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On the existence and uniqueness of solutions of a certain integral-functional equation

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Abstract. In the paper we deal with the integral-functional equation

$$(*)\qquad x(t) = F\Big(t, \int_0^{a(t)} f(t, s, x(s))\,ds, x(\beta(t))\Big)$$

considered in a Banach space. The first part of the paper contains a theorem on the existence and uniqueness and on the convergence of successive approximations proved by the comparative method. It is supposed that the operator defined by the right-hand side of (*) satisfies an inequality similar to the Lipschitz condition. The conditions given in the paper involve some relations between the functions a, β and the Lipschitz coefficients of the functions F and f.

In the second part of the paper equation (*) is considered in a finite dimensional Banach space. The theorem on the existence of a solution of equation (*) is established under the assumptions involving the relations between the functions a, β, the Lipschitz coefficient of the function F with respect to the last variable and the continuity modulus of the functions f and F.

Introduction. Let $C(X, Y)$ denote the class of continuous functions defined in $X \subset B_1$ with range in $Y \subset B_2$, B_1, B_2 being arbitrarily fixed Banach spaces. Put $I = [0, a]$, where a is a fixed positive real number.

Let B be a Banach space with the norm $\|\cdot\|$, and let functions $F \in C(I \times B^2, B)$, $f \in C(I^2 \times B, B)$, $a, \beta \in C(I, I)$ be given. We shall always assume that $0 \leqslant a(t) \leqslant t$, $0 \leqslant \beta(t) \leqslant t$, $t \in I$.

We consider the integral-functional equation

$$(1)\qquad x(t) = F\Big(t, \int_0^{a(t)} f\big(t, s, x(s)\big)\,ds,\ x\big(\beta(t)\big)\Big), \qquad t \in I.$$

There exists an extensive literature on the existence and uniqueness problems for equations of this type and also for equations of the more general form

$$(2)\qquad x(t) = G\big(t, x(\cdot)\big),$$

where G is an operator defined in the suitable space.

For more detailed information on the existence and uniqueness problems for such general equations see [1], [4]–[6], [9], [12], [13].

The fundamental idea in treatement of equation (2) is to associate the operator G with a non-decreasing operator Ω by the inequality

$$\varrho\big(G(t, x_1(\cdot)),\ G(t, x_2(\cdot))\big) \leqslant \Omega\big(t, \varrho(x_1(\cdot), x_2(\cdot))\big)$$

with suitably chosen (not necessarily real valued) distance ϱ. If the operator Ω is "good", i.e. such that the sequence $\{u_n\}$ produced by the recurrent formula

$$u_{n+1}(t) = \Omega\big(t, u_n(\cdot)\big)$$

(with suitably chosen u_0) is convergent to $u = 0$, and for the sequence $\{x_n\}$ defined by the relation

$$x_{n+1}(t) = G\big(t, x_n(\cdot)\big)$$

(with suitably chosen x_0) the estimations $\varrho\big(x_{n+p}(t), x_n(t)\big) \leqslant u_n(t)$, $n, p = 0, 1, \ldots,$ hold, then the sequence $\{x_n\}$ is convergent to the unique solution $\bar{x}$ of equation (2). The abstract presentation of this idea can be found in [14].

Now the following question arises: for a given operator G (for instance for the operator given by the right-hand side of equation (1)), what operator Ω is "good"? Always we want to have an effective answer to this question.

We are not going to deal in this paper with the general equation (2). It seems that it is not possible to give more precise conditions for the given operator Ω to be "good" if we consider the general equation.

Recently effective conditions for suitable operators Ω associated with operators involved by neutral-differential equations were given in [3], [15].

It is well known that if the function F satisfies the Lipschitz condition with respect to the last two variables with constants k_1, l respectively and the function f fulfils the Lipschitz condition with respect to the third variable with the constant k_2, and if $l < 1$, then the operator

$$\Omega\big(t, u(\cdot)\big) = k_1 k_2 \int_0^{\alpha(t)} u(s)\, ds + l u\big(\beta(t)\big) \tag{3}$$

is "good" for the operator given by the right-hand side of equation (1) (short: "good" for equation (1)).

