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Asymptotic properties of some series of polynomials

by Z. POLNIAKOWSKI (Poznań)

1. Let us consider the following problem: Suppose that $\sum_{n=0}^{\infty} a_n f_n(x) = f(x)$ and $\sum_{n=0}^{\infty} \mu_n a_n f_n(x) = F(x)$ for $x \geqslant x_0$, where $\{a_n\}$, $\{\mu_n\}$ and $\{f_n(x)\}$ are complex. We ask under what conditions concerning the sequence $\{\mu_n\}$ the relation $\overline{\lim\limits_{x\to\infty}} \left|\frac{f(x)}{\psi(x)}\right| = M < \infty$ and $f(x) \sim s\psi(x)$ $(x \to \infty)$, with some given function $\psi(x)$, implies the relation $\overline{\lim\limits_{x\to\infty}} \left|\frac{F(x)}{\psi(x)}\right| \leqslant MK$ (where the constant K depends only on the choice of the sequence $\{\mu_n\}$ and of the function $\psi(x)$) and $F(x) \sim s_1\psi(x)$ for $x \to \infty$ $(|s| < \infty, |s_1| < \infty)$ respectively.

The purpose of this paper is the investigation of this problem in the case where $\psi(x) = x^a$ (a complex), $\mu_n = \frac{W_1(n)}{W(n)}$ ($W(x)$ and $W_1(x)$ are polynomials) and where $\{f_n(x)\}$ is one of the following families of polynomials: $\{x^n\}$, $\left\{\binom{x}{n}\right\}$, the polynomials of Laguerre, the polynomials of Hermite and, if $\operatorname{re} a > -1$, the polynomials of Chebyshev.

2. Suppose that

1) $W(x)$ and $W_1(x)$ are polynomials of degrees k and $l \leqslant k$ respectively. Moreover, these polynomials have no common zeros and $W(n) \neq 0$ for $n = 0, 1, 2, \ldots$,

2) $\{f_n(x)\}$ are functions defined for $x \geqslant x_0$,

3) $\sum_{n=0}^{\infty} a_n f_n(x) = f(x)$ and $\sum_{n=0}^{\infty} \frac{W_1(n)}{W(n)} a_n f_n(x) = F(x)$ for $x \geqslant x_0$.

We shall say that the *series* $\sum_{n=0}^{\infty} a_n f_n(x)$ *has the property* S_a if for some complex a we have:

1a) $\overline{\lim\limits_{n\to\infty}}|f(x)x^{-a}| = M < \infty$, resp.

1b) $f(x)\sim sx^a$ for $x\to\infty$,

2) supposing that $W(a)\neq 0$ the relations $\overline{\lim\limits_{x\to\infty}}|F(x)x^{-a}|\leqslant MK$ (where the constant K depends only on a, $W(x)$ and $W_1(x)$) in the case 1a), and $F(x)\sim s\dfrac{W_1(a)}{W(a)}x^a$ for $x\to\infty$ in the case 1b) hold respectively if and only if $\operatorname{re}a > \max\limits_{\nu}\operatorname{re}r_\nu$, where r_ν are zeros of the polynomial $W(x)$.

THEOREM 1. *For every complex a the series $\sum\limits_{n=0}^{\infty}a_nx^n = f(x)$, convergent for all x, and $\sum\limits_{n=0}^{\infty}b_n\binom{x}{n} = f_1(x)$, convergent for $x\geqslant x_0$, have the property* S_a.

This theorem is a generalization of Theorems 1C and 1B ([1], p. 23 and 18). In order to prove it we observe that the function

$$F(x) = \sum_{n=0}^{\infty} a_n\frac{W_1(n)}{W(n)}x^n \qquad (|x|<\infty)$$

satisfies the differential equation

$$L(y) = L^*(f(x))$$

and the function

$$F_1(x) = \sum_{n=0}^{\infty} b_n\frac{W_1(n)}{W(n)}\binom{x}{n} \qquad (x\geqslant x_0)$$

satisfies the difference equation

$$L_1(y) = L_1^*(f_1(x)),$$

where

$$L(y) = \sum_{\nu=0}^{k}\Delta^\nu W(0)\frac{x^\nu}{\nu!}y^{(\nu)}(x), \qquad L^*(y) = \sum_{\nu=0}^{l}\Delta^\nu W_1(0)\frac{x^\nu}{\nu!}y^{(\nu)}(x),$$

$$L_1(y) = \sum_{\nu=0}^{k}\Delta^\nu W(0)\binom{x}{\nu}\Delta^\nu y(x-\nu), \qquad L_1^*(y) = \sum_{\nu=0}^{l}\Delta^\nu W_1(0)\binom{x}{\nu}\Delta^\nu y(x-\nu).$$

Now, we apply Theorems 2C and 2B ([2], p. 131 and 125), and use the counter-examples given in the proofs of Theorems 1C and 1B ([1], p. 23 and 19).

2.1. *If* $a_n \sim n^p$ *(p complex), then*

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \sim e^x x^p \quad \textit{for} \quad x \to \infty .$$

For the proof we use the Gudermann transformation ([1], p. 22)

$$\sum_{n=0}^{\infty} \varphi(n) f^{(n)}(0) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \Delta^n \varphi(0) \frac{x^n}{n!} f^{(n)}(x)$$

for $\varphi(x) = 1/(x-p)$ and $f(x) = e^{-x}$ and obtain for $p \neq 0, 1, 2 \ldots$ by 3.4.3. ([1], p. 22)

$$\sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n-p+1)} \sim e^x x^p \quad \text{for} \quad x \to \infty .$$

For $p = m = 0, 1, 2, \ldots$ we have $\sum_{n=m}^{\infty} \frac{x^n}{(n-m)!} = e^x x^m$.

