]\text{sign Det } A ,$$

where $L(X) = AXA^{-1}$ is the second normal form of X and

$$\Phi[L(X)] \stackrel{\text{def}}{=} \Psi[L(X)] .$$

Now \tilde{X} can be transformed to $L(X)$ by several matrices A . If all matrices A transforming X into the second normal form $L(X)$ have determinants of the same sign (all positive or all negative), then (21) is the uniquely determined general solution of the functional equation (20). If for the same X there exist A 's of positive and of negative determinants, then the trivial biscalar concomitant $\Psi(X) \equiv 0$ is the only solution of (20).

Both cases can occur. E.g. in spaces of odd dimensions we always have $\Psi(X) \equiv 0$, as can be seen by the same argument as that used for the solution of (19) but with $d < 0$. Also $\Psi(X) \equiv 0$ (in even dimensional spaces as well) if $\text{Det } X < 0$, as can be seen directly by putting $A = X$ into (20) (one gets $\Psi(X) = -\Psi(X) = 0$). On the other hand,

$$X = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

is an example of a matrix where all matrices A transforming it into the second normal form have determinants of the same (positive) sign, and thus (21) gives its general biscalar concomitants. In fact, one can easily check that in this case necessarily

$$A = \begin{vmatrix} A_1^1 & A_2^1 \\ -A_2^1 & A_1^1 \end{vmatrix}, \quad \text{Det } A = (A_1^1)^2 + (A_2^1)^2 > 0 .$$

Also in (21) $\Phi[L(X)]$ can be replaced by the right-hand side of (17).

On the other hand, let \mathbf{a} be an arbitrary geometric object (with one or more components) having a transformation formula

$$\mathbf{a} = \Phi(\mathbf{a}, \tau)$$

(τ represents the transformation $\bar{\xi}^* = \varphi^*(\xi^k)$ of coordinates). Then every density concomitant $\Psi(\mathbf{a})$ of \mathbf{a} must fulfil the equation

$$(22) \quad \Psi[\Phi(\mathbf{a}, \tau)] = \Psi(\mathbf{a}) |\text{Det } A_i^k|^p \varepsilon \quad (\varepsilon = 1 \text{ or } \varepsilon = \text{sign Det } A_i^k).$$

If we know a particular solution $\Psi_0 \neq 0$ of (22), i.e. a nonvanishing density-concomitant of \mathbf{a} , then the most general one can be determined by multiplying Ψ_0 by the general scalar concomitant of \mathbf{a} .

In fact, dividing (22) by

$$\Psi_0[\Phi(\mathbf{a}, \tau)] = \Psi_0(\mathbf{a}) |\text{Det } A_i^k|^p \varepsilon$$

one gets

$$\frac{\Psi[\Phi(\mathbf{a}, \tau)]}{\Psi_0[\Phi(\mathbf{a}, \tau)]} = \frac{\Psi(\mathbf{a})}{\Psi_0(\mathbf{a})},$$

i.e. Ψ/Ψ_0 is a scalar, q.e.d.

E.g. S. Gołab gives in [3] (p. 115-117) the general derivable density-concomitant of non-symmetric twice covariant tensors in 2-dimensional spaces in the form

$$\Psi(a_{kl}) = |a_{12} - a_{21}|^p \varepsilon \Theta \left[\frac{\text{Det } a_{kl}}{(a_{12} - a_{21})^2} \right] \quad (\varepsilon = 1 \text{ or } \varepsilon = \text{sign}(a_{12} - a_{21})).$$

Here $\Psi_0(a_{kl}) = |a_{12} - a_{21}|^p \varepsilon$ is a particular density concomitant and $\Theta \left[\frac{\text{Det } a_{kl}}{(a_{12} - a_{21})^2} \right]$ is the general scalar concomitant of a_{kl} ($\neq a_{lk}$).

By the same argument as that used above for (22), from the general scalar concomitant and from certain particular concomitants we can determine also those general concomitants which have a linear transformation formula:

$$(23) \quad \Psi[\Phi(\mathbf{a}, \tau)] = \Psi(\mathbf{a}) \Lambda(\tau) + \Theta(\tau).$$

In fact, let Ψ_0 be a particular solution of (23):

$$\Psi_0[\Phi(\mathbf{a}, \tau)] = \Psi_0(\mathbf{a}) \Lambda(\tau) + \Theta(\tau);$$

by comparing this with (23) we have

$$\Psi[\Phi(\mathbf{a}, \tau)] - \Psi_0[\Phi(\mathbf{a}, \tau)] = [\Psi(\mathbf{a}) - \Psi_0(\mathbf{a})] \Lambda(\tau).$$

If $\tilde{\Psi}_0$ is a particular solution of

$$(24) \quad \tilde{\Psi}[\Phi(\mathbf{a}, \tau)] = \tilde{\Psi}(\mathbf{a}) \Lambda(\tau),$$

then, by the same argument,

$$\tilde{\Psi}(\mathbf{a}) = \tilde{\Psi}_0(\mathbf{a})\Xi(\mathbf{a})$$

is the general solution of (24) and

$$\Psi(\mathbf{a}) = \tilde{\Psi}_0(\mathbf{a})\Xi(\mathbf{a}) + \Psi_0(\mathbf{a})$$

is the general solution of (23), where Ψ_0 and $\tilde{\Psi}_0$ are particular solutions of (23), resp. (24), and $\Xi(\mathbf{a})$ is the general scalar concomitant of \mathbf{a} .

Nevertheless, this generalization is an essential one only in 1-dimensional spaces, as in [4], [8] it was proved that all objects in more than 1-dimensional spaces of linear transformation formulae

$$\bar{\Omega} = \Omega \Lambda(\tau) + \Theta(\tau)$$

with measurable Λ , Θ are essentially equivalent to densities.

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