From the results of papers [3], [15] it follows that so defined Ω is also "good" for equation (1) if $\beta(t) \leqslant \beta \cdot t$, $\beta = \text{const}$ and $l \cdot \beta < 1$ (now it may happen that $l > 1$). We now ask: if Lipschitz coefficients k_1, k_2, l depend continuously on t, $t \in I$, what conditions involving α, β, k_1, k_2, l are sufficient for operator (3) to be "good" for equation (1)?

The first part of this paper deals with this question, and also with the problem of continuous dependence of the solution of equation (1) on the right-hand side.

The second part of the paper concerns equation (1) considered in a finite dimensional Banach space B, i. e. $B = R^n$.

It is known that if the function F is bounded (this condition can be slightly weakened) and fulfils the Lipschitz condition with respect to the third variable with a constant l, $l < 1$, then there exists at least one solution of equation (1).

Results of this type for suitable differential equations can be found in [7], [8], [16]. The result pointed out here is contained also in [2], [10], [11].

We are interested in the question: can the condition $l < 1$ be weakened? The positive answer to this question for suitable differential equation is contained in paper [16]. The result of the present paper is non-local and more precise than that of [16].

I.1. The main lemma. We have

LEMMA 1. *If* $h, l, L \in C(I, R_+)$, $R_+ \overset{\text{df}}{=} [0, +\infty)$, $\alpha, \beta \in C(I, I)$, $\alpha(t)$ $\beta(t) \in [0, t]$, $t \in I$,

$$(4) \qquad m(t) \overset{\text{df}}{=} \sum_{i=0}^{\infty} l_i(t) h\big(\beta_i(t)\big) < +\infty, \qquad t \in I,$$

$$(5) \qquad \overline{m}(t) \overset{\text{df}}{=} \sum_{i=0}^{\infty} L\big(\beta_i(t)\big) l_i(t) \alpha\big(\beta_i(t)\big) < +\infty, \qquad t \in I, \qquad \sup_I \frac{\overline{m}(t)}{t} < +\infty$$

and $m, \overline{m} \in C(I, R_+)$, *where*

$$\beta_0(t) = t, \quad \beta_{i+1}(t) = \beta\big(\beta_i(t)\big), \qquad i = 0, 1, \ldots, \; t \in I,$$

$$l_0(t) = 1, \quad l_{i+1}(t) = \prod_{j=0}^{i} l\big(\beta_j(t)\big), \qquad i = 0, 1, \ldots,$$

$t \in I$, *then*

1° *there exists a function* $z_0 \in C(I, R_+)$ *which is the unique solution of the equation*

$$(6) \qquad z(t) = \sum_{i=0}^{\infty} L\big(\beta_i(t)\big) l_i(t) \int_0^{\alpha(\beta_i(t))} z(s)\,ds + \sum_{i=0}^{\infty} l_i(t) h\big(\beta_i(t)\big), \qquad t \in I,$$

in the class of bounded non-negative and measurable functions defined in I. *(This class will be denoted by* $M(I, R_+)$.*)*

2° *The function* z_0 *is the unique solution of the equation*

$$z(t) = L(t)\int_0^{\alpha(t)} z(s)\,ds + l(t)z\big(\beta(t)\big) + h(t) \tag{7}$$

in the class

$$M_0(I, R_+) \overset{\mathrm{df}}{=} \bigcup_{c\geqslant 0} M_c(I, R_+),$$

where

$$M_c(I, R_+) \overset{\mathrm{df}}{=} [z: z\in M(I, R_+),\ 0 \leqslant z(t) \leqslant c\cdot z_0(t)], \qquad c\in R_+.$$

3° *The function* z, $z(t)\equiv 0$, *is the unique solution of the inequality*

$$z(t) \leqslant L(t)\int_0^{\alpha(t)} z(s)\,ds + l(t)z\big(\beta(t)\big), \qquad t\in I, \tag{8}$$

in the calss $M_0(I, R_+)$.