Since we have $\Gamma(n-\mathrm{re}p+1) \sim |\Gamma(n-p+1)| \sim n! n^{-\mathrm{re}p}$, then for given $\varepsilon > 0$ there exists an $N \geqslant \mathrm{re}p$ such that

$$\left| \frac{1}{|\Gamma(n-p+1)|} - \frac{1}{\Gamma(n-\mathrm{re}p+1)} \right| \leqslant \frac{\varepsilon}{\Gamma(n-\mathrm{re}p+1)} \quad \text{for} \quad n \geqslant N,$$

$$\left| \sum_{n=N}^{\infty} \frac{x^n}{|\Gamma(n-p+1)|} - \sum_{n=N}^{\infty} \frac{x^n}{\Gamma(n-\mathrm{re}p+1)} \right| \leqslant \varepsilon \sum_{n=N}^{\infty} \frac{x^n}{\Gamma(n-\mathrm{re}p+1)}$$

for $n \geqslant N$ and $x \geqslant 0$.

Dividing the above relation by $e^x x^{\mathrm{re}p}$ we obtain

$$\sum_{n=0}^{\infty} \frac{x^n}{|\Gamma(n-p+1)|} \sim \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n-\mathrm{re}p+1)} \sim e^x x^{\mathrm{re}p} \quad \text{for} \quad x \to \infty .$$

Since $n!/\Gamma(n-p+1) \sim n^p$, 2.1 follows from the following Cesàro theorem ([4], p. 16 for $b_n > 0$): If $a_n \sim b_n$, $b_n \neq 0$ for $n \geqslant N$, $\sum_{n=0}^{\infty} b_n x^n$ is convergent for every x and $\sum_{n=0}^{\infty} |b_n| x^n \leqslant K \left| \sum_{n=0}^{\infty} b_n x^n \right|$ for $x \geqslant 0$, then $\sum_{n=0}^{\infty} a_n x^n \sim \sum_{n=0}^{\infty} b_n x^n$ for $x \to \infty$.

2.2. *The differential equations* (*r and a complex*)

(1) $$xy'' - (x-a-1)y' + ry = 0 \quad (x > 0)$$

and

(1*) $$\tfrac{1}{2}y'' - xy' + ry = 0$$

have solutions $\varphi_1(x)$ *resp.* $\varphi_1^*(x)$ *such that* $\varphi_1(x) \sim \varphi_1^*(x) \sim x^r$ *for* $x \to \infty$. *In the case* $r \neq 0$ *we have* $\varphi_1'(x) \sim \varphi_1^{*\prime}(x) \sim rx^{r-1}$. *The differential equation* (1) *has a second solution* $\varphi_2(x)$ *such that* $\varphi_2(x) \sim \varphi_2'(x) \sim e^x x^{-a-r-1}$ *and* (1*) *has a solution* $\varphi_2^*(x)$ *such that* $\varphi_2^*(x) \sim e^{x^2} x^{-r-1}$ *and* $\varphi_2^{*\prime}(x) \sim 2e^{x^2} x^{-r}$.

We observe that the differential equation (1) has for $r \neq 0, 1, 2, \ldots$ a solution

$$\varphi_2(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n-r)}{\Gamma(n+a+1)} \cdot \frac{x^n}{n!} \sim e^x x^{-a-r-1}$$

by 2.1 since $\Gamma(n-r)/\Gamma(n+a+1) \sim n^{-a-r-1}$ for $n \to \infty$. (We take here $\Gamma(n-r)/\Gamma(n+a+1) = 0$ if $n+a+1$ is a negative integer.) Similarly we obtain

$$\varphi_2'(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n-r+1)}{\Gamma(n+a+2)} \cdot \frac{x^n}{n!} \sim e^x x^{-a-r-1}.$$

Next, if $\varphi_1(x)$ is chosen so that $W(\varphi_1, \varphi_2) = e^x x^{-a-1}$, we have

$$\frac{\varphi_1(x)}{\varphi_2(x)} = \int_x^{\infty} \frac{W(\varphi_1, \varphi_2)}{\varphi_2^2(t)} dt = \int_x^{\infty} \frac{e^t t^{-a-1}}{\varphi_2^2(t)} dt.$$

The function $e^{-x} x^{a+2r+1}$ has the property H (1) at the point $\xi = \infty$ with every constant $K > 1$ by 1.2 ([3], p. 170). Using l'Hospital's rule in the formulation of Theorem C ([1], p. 20), we obtain

$$\lim_{x\to\infty} \frac{\varphi_1(x)}{\varphi_2(x) e^{-x} x^{a+2r+1}} = \lim_{x\to\infty} \frac{1}{e^{-x} x^{a+2r+1}} \int_x^{\infty} \frac{e^t t^{-a-1}}{\varphi_2^2(t)} dt = 1.$$

Similarly

$$\frac{\varphi_1'(x)}{\varphi_2'(x)} = \int_x^{\infty} \frac{W(\varphi_1', \varphi_2')}{\varphi_2'^2(t)} dt = r \int_x^{\infty} \frac{e^t t^{-a-2}}{\varphi_2'^2(t)} dt,$$

$$\lim_{x\to\infty} \frac{\varphi_1'(x)}{\varphi_2'(x) e^{-x} x^{a+2r}} = \lim_{x\to\infty} \frac{r}{e^{-x} x^{a+2r}} \int_x^{\infty} \frac{e^t t^{-a-2}}{\varphi_2'^2(t)} dt = r.$$

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(1) We shall say that the function $g(x)$ *has the property* H with the constant K (≥ 1) at the point $x = \xi$ ($|\xi| \leq \infty$) if it is defined and differentiable in some neighbourhood I of ξ, except the point ξ at most, and if there exists a constant K such that for each pair of points $x, x_0 \in I$ we have

$$\lim_{x\to\xi} |g(x)| = \infty \quad \text{and} \quad \left| \int_{x_0}^{x} |g'(t)| dt \right| \leq K |g(x)| \quad (x_0 < x < \xi \text{ or } \xi < x < x_0)$$

or

$$\lim_{x\to\xi} g(x) = 0 \quad \text{and} \quad \left| \int_x^{\xi} |g'(t)| dt \right| \leq K |g(x)| \quad (x \neq \xi).$$

This implies

$$\varphi_1(x) \sim x^r, \quad \varphi_1'(x) \sim r x^{r-1} \quad \text{as} \quad x \to \infty .$$

The differential equation (1) has for $r = n = 0, 1, 2, \ldots$ a solution $\varphi_1(x) = L_n^{(a)}(x)$, where

$$L_n^{(a)}(x) = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n+a}{n-\nu} \frac{x^\nu}{\nu!}$$

is a Laguerre polynomial. Similarly we prove that in this case $\varphi_2(x) \sim \varphi_2'(x) \sim e^x x^{-a-n-1}$.

The differential equation (1*) has by 2.1 for $r \neq 0, 1, 2 \ldots$ a solution

$$\varphi_2^*(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2}r)}{\Gamma(n+\frac{1}{2})} \cdot \frac{x^{2n}}{n!} \sim e^{x^2} x^{-r-1} .$$

We have $\varphi_2^{*\prime}(x) \sim 2e^{x^2} x^{-r}$. For $r = n = 0, 1, 2 \ldots$ (1*) has a solution $\varphi_1^*(x) = H_n(x)$, where

$$H_n(x) = \sum_{\nu=0}^{[n/2]} (-1)^\nu \frac{1}{2^{2\nu}\nu!} \cdot \frac{x^{n-2\nu}}{(n-2\nu)!}$$

is a Hermite polynomial. We complete the proof of 2.2 in the same way as in the case of differential equation (1).

2.3. *Let the functions* $f(x)$, $w_\nu(x)$, $\psi_\nu(x)$, $\nu = 1, 2, \ldots, k$ *and* $\psi_{k+1}(x)$ *be defined in the interval* $\langle a, b\rangle$. *Suppose that for* $\nu = 1, 2, \ldots, k$

1) *the function* $\psi_\nu(x) w_\nu(x)$ *has the property* H (*with the constant* K_ν) *at the point* $\xi \in (a, b)$,

2) $w_\nu'(x) \neq 0$, $\psi_{k+1}(x) \neq 0 \quad (a \leqslant x \leqslant b)$,

3) $$\frac{\frac{d}{dx}\psi_\nu(x) w_\nu(x)}{w_\nu'(x)} = \psi_\nu(x) + \frac{w_\nu(x)}{w_\nu'(x)} \psi_\nu'(x) \sim \psi_{\nu+1}(x) \quad (x \to \xi),$$

4) *the functions* $w_\nu'(x)$, $\psi_\nu'(x)$ *and* $f(x)$ *are continuous in* $\langle a, b\rangle$; *furthermore we have*

5a) $\overline{\lim\limits_{x\to\xi}} \left| \frac{f(x)}{\psi_{k+1}(x)} \right| = M < \infty$, *or*

5b) $f(x) \sim \psi_{k+1}(x) \quad (x \to \xi)$.

Then the system of differential equations

$$(2) \qquad y_\nu(x) + \frac{w_\nu(x)}{w_\nu'(x)} y_\nu'(x) = y_{\nu+1}(x) \quad (a \leqslant x \leqslant b, \ \nu = 1, 2, \ldots, k)$$

(cf. 3)), *where* $y_{k+1}(x) = f(x)$, *has in* $\langle a, b\rangle$ *an integral* $\{\bar{y}_\nu(x)\}$ *such that* $\overline{\lim}_{x\to\xi}\left|\frac{\bar{y}_\nu(x)}{\psi_\nu(x)}\right| \leqslant M\prod_{j=\nu}^{k} K_j$ *in case* 5a), *or* $\bar{y}_\nu(x)\sim\psi_\nu(x)$ *for* $x\to\xi$ *in case* 5b). *Under the hypothesis* $\lim_{x\to\xi}|\psi_\nu(x)w_\nu(x)| = \infty$, $\nu = 1, 2, \ldots, k$, *every integral* $\{y_\nu(x)\}$ *of* (2) *has at the point* ξ *one of the asymptotic properties given above* (cf. [2], Theorem 1C, p. 130).

From 2), 3) and 1) it follows, that in some interval $\langle a_1, b_1\rangle$ ($a \leqslant a_1 \leqslant \xi \leqslant b_1 \leqslant b$) with the possible exclusion of the point ξ we have

$$[\psi_\nu(x)w_\nu(x)]' \neq 0, \quad \psi_\nu(x) \neq 0, \quad w_\nu(x) \neq 0, \quad \nu = 1, 2, \ldots, k.$$

Substituting $y_\nu(x) = \psi_\nu(x)z_\nu(x)$ ($a_1 \leqslant x \leqslant b_1$, $\nu = 1, 2, \ldots, k+1$) in (2) we get the system

$$z_\nu + \frac{\psi_\nu w_\nu}{(\psi_\nu w_\nu)'}z_\nu' = \frac{\psi_{\nu+1}w_\nu'}{(\psi_\nu w_\nu)'}z_{\nu+1} \qquad (a_1 \leqslant x \leqslant b_1,\ \nu = 1, 2, \ldots, k).$$

(Let us observe that $\overline{\lim}_{x\to\xi}|z_{k+1}(x)| = M$ in the case 5a), or $\lim_{x\to\xi} z_{k+1}(x) = 1$ in the case 5b).) We find from this system succesively for $\nu = k$, $k-1, \ldots, 1$ by 3) and analogously to 3.2.2 ([2], p. 130) resp. 3.3.1 ([1], p. 20) (for $g(x) = \psi_\nu(x)w_\nu(x)$) that there exists an integral $\{\bar{z}_\nu(x)\}$ of this system such that $\overline{\lim}_{x\to\xi}|\bar{z}_\nu(x)| \leqslant M\prod_{j=\nu}^{k} K_j$ in the case 5a), or $\lim_{x\to\xi}\bar{z}_\nu(x) = 1$ in the case 5b), $\nu = 1, 2, \ldots, k$. We set $\bar{y}_\nu(x) = \psi_\nu(x)\bar{z}_\nu(x)$ ($a_1 \leqslant x \leqslant b_1$, $\nu = 1, 2, \ldots, k$). We complete the proof by using the second part of the assertion of 3.2.2 ([2], p. 130) and 3.3.1 ([1], p. 20).