The function z_0 *is non-decreasing if the functions* h, l, L, α, β, *are so.*

Proof. At first we prove 1°. Let us not that if $z\in M(I, R_+)$ is a solution of equation (6), then $z\in C(I, R_+)$. This fact follows from the uniform convergence of the series (5) which is implied by the assumed continuity of $\overline{m}$ and Dini's theorem. Thus we shall prove that equation (6) has a unique solution z_0 in $C(I, R_+)$.

Put

$$\|z\|_0 = \max_I e^{-\lambda t}|z(t)|, \qquad z\in C(I, R_+),$$

with $\lambda > \Lambda \overset{\mathrm{df}}{=} \sup\limits_I \dfrac{\overline{m}(t)}{t}$.

Now we can prove that the operator A defined by the right-hand side of equation (6) is a contraction. Indeed, we have

$$\begin{aligned}\|Az - Aw\|_0 &\leqslant \max_I e^{-\lambda t}\sum_{i=0}^{\infty} L\big(\beta_i(t)\big)l_i(t)\int_0^{\alpha(\beta_i(t))} |z(s)-w(s)|\,ds\\ &\leqslant \max_I e^{-\lambda t}\sum_{i=0}^{\infty} L\big(\beta_i(t)\big)\cdot l_i(t)\int_0^{\alpha(\beta_i(t))} e^{\lambda s}\max_I e^{-\lambda s}|z(s)-w(s)|\,ds\\ &\leqslant \frac{1}{\lambda}\|z-w\|_0 \max_I \sum_{i=0}^{\infty} L\big(\beta_i(t)\big)l_i(t)e^{-\lambda t}\left(\exp\lambda t\frac{\alpha\big(\beta_i(t)\big)}{t} - 1\right)\\ &\leqslant \frac{\Lambda}{\lambda}\|z-w\|_0,\end{aligned}$$

in view of the inequality

$$e^{\gamma t} - 1 \leqslant \gamma e^{t} \qquad \text{for } \gamma\in[0, 1],\ t\geqslant 0.$$

Thus assertion 1° of lemma is implied by the Banach fixed point theorem.

Now we prove 2°. At first we show that any solution of equation (6) is a solution of equation (7).

Indeed, if z^* is a solution of equation (6), then we have

$$z^*(t)-L(t)\int_0^{\alpha(t)} z^*(s)\,ds - l(t)z^*\big(\beta(t)\big)-h(t) = z^*(t)-L(t)\int_0^{\alpha(t)} z^*(s)\,ds -$$

$$-l(t)\Big[\sum_{i=0}^{\infty} L\big(\beta_{i+1}(t)\big)l_i\big(\beta(t)\big)\int_0^{\alpha(\beta_{i+1}(t))} z^*(s)\,ds+\sum_{i=0}^{\infty} l_i\big(\beta(t)\big)h\big(\beta_{i+1}(t)\big)\Big]-h(t)$$

$$= z^*(t)-L(t)\int_0^{\alpha(t)} z^*(s)\,ds-\sum_{i=0}^{\infty} L\big(\beta_{i+1}(t)\big)l_{i+1}(t)\int_0^{\alpha(\beta_{i+1}(t))} z^*(s)\,ds -$$

$$-\sum_{i=0}^{\infty} l_{i+1}(t)h\big(\beta_{i+1}(t)\big)-h(t)$$

$$= z^*(t)-\sum_{i=0}^{\infty} L\big(\beta_i(t)\big)l_i(t)\int_0^{\alpha(\beta_i(t))} z^*(s)\,ds-\sum_{i=0}^{\infty} l_i(t)h\big(\beta_i(t)\big)\equiv 0.$$