2.4. *The series*

$$\sum_{n=0}^{\infty}\lambda^n L_n^{(a)}(x), \qquad \sum_{n=0}^{\infty}\lambda^n \frac{d}{dx}L_n^{(a)}(x) \qquad \textit{and} \qquad \sum_{n=0}^{\infty}\lambda^n \frac{d^2}{dx^2}L_n^{(a)}(x)$$

(*see* 2.2), (*for* $|\lambda| < 1$ *and arbitrary real* a) *are almost uniformly convergent for* $x > 0$; *the series*

$$\sum_{n=0}^{\infty}\lambda^n H_n(x), \qquad \sum_{n=0}^{\infty}\lambda^n H_n'(x) \qquad \textit{and} \qquad \sum_{n=0}^{\infty}\lambda^n H_n''(x)$$

(*for arbitrary* λ) *are almost uniformly convergent for* $-\infty < x < \infty$.

Let us observe that

$$\frac{d}{dx}L_n^{(a)}(x) = L_{n-1}^{(a+1)}(x) \qquad \text{and} \qquad H_n'(x) = H_{n-1}(x) \qquad (n \geqslant 1).$$

Lemma 2.4. then follows from the formula

$$L_n^{(a)}(x) = (-1)^n \pi^{-\frac{1}{2}} e^{\frac{x}{2}} x^{-\frac{a}{2}-\frac{1}{4}} n^{\frac{a}{2}-\frac{1}{4}} \cos\left\{2\sqrt{nx} - \frac{a\pi}{2} - \frac{\pi}{4}\right\} + O(n^{\frac{a}{2}-\frac{3}{4}}),$$

which holds almost uniformly for $x > 0$, and from the formula

$$2^n \Gamma\left(\frac{n}{2}+1\right) e^{-\frac{x^2}{2}} H_n(x)$$

$$= \cos\left(\sqrt{2n+1}\,x - \frac{n\pi}{2}\right) + \frac{x^3}{6} \cdot \frac{1}{\sqrt{2n+1}} \sin\left(\sqrt{2n+1}\,x - \frac{n\pi}{2}\right) + O\left(\frac{1}{n}\right),$$

which holds almost uniformly for $-\infty < x < \infty$. See Szegö [5], 8.22.1, p. 192, 8.22.8, p. 194. (We use here the notation $(-1)^n L_n^{(a)}(x)$ and $2^n n! H_n(x)$ instead of $L_n^{(a)}(x)$ and $H_n(x)$.)

2.5. *If $w_k(x)$ is a polynomial of degree k, then we have*

$$\sum_{n=0}^{\infty} (-1)^n w_k(n) \lambda^n L_n^{(a)}(x) = w_k^*(x) e^{-\frac{\lambda x}{1-\lambda}} \qquad (|\lambda| < 1,\ a \text{ real},\ x > 0),$$

$$\sum_{n=0}^{\infty} (-1)^n w_k(n) \lambda^n H_n(x) = w_k^{**}(x) e^{-\lambda x} \qquad (|\lambda| < \infty,\ |x| < \infty),$$

where $w_k^*(x)$ and $w_k^{**}(x)$ are polynomials of degrees at most k ($k \geqslant 0$) depending on λ.

Obviously it suffices for the proof to assume that $w_k(x) = x^k$. This can be proved by induction. In the case of the first series for $k = 0$ the equality is well known. Suppose that the equality

$$\sum_{n=0}^{\infty} (-1)^n n^k \lambda^n L_n^{(a)}(x) = \overline{w}_k(x) e^{-\frac{\lambda x}{1-\lambda}} \qquad (k \geqslant 0,\ x > 0)$$

holds, where $\overline{w}_k(x)$ is a polynomial of degree at most k. Since

$$nL_n^{(a)}(x) = (x-a-1)\frac{d}{dx} L_n^{(a)}(x) - x\frac{d^2}{dx^2} L_n^{(a)}(x),$$

we obtain by 2.4

$$\sum_{n=0}^{\infty} (-1)^n n^{k+1} \lambda^n L_n^{(a)}(x)$$

$$= (x-a-1)\sum_{n=1}^{\infty} (-1)^n n^k \lambda^n \frac{d}{dx} L_n^{(a)}(x) - x \sum_{n=2}^{\infty} (-1)^n n^k \lambda^n \frac{d^2}{dx^2} L_n^{(a)}(x)$$

$$= (x-a-1)\frac{d}{dx}\overline{w}_k(x) e^{-\lambda x/(1-\lambda)} - x\frac{d^2}{dx^2}\overline{w}_k(x) e^{-\lambda x/(1-\lambda)} = \overline{w}_{k+1}(x) e^{-\lambda x/(1-\lambda)},$$

where $\overline{w}_{k+1}(x)$ is a polynomial of degree at most $k+1$.

In the case of the second relation in 2.5 we have a similar proof by using the relations

$$\sum_{n=0}^{\infty}(-1)^n\lambda^n H_n(x) = e^{-\lambda x-\frac{1}{4}\lambda^2}, \qquad nH_n(x) = xH_n'(x)-\tfrac{1}{2}H_n''(x)$$

and 2.4.

2.6. *Suppose that*

1) $K(x,t)$ *is an integrable function of the variable* t *and for some* M *we have* $|K(x,t)| \leqslant M$ $(x \geqslant a,\ t \geqslant b)$,

2) $\lim_{x\to\infty} K(x,t) = 1$ *almost uniformly for* $t \geqslant b$,

3) $\int_b^\infty F(t)\,dt = s$ *and* $\int_b^\infty |F(t)|\,dt = A < \infty$.

Then

$$\lim_{x\to\infty}\int_b^\infty K(x,t)F(t)\,dt = s.$$

We choose for a given $\varepsilon > 0$ the number $t_0 \geqslant b$ such that the inequality

$$\int_{t_0}^\infty |F(t)|\,dt \leqslant \frac{\varepsilon}{2(M+1)}$$

holds and we choose $x_0 \geqslant a$ such that

$$|K(x,t)-1| \leqslant \varepsilon/2A \qquad (x \geqslant x_0,\ b \leqslant t \leqslant t_0)$$

is satisfied. We obtain

$$\Big|\int_b^\infty \{K(x,t)F(t)-F(t)\}\,dt\Big|$$
$$\leqslant \int_b^{t_0} |F(t)|\,|K(x,t)-1|\,dt + \int_{t_0}^\infty |F(t)|\,|K(x,t)-1|\,dt \leqslant \varepsilon \qquad (x \geqslant x_0).$$

2.7.1. *If* $x\to\infty$, *then*

$$\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n}{n-r}L_n^{(a)}(x) \sim \lambda^r\Gamma(-r)x^r$$

for all complex $r \neq 0,1,2,\ldots$ *arbitrary complex* a *and for some* $\lambda \in (0,1)$.

We put

$$f(x,t)=\sum_{n=0}^{\infty}(-1)^n L_n^{(a)}(x)t^n=(1-t)^{-a-1}e^{-tx/(1-t)},$$

$$g(x,t)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n-r}L_n^{(a)}(x)t^n \qquad (|x|<\infty,\ |t|<1).$$

Then we get

$$\frac{\partial}{\partial t}[g(x,t)t^{-r}]=f(x,t)t^{-r-1} \qquad (|x|<\infty,\ 0<t<1)$$

and

$$g(x,\lambda)\lambda^{-r}=\left\{g\left(x,\frac{1}{x}\right)+J(x)\right\}x^r \qquad (x\geqslant 1/\lambda),$$

where

$$J(x)=x^{-r}\int_{1/x}^{\lambda}f(x,t)t^{-r-1}dt=\int_{1}^{\lambda x}f(x,u/x)u^{-r-1}du.$$

We shall show that the hypotheses of 2.6 are satisfied in the case of the functions

$$K(x,u)=\begin{cases} f(x,u/x)e^u & (1\leqslant u\leqslant \lambda x),\\ 0 & (1\leqslant \lambda x<u)\end{cases}$$

and $F(u)=e^{-u}u^{-r-1}$. We have for $1\leqslant u\leqslant \lambda x$:

$$|K(x,u)|\leqslant\begin{cases} 1 & (\mathrm{re}\,a\leqslant -1),\\ (1-\lambda)^{-\mathrm{re}\,a-1} & (\mathrm{re}\,a>-1).\end{cases}$$

Then the hypothesis 1) of 2.6 is fulfilled.

Next, for fixed $u_0\geqslant 1$ and $\varepsilon>0$ we choose $x_0\geqslant u_0/\lambda$ such that the following inequalities are satisfied:

$$\left|\left(1-\frac{u}{x}\right)^{-a-1}-1\right|\leqslant\frac{\varepsilon}{2} \quad\text{and}\quad \frac{u^2}{x-u}\leqslant\frac{\varepsilon}{2} \qquad (x\geqslant x_0,\ 1\leqslant u\leqslant u_0).$$

Applying the inequality $1-e^{-a}\leqslant a$ we obtain

$$|K(x,u)-1|\leqslant e^{-u^2/(x-u)}\left|\left(1-\frac{u}{x}\right)^{-a-1}-1\right|+|e^{-u^2/(x-u)}-1|\leqslant\varepsilon$$
$$(x\geqslant x_0,\ 1\leqslant u\leqslant u_0),$$

whence the hypothesis 2) of 2.6 is also satisfied. Therefore we obtain the equality

$$\lim_{x\to\infty}J(x)=\int_{1}^{\infty}e^{-u}u^{-r-1}du$$

by 2.6. Next, we have

$$g(x,t)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n-r}L_n^{(a)}(x)t^n=\sum_{\nu=0}^{\infty}(-1)^\nu\frac{x^\nu}{\nu!}\sum_{n=\nu}^{\infty}\binom{n+a}{n-\nu}\frac{t^n}{n-r}$$

$$=\sum_{\nu=0}^{\infty}(-1)^\nu\frac{(tx)^\nu}{\nu!}\sum_{n=0}^{\infty}\binom{n+\nu+a}{n}\frac{t^n}{n+\nu-r}\qquad(|x|<\infty,\ 0<t<1).$$

We obtain from this

$$g\left(x,\frac{1}{x}\right)=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!}\sum_{n=0}^{\infty}\binom{n+\nu+a}{n}\frac{x^{-n}}{n+\nu-r}=\sum_{n=0}^{\infty}c_nx^{-n},$$

where

$$c_n=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!(n+\nu-r)}\binom{n+\nu+a}{n}$$

and the above iterated double series is absolutely convergent. It follow that

$$\lim_{x\to\infty}g\left(x,\frac{1}{x}\right)=c_0=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(\nu-r)\nu!}.$$

Finally, we obtain

$$\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n}{n-r}L_n^{(a)}(x)=g(x,\lambda)=\left\{g\left(x,\frac{1}{x}\right)+J(x)\right\}\lambda^rx^r$$

$$\sim\left\{\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(\nu-r)\nu!}+\int_1^{\infty}e^{-u}u^{-r-1}du\right\}\lambda^rx^r=\lambda^r\Gamma(-r)x^r\qquad(x\to\infty).$$

2.7.2. *Suppose that*

1) $\sum_{n=0}^{\infty}a_nx^n=\varphi(x),\ \sum_{n=0}^{\infty}|a_n|\lambda^n<\infty\quad(\lambda>0,\ a_0\neq0,\ |x|\leqslant\lambda),$

2) $\sum_{n=0}^{\infty}b_nx^n=\psi(x)\quad(|x|<\infty),$

3) $\int_1^{\infty}|\psi(-x)x^{-r-1}|dx<\infty\quad(|r|<\infty).$

Then

$$\sum_{n=0}^{\infty}(-1)^n\lambda^nW_n(x)=\varphi(-\lambda)\psi(-\lambda x)\qquad(|x|<\infty),$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n}{n-r}W_n(x)\sim A(r)x^r\qquad(x\to\infty),$$

where

$$W_n(x) = \sum_{\nu=0}^{n} a_\nu b_{n-\nu} x^{n-\nu}, \qquad A(r) = a_0\lambda^r\Big\{\sum_{n=0}^{\infty}(-1)^n \frac{b_n}{n-r} + \int_1^\infty \psi(-t)t^{-r-1}dt\Big\}.$$