To finish the proof of 2° we observe that

$$l_n(t)l_i\big(\beta_n(t)\big)=l_{n+i}(t), \qquad \beta_i\big(\beta_n(t)\big)=\beta_{n+i}(t)$$

and

$$l_n(t)z_0\big(\beta_n(t)\big)=l_n(t)\sum_{i=0}^{\infty} L\big(\beta_{n+i}(t)\big)l_i\big(\beta_n(t)\big)\int_0^{\alpha(\beta_{n+i}(t))} z_0(s)\,ds+$$

$$+\sum_{i=0}^{\infty} l_n(t)l_i\big(\beta_n(t)\big)h\big(\beta_{n+i}(t)\big)=\sum_{i=n}^{\infty} L\big(\beta_i(t)\big)l_i(t)\int_0^{\alpha(\beta_i(t))} z_0(s)\,ds+$$

$$+\sum_{i=n}^{\infty} l_i(t)h\big(\beta_i(t)\big)\leqslant \max_I z_0(t)\cdot\sum_{i=n}^{\infty} L\big(\beta_i(t)\big)l_i(t)\,\alpha\big(\beta_i(t)\big)+$$

$$+\sum_{i=n}^{\infty} l_i(t)h\big(\beta_i(t)\big).$$

Hence we get

$$l_n(t)z_0\big(\beta_n(t)\big)\xrightarrow{u} 0 \quad \text{if } n\to\infty \tag{9}$$

($g_n(t)\xrightarrow{u} g(t)$ denotes the uniform convergence in I).

Further, by induction, we easily obtain for any solution $\bar{z} \in M_0(I, R_+)$ of (7) the following relation:

$$(10) \qquad \bar{z}(t) = \sum_{i=0}^{n-1} L(\beta_i(t)) l_i(t) \int_0^{a(\beta_i(t))} \bar{z}(s)\, ds + \sum_{i=0}^{n-1} l_i(t) h(\beta_i(t)) + $$
$$+ l_n(t) \bar{z}(\beta_n(t)), \quad n = 1, 2, \ldots, \ t \in I.$$

If $\bar{z} \in M_0(I, R_+)$, then for some $c \geqslant 0$ we have

$$0 \leqslant \bar{z}(t) \leqslant c \cdot z_0(t), \quad t \in I;$$

now, according to (9), we infer that

$$l_n(t) \bar{z}(\beta_n(t)) \xrightarrow{u} 0 \quad \text{as } n \to \infty.$$

If we let $n \to \infty$ in (10), we get

$$\bar{z}(t) = \sum_{i=0}^{\infty} L(\beta_i(t)) l_i(t) \int_0^{a(\beta_i(t))} \bar{z}(s)\, ds + \sum_{i=0}^{\infty} l_i(t) h(\beta_i(t)),$$

i. e. $\bar{z}$ is a solution of equation (6), but this equation has only one solution z_0, thus it results $\bar{z} = z_0$, and 2° is proved.

Finally we prove 3°. Put

$$z_{n+1}(t) = \sum_{i=0}^{\infty} L(\beta_i(t)) l_i(t) \int_0^{a(\beta_i(t))} z_n(s)\, ds, \quad n = 0, 1, \ldots, \ t \in I.$$

Since z_0 is the solution of (6) and $m(t) \geqslant 0$, we easily infer that

$$0 \leqslant z_{n+1}(t) \leqslant z_n(t), \quad n = 0, 1, \ldots, \ t \in I,$$

and $z_n \xrightarrow{u} 0$ as $n \to \infty$. In fact, if

$$z(t) = \lim_{n \to \infty} z_n(t), \quad t \in I,$$

then the function z satisfies the homogeneous equation corresponding to equation (6), but such equation according to 1° has only the trivial solution $z(t) \equiv 0$.

Now if $\tilde{z} \in M_0(I, R_+)$ is a solution of inequality (8), we get by induction

$$\tilde{z}(t) \leqslant \sum_{i=0}^{n-1} L(\beta_i(t)) l_i(t) \int_0^{a(\beta_i(t))} \tilde{z}(s)\, ds + l_n(t) \tilde{z}(\beta_n(t)),$$
$$n = 1, 2, \ldots, \ t \in I.$$

We have for some $c \in R_+$, $\tilde{z}(t) \leqslant c \cdot z_0(t)$, $t \in I$.