In case $\operatorname{re} r < 0$ *it is*

$$A(r) = a_0\lambda^r \int_0^\infty \psi(-t)t^{-r-1}dt\,.$$

We obtain by 1) and 2)

$$\varphi(t)\psi(tx) = \sum_{\nu=0}^{\infty} a_\nu t^\nu \sum_{n=\nu}^{\infty} b_{n-\nu}(tx)^{n-\nu}$$

$$= \sum_{n=0}^{\infty} t^n \sum_{\nu=0}^{n} a_\nu b_{n-\nu} x^{n-\nu} = \sum_{n=0}^{\infty} W_n(x)t^n \qquad (|x| < \infty,\ |t| \leqslant \lambda)\,,$$

whence follows the first of the relations 2.7.2. Next we take, as in 2.7.1,

$$f(x,t) = \sum_{n=0}^{\infty} W_n(x)t^n = \varphi(t)\psi(tx)\,,$$

$$g(x,t) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n-r} W_n(x)t^n \qquad (|x| < \infty,\ |t| \leqslant \lambda)\,.$$

We have

$$g(x,\lambda)\,\lambda^{-r} = \Big\{g\Big(x,\frac{1}{x}\Big) + J(x)\Big\}x^r \qquad \Big(|x| \geqslant \frac{1}{\lambda}\Big),$$

where

$$J(x) = x^{-r}\int_{1/x}^{\lambda} f(x,-t)t^{-r-1}dt = \int_1^{\lambda x} f\Big(x,-\frac{u}{x}\Big)u^{-r-1}du$$

$$= \int_1^{\lambda x} \varphi\Big(-\frac{u}{x}\Big)\psi(-u)u^{-r-1}du\,.$$

We shall show that the hypotheses of 2.6 are fulfilled in case of the functions

$$K(x,u) = \begin{cases} \dfrac{1}{a_0}\varphi\Big(-\dfrac{u}{x}\Big) & (1 \leqslant u \leqslant \lambda x)\,, \\ 0 & (1 \leqslant \lambda x < u) \end{cases}$$

and $F(u) = a_0\psi(-u)u^{-r-1}$. Notice that

$$|K(x,u)| = \Big|\frac{1}{a_0}\varphi\Big(-\frac{u}{x}\Big)\Big| \leqslant \frac{1}{|a_0|}\sum_{n=0}^{\infty}|a_n|\lambda^n \qquad (1 \leqslant u \leqslant \lambda x)\,.$$

For given $u_0 > 1$ and $\varepsilon > 0$ there exists an $x_0 \geqslant u_0/\lambda$ such that

$$|K(x,u)-1| = \left|\frac{1}{a_0}\sum_{n=0}^{\infty} a_n\left(\frac{u}{x}\right)^n - 1\right| = \frac{1}{|a_0|}\left|\sum_{n=1}^{\infty} a_n\left(\frac{u}{x}\right)^n\right|$$
$$\leqslant \frac{1}{|a_0|}\sum_{n=1}^{\infty}|a_n|\left(\frac{u_0}{x}\right)^n \leqslant \varepsilon \qquad (x \geqslant x_0,\ 1 \leqslant u \leqslant u_0).$$

We obtain by 3) and 2.6

$$\lim_{x\to\infty} J(x) = a_0\int_1^{\infty}\psi(-u)u^{-r-1}du.$$

Thus we have

$$g\left(x,\frac{1}{x}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{-n}}{n-r}\sum_{\nu=0}^{\infty}a_\nu b_{n-\nu}x^{n-\nu}$$
$$= \sum_{\nu=0}^{\infty}(-1)^\nu a_\nu x^{-\nu}\sum_{n=0}^{\infty}(-1)^n\frac{b_n}{n+\nu-r} = \sum_{\nu=0}^{\infty}(-1)^\nu a_\nu c_\nu x^{-\nu} \qquad \left(x \geqslant \frac{1}{\lambda}\right),$$

where the sequence $\{c_\nu\}$ with $c_\nu = \sum_{n=0}^{\infty}(-1)^n\frac{b_n}{n+\nu-r}$ is bounded. Therefore

$$\lim_{x\to\infty} g\left(x,\frac{1}{x}\right) = a_0c_0 = a_0\sum_{n=0}^{\infty}(-1)^n\frac{b_n}{n-r},$$

and

$$\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n}{n-r}W_n(x) \sim A(r)x^r \qquad (x\to\infty).$$

2.7.3. *From* 2.7.2 *for* $a_{2n} = (-1)^n\frac{1}{2^{2n}n!}$, $a_{2n+1} = 0$ *and* $b_n = \frac{1}{n!}$ *we obtain the following asymptotic formula*:

$$\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n}{n-r}H_n(x) \sim \lambda^r\Gamma(-r)x^r \qquad (0 < \lambda < \infty,\ x\to\infty).$$

THEOREM 2. *Suppose that for real* a:

1) *the series of Laguerre polynomials* $\sum_{n=0}^{\infty} a_nL_n^{(a)}(x) = f(x)$ *is convergent almost uniformly for* $x \geqslant x_0 > 0$,

2) $\sum_{n=0}^{\infty} a_n \frac{d}{dx} L_n^{(a)}(x) = f'(x)$ *almost uniformly for* $x \geqslant x_0$,

3) $\sum_{n=0}^{\infty} \frac{a_n}{n-r} L_n^{(a)}(x) = o(e^x x^{-a-r-1})$ *for* $x \to \infty$ *and* $r \neq 0, 1, 2, \ldots$

Then for every complex a *the series given in* 1) *has the property* S_a.

Proof. a) Suppose first that the following inequalities are satisfied: $\operatorname{re} a > \max_{\nu} \operatorname{re} r_\nu$ and $\overline{\lim_{x\to\infty}} |f(x) x^{-a}| = M < \infty$, where r_ν are zeros of $W(x)$.