Hence and by (9) we find that $\tilde{z}$ satisfies the inequality

$$\tilde{z}(t) \leqslant \sum_{i=0}^{\infty} L(\beta_i(t)) l_i(t) \int_0^{a(\beta_i(t))} \tilde{z}(s)\,ds \leqslant c\cdot z_0(t).$$

Consequently we get

$$0 \leqslant \tilde{z}(t) \leqslant c\cdot z_n(t), \qquad n = 0, 1, \ldots, \quad t \in I.$$

Finally, leting $n \to \infty$, we conclude $\tilde{z}(t) = 0$.

The last assertion of the lemma is obvious.

Remark 1. If the assumptions of Lemma 1 are fulfilled also for $\tilde{h} \in C(I, R_+)$ and $\tilde{h}(t) \leqslant h(t)$, $t \in I$, then the suitable solution $\tilde{z}_0$ of equation (6) with $\tilde{h}$ instead of h established in Lemma 1 is the only solution of equation (7) with h replaced by $\tilde{h}$ in the class $M_0(I, R_+)$ produced by z_0.

This fact follows immediately from the proof of assertion 2° of Lemma 1.

2. Some remarks and further lemmas.

(a) If we assume that

$$(11) \qquad l(t) \leqslant l = \text{const}, \quad L(t) \leqslant L = \text{const}, \quad a(t) \leqslant a\cdot t,$$
$$\beta(t) \leqslant \beta\cdot t, \quad t \in I, \quad a, \beta \in [0, 1],$$

then conditions (4), (5) in Lemma 1 can be replaced by the following ones:

$$(4') \qquad h \text{ is non-decreasing and } \sum_{i=0}^{\infty} l^i h(\beta^i\cdot t) < +\infty,$$

$$(5') \qquad l\beta < 1.$$

We note that now $\beta_i(t) \leqslant \beta^i\cdot t$, $l_i(t) \leqslant l^i$.

(b) If we assume that

$$l(t) \leqslant l, \quad L(t) \leqslant L\cdot t, \quad a(t) \leqslant a\cdot t, \quad \beta(t) \leqslant \beta\cdot t,$$

then conditions (4), (5) can be replaced by (4′) and

$$(5'') \qquad l\beta^2 < 1.$$

(c) If we assume that

$$l(t) \leqslant l\cdot t, \quad L(t) \leqslant L, \quad a(t) \leqslant a\cdot t \quad \beta(t) \leqslant \beta\cdot t,$$

then conditions (4), (5) can be replaced by

$$(4'') \qquad h \text{ is non-decreasing and } \sum_{i=0}^{\infty} (l\beta t)^i h(\beta^i\cdot t) < +\infty,$$

$$(5''') \qquad l\cdot\beta^2\cdot a < 1.$$

It is clear that both conditions (4''), (5''') hold if $l\beta a < 1$.

(d) If we assume that

$$l(t) \leqslant l, \quad L(t) \leqslant Lt, \quad \beta(t) \leqslant \beta t \quad \text{and} \quad a(t) \leqslant at^2,\ a < 1,$$

then (4), (5) can be replaced by (4') and

$$l\beta^3 < 1. \tag{5''''}$$

(e) Finally, if we assume

$$l(t) \leqslant l, \quad L(t) \leqslant L, \quad \beta(t) \leqslant t^2, \quad a(t) \leqslant at, \quad a < 1,$$

then (4), (5) can be replaced solely by

$$h \text{ is non-decreasing and } \sum_{i=0}^{\infty} l^i h(t^{2^i}) < +\infty.$$

(f) If we suppose (11) and $h(t) \leqslant H \cdot t^p$, $t \in I$, H — positive constant, then (4) and (5) are both fulfilled if $l\beta^\nu < 1$ with $\nu = \min(1, p)$.