In the case $W(x) = x-r$, $W_1(x) = 1$ $(r \neq 0, 1, 2, \ldots)$ according to 1), 2), Dirichlet's theorem and the equality $P_n(L_n^{(a)}(x)) = 0$ for $n = 0, 1, 2, \ldots$, where $P_r(y) = -xy'' + (x-a-1)y' - ry$, the series

$$\sum_{n=0}^{\infty} \frac{a_n}{n-r} L_n^{(a)}(x) , \quad \sum_{n=0}^{\infty} \frac{a_n}{n-r} \cdot \frac{d}{dx} L_n^{(a)}(x) \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n-r} \cdot \frac{d^2}{dx^2} L_n^{(a)}(x)$$

are almost uniformly convergent for $x \geqslant x_0$. The function

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n-r} L_n^{(a)}(x)$$

satisfies the differential equation

$$P_r(y) = f(x) \quad (x \geqslant x_0) , \tag{3}$$

since we have

$$P_r(F(x)) = \sum_{n=0}^{\infty} a_n P_r\left(\frac{L_n^{(a)}(x)}{n-r}\right) = \sum_{n=0}^{\infty} a_n L_n^{(a)}(x) \quad (x \geqslant x_0) .$$

Let $\varphi_1(x)$ and $\varphi_2(x)$ be linearly independent integrals of the differential equation $P_r(y) = 0$ (see 2.2).

Supposing that $r \neq 0$ we have for some $x_1 \geqslant x_0$: $\varphi_1(x) \neq 0$ and $\varphi_1'(x) \neq 0$ for $x \geqslant x_1$, and (3) is then by 3.2.1 ([2], p. 129) equivalent to the system

$$y_1 + \frac{w_1}{w_1'} y_1' = y_2 ,$$

$$y_2 + \frac{w_2}{w_2'} y_2' = -\frac{1}{r} f(x) \quad (x \geqslant x_1) ,$$

where $y_1 = y(x)$, $w_1 = 1/\varphi_1(x)$, $w_2 = \varphi_1'/W(\varphi_1, \varphi_2)$. Applying 2.2, we get

$$\frac{w_1'}{w_1} = -\frac{\varphi_1'(x)}{\varphi_1(x)} \sim -\frac{r}{x} ,$$

$$\frac{w_2'}{w_2} = -\frac{\varphi_1(x)}{\varphi_1'(x)} \cdot \frac{W(\varphi_1', \varphi_2')}{W(\varphi_1, \varphi_2)} = -\frac{\varphi_1(x)}{\varphi_1'(x)} \cdot \frac{r}{x} \to -1 \quad (x \to \infty) .$$

Since $\frac{(x^a w_1)'}{x^a w_1} \sim \frac{a-r}{x}$ and $\frac{(x^a w_2)'}{x^a w_2} \to -1$ for $x\to\infty$, we infer that the functions $x^a w_1(x)$ and $x^a w_2(x)$ have the property H (at $\xi=\infty$ with the constants $K_1 = \frac{|a-r|}{\operatorname{re}(a-r)}+\varepsilon$, or $K_2 = 1+\varepsilon$ for arbitrary $\varepsilon > 0$) by 1.2 ([3], p. 170), and in virtue of 2.3 with $\xi=\infty$, $\psi_1(x) = \frac{x^a}{a-r}$, $\psi_2(x) = \psi_3(x) = -\frac{x^a}{r}$ and $-\frac{1}{r}f(x)$ instead of $f(x)$, the differential equation (3) has for $x \geqslant x_1$ an integral $\bar{y}(x)$ satisfying the inequality $\overline{\lim\limits_{x\to\infty}}|\bar{y}(x)x^{-a}| \leqslant \frac{M}{\operatorname{re}(a-r)}$.

In the case $r=0$ the differential equation (3) is equivalent to the system

$$y_1' = y_2\,,$$

$$y_2 + \frac{w_2}{w_2'}y_2' = \frac{f(x)}{x-a-1} \qquad (x > \max(x_0, a+1))\,,$$

where $w_2(x)$ satisfies the relation $\frac{w_2(x)}{w_2'(x)} = \frac{-x}{x-a-1}$. We prove, as before, that there exists for large x an integral $\bar{y}_2(x)$ of the second differential equation of the above system, such that $\overline{\lim\limits_{x\to\infty}}|\bar{y}_2(x)x^{-a+1}| \leqslant M$. Then the function $\bar{y}(x) = \int\limits_{x_2}^{x}\bar{y}_2(t)\,dt$ (for $x \geqslant x_2$ with sufficiently large $x_2 \geqslant x_0$) satisfies (3) for $x \geqslant x_2$ and by l'Hospital's rule $\overline{\lim\limits_{x\to\infty}}|\bar{y}(x)x^{-a}| \leqslant \frac{M}{\operatorname{re} a}$.

The solution $\bar{y}(x)$ may be extended to the value x_0. Then

$$F(x) = \bar{y}(x) + c_1\varphi_1(x) + c_2\varphi_2(x) \qquad (x \geqslant x_0)\,,$$

where c_1 and c_2 are constants. In virtue of 3) we obtain $c_2 = 0$ and $\overline{\lim\limits_{x\to\infty}}|F(x)x^{-a}| \leqslant \frac{M}{\operatorname{re}(a-r)}$.

In case $W(x) = (x-r)^p$, $p = 2, 3, \ldots$ and $W_1(x) = 1$, applying the above reasoning successively to the functions $F_\nu(x)$ and $F_{\nu-1}(x)$ (where $F_\nu(x) = \sum\limits_{n=0}^{\infty}\frac{a_n}{(n-r)^\nu}L_n^{(a)}(x)$, $\nu = 2, 3, \ldots, p$) instead of $F(x)$ and $f(x)$, we obtain the inequality $\overline{\lim\limits_{x\to\infty}}|F(x)x^{-a}| \leqslant \frac{M}{[\operatorname{re}(a-r)]^p}$.