However, under such assumptions we can obtain a better result. We have

LEMMA 2. *If $h, L \in C(I, R_+)$, $a, \beta \in C(I, I)$, and conditions* (11) *are fulfilled, moreover, if $h(t) \leqslant H \cdot t^p$, $t \in I$, $p > 0$ and $l\beta^p < 1$, then the assertion of Lemma* 1 *holds if the classes $M(I, R_+)$ and $M_0(I, R_+)$ are both replaced by the class $V_p(I, R_+)$,*

$$V_p(I, R_+) \stackrel{\text{df}}{=} [z : z \in M(I, R_+),\ \|z\|_* < +\infty],$$

where

$$\|z\|_* = \max_I \big(t^{-p} |z(t)|\big).$$

Proof. The proof of this lemma is similar to that of Lemma 1. The only difference is that now we consider equation (6) in the space $V_p^c \stackrel{\text{df}}{=} V_p(I, R_+) \cap C(I, R_+)$.

We introduce the norm

$$\|z\|_{**} \stackrel{\text{df}}{=} \max_I \left(e^{-\lambda t} \frac{|z(t)|}{t^p}\right)$$

and prove that the operator A (defined by the right-hand side of equation (6)) is a contraction in V_p^c if $\lambda > L/(1 - l\beta^p)$. Indeed, if $z \in V_p^c$ and $w = Az$, we get

$$\max_I \big(e^{-\lambda t} t^{-p} w(t)\big) \leqslant \max_I \Big\{\sum_{i=0}^{\infty} e^{-\lambda t} t^{-p} L l^i \int_0^{a \cdot \beta^i t} e^{\lambda s} t^s \max_I \big(e^{-\lambda s} t^{-s} z(s)\big) ds + \\ + \sum_{i=0}^{\infty} l^i H \beta^{pi} t^p e^{-\lambda t} t^{-p}\Big\}$$

$$\leqslant \|z\|_{**}\max_I \sum_{i=0}^{\infty} e^{-\lambda t}t^{-p}Ll^i\alpha^p\beta^{ip}t^p\int_0^t e^{\lambda s}ds+\frac{H}{1-l\beta^p}$$

$$\leqslant L\|z\|_{**}\max_I \sum_{i=0}^{\infty}(l\beta^p)^i e^{-\lambda t}\int_0^t e^{\lambda s}ds+\frac{H}{1-l\beta^p}$$

$$\leqslant L\|z\|_{**}\frac{1}{1-l\beta^p}+\frac{H}{1-l\beta^p}=\frac{1}{1-l\beta^p}(H+L\|z\|_{**}).$$

Thus $w\in V_p^c$. Further we get

$$\|Az-Av\|_{**}\leqslant \max_I t^{-p}e^{-\lambda t}\sum_{i=0}^{\infty}Ll^i\int_0^{\alpha(\beta_i(t))} e^{\lambda s}s^p\max_I\left(e^{-\lambda s}s^{-p}|v(s)-z(s)|\right)ds$$

$$\leqslant \max_I t^{-p}e^{-\lambda t}\sum_{i=0}^{\infty}Ll^i\alpha^p\beta^{pi}t^p\|z-v\|_{**}\int_0^t e^{\lambda s}ds$$

$$\leqslant L\|z-v\|_{**}\frac{1}{1-l\beta^p}\max_I e^{-\lambda t}\int_0^t e^{\lambda s}ds\leqslant \frac{L}{\lambda(1-l\beta^p)}\|z-v\|_{**}.$$

We see that A is a contraction in V_p^c.

The remaining argument is the same as in the proof of Lemma 1.

We want to point out the case when both conditions (4) and (5) in Lemma 1 are superfluous.