In the general case we take

$$\frac{W_1(x)}{W(x)} = l_0 + \sum_{\nu=1}^{k}\frac{l_\nu}{(x-r_\nu)^{p_\nu}}\,,$$

$$F_\nu(x) = \sum_{n=0}^{\infty}\frac{a_n}{(n-r_\nu)^{p_\nu}}L_n^{(a)}(x) \qquad (x \geqslant x_0,\ \nu = 1, 2, \ldots, k)\,.$$

We obtain

$$F(x) = l_0 f(x) + \sum_{\nu=1}^{k} l_\nu F_\nu(x) \qquad (x \geqslant x_0)\,,$$

$$\overline{\lim_{x\to\infty}} |F(x) x^{-a}| \leqslant MK\,, \quad \text{where} \quad K = |l_0| + \sum_{\nu=1}^{k} \frac{|l_\nu|}{[\mathrm{re}(a - r_\nu)]^{p_\nu}}\,.$$

b) Under the hypothesis $\mathrm{re}\, a > \max_\nu \mathrm{re}\, r_\nu$, $W(x) = x - r$, $W_1(x) = 1$ and $f(x) \sim s x^a$ for $x \to \infty$, we prove, as in a), that the relation $\sum_{n=0}^{\infty} \frac{a_n}{n-r} L_n^{(a)}(x) \sim \frac{s}{a-r} x^a$ holds and then we prove the general case.

c) Suppose for example that $\mathrm{re}\, a \leqslant \mathrm{re}\, r_1$. We set

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{W(n)}{n - r_1} \lambda^n L_n^{(a)}(x) \qquad (0 < \lambda < 1,\ |x| < \infty)\,. \tag{4}$$

We have $f(x) = w(x) e^{-\lambda x/(1-\lambda)}$ by 2.5, where $w(x)$ is a polynomial of degree at most $k-1$. However,

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{W_1(n)}{n - r_1} \lambda^n L_n^{(a)}(x)$$

$$= w_1(x) e^{-\lambda x/(1-\lambda)} + W_1(r_1) \sum_{n=0}^{\infty} (-1)^n \frac{1}{n - r_1} \lambda^n L_n^{(a)}(x)\,,$$

where $w_1(x)$ is a polynomial of degree at most $l-1$ if $l \geqslant 1$ and $w_1(x) = 0$ in the case $l = 0$. We then have $\lim_{x\to\infty} f(x) x^{-a} = 0$ and

$$\overline{\lim_{x\to\infty}} |F(x) x^{-a}| \geqslant |W_1(r_1) \lambda^{r_1} \Gamma(-r_1)| > 0$$

in virtue of 2.7.1 and the series (4) does not have the property S_a.

THEOREM 3. *Suppose that*

1) *the series of Hermite polynomials* $\sum_{n=0}^{\infty} a_n H_n(x) = f(x)$ *ist almost uniformly convergent for* $x \geqslant x_0$,

2) $\sum_{n=0}^{\infty} a_n H'_n(x) = f'(x)$ *almost uniformly for* $x \geqslant x_0$,

3) $\sum_{n=0}^{\infty} \frac{a_n}{n-r} H_n(x) = o(e^{x^2} x^{-r-1})$ *for* $x \to \infty$ *and* $r \neq 0, 1, 2, \ldots$

Then for every complex a *the series given in* 1) *has the property* S_a.

We prove this theorem in the same way as Theorem 2 by observing that the function $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n-r} H_n(x)$ satisfies the differential equation

$$-\tfrac{1}{2}y'' + xy' - ry = f(x) \qquad (x \geqslant x_0),$$

and that there exist by 2.2 the integrals $\varphi_1(x)$ and $\varphi_2(x)$ of (1*) satisfying the asymptotic relations $\varphi_1(x) \sim x^r$ and $\varphi_2(x) \sim e^{x^2} x^{-r-1}$ for $x \to \infty$, using 3) and 2.7.3.

2.8. *Suppose that*

1) *the series of Chebyshev polynomials* $\sum_{n=0}^{\infty} a_n T_n(x) = f(x)$ *is convergent for every* x, *where* $T_0(x) = 1$, $T_n(x) = \frac{1}{2^n}(u^n + u^{-n})$, $n \geqslant 1$, $u = x + \sqrt{x^2-1}$,

2) $\operatorname{re} a > -1$.

Then

$$\overline{\lim_{x\to\infty}} |f(x)x^{-a}| = \overline{\lim_{x\to\infty}} \left| x^{-a} \sum_{n=0}^{\infty} a_n x^n \right|$$

and the relation $f(x) \sim sx^a$ *is equivalent to the relation* $\sum_{n=0}^{\infty} a_n x^n \sim sx^a$ $(x \to \infty)$.

Let us observe for the proof that the series $\sum_{n=0}^{\infty} a_n \frac{u^n}{2^n} = f_1(x)$ and $\sum_{n=1}^{\infty} a_n \frac{u^{-n}}{2^n} = f_2(x)$ are convergent for every x and $\lim_{x\to\infty} f_2(x) x^{-a} = 0$ by 2).

Then

$$\overline{\lim_{x\to\infty}} |f(x)x^{-a}| = \overline{\lim_{x\to\infty}} |f_1(x)x^{-a}| = \overline{\lim_{t\to\infty}} \left| t^{-a} \sum_{n=0}^{\infty} a_n t^n \right|, \quad \text{where} \quad t = \tfrac{1}{2}u.$$

In the case $f(x) \sim sx^a$ the proof is quite similar.

From 2.8 we obtain in virtue of Theorem 1 the following

THEOREM 4. *Suppose that*

1) $\sum_{n=0}^{\infty} a_n T_n(x) = f(x)$ *for every* x,

2) $\operatorname{re} a > -1$.

Then the series given in 1) *has the property* S_a.

References

[1] Z. Polniakowski, *Polynomial Hausdorff transformations I*, Ann. Polon. Math. 5 (1958), pp. 1-24.

[2] — *Polynomial Hausdorff transformations II*, Ann. Polon. Math. 6 (1959), pp. 111-133.

[3] — *Asymptotic properties of solutions of some integral equations and second order differential equations*, Ann. Polon. Math. 16 (1965), pp. 169-183.

[4] G. Pólya und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis I*, Berlin 1925.

[5] G. Szegö *Orthogonal polynomials*, Amer. Coll. Publ.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

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