We have

LEMMA 3. *If $h, l, L\in C(I, R_+)$, h non-decreasing, $\alpha, \beta\in C(I, I)$, $\alpha(t)\leqslant t$, $\beta(t)\leqslant\gamma(t)\cdot t$, $0\leqslant\gamma(t)<1$, $t\in I$, $\gamma(t)\to 0$ for $t\to 0$ and $h(\nu t)\leqslant\nu^q\cdot h(t)$, $q>0$, $t\in I$, then the assertion of Lemma* 1 *holds.*

Proof. We prove that under assumptions of Lemma 3 both series (4) and (5) are convergent.

Let $l=\max_I l(t)$; then we have

$$l_i(t)\leqslant l^i \quad \text{and} \quad \beta_{i+1}(t)\leqslant\gamma(\beta_i(t))\beta_i(t).$$

Since $\gamma(t)<1$ in I, we get $\beta_n(t)\xrightarrow{u}0$ as $n\to+\infty$, and therefore there exists n_0 such that

$$\gamma(\beta_n(t))<\omega\overset{\mathrm{df}}{=}\min\left(\frac{1}{1+l}, \left(\frac{1}{1+l}\right)^{1/q}\right)$$

for $n\geqslant n_0$ and $t\in I$.

Hence we have

$$\beta_{n+1}(t)\leqslant\omega\cdot\beta_n(t), \qquad n\geqslant n_0, \ t\in I,$$

and consequently

$$\beta_{n_0+j}(t)\leqslant\omega^j\beta_{n_0}(t), \qquad j=1, 2, \ldots, \ t\in I.$$

Finally we get

$$\sum_{i=0}^{\infty} l_i(t)h\big(\beta_i(t)\big) = \sum_{i=0}^{n_0} l_i(t)h\big(\beta_i(t)\big) + \sum_{i=n_0+1}^{\infty} l_i(t)h\big(\beta_i(t)\big)$$
$$= \sum_{i=0}^{n_0} l_i(t)h\big(\beta_i(t)\big) + \sum_{j=1}^{\infty} l_{n_0+j}(t)h\big(\beta_{n_0+j}(t)\big),$$

but

$$\sum_{j=1}^{\infty} l_{n_0+j}(t)h\big(\beta_{n_0+j}(t)\big) \leqslant \sum_{j=1}^{\infty} l^{n_0+j}h\big(\omega^j\beta_{n_0}(t)\big)$$
$$\leqslant l^{n_0}\sum_{j=1}^{\infty} l^j(\omega^j)^q h\big(\beta_{n_0}(t)\big) \leqslant l^{n_0}h\big(\beta_{n_0}(t)\big)\sum_{j=1}^{\infty} l^j\left(\frac{1}{1+l}\right)^j$$
$$= l^{n_0+1}h\big(\beta_{n_0}(t)\big) \leqslant l^{n_0+1}\max_I h\big(\beta_{n_0}(t)\big).$$

Similarly we prove that

$$\sum_{i=n_0+1}^{\infty} L\big(\beta_i(t)\big)l_i(t)\,a\big(\beta_i(t)\big) \leqslant \max_I L(t)\cdot \sum_{i=n_0+1}^{\infty} l_i(t)\beta_i(t)$$
$$\leqslant \max_I L(t)\cdot l^{n_0+1}\beta_{n_0}(t) \leqslant \max_I L(t)\cdot \max_I \beta_{n_0}(t)\cdot l^{n_0+1}.$$

Thus the proof of the lemma is finished.

3. Theorem on the existence and uniqueness. In order to formulate the theorem on the existence and uniqueness of the solution of equation (1) we assume the following:

ASSUMPTION H_1. *There exist $k_1, k_2, l \in C(I, R_+)$ such that*

(12) $$\|F(t, u, v) - F(t, \bar{u}, \bar{v})\| \leqslant k_1(t)\|u-\bar{u}\| + l(t)\|v-\bar{v}\|,$$

(13) $$\|f(t, s, u) - f(t, s, \bar{u})\| \leqslant k_2(t)\|u-\bar{u}\|$$

for $t \in I$, $s \in [0, a(t)]$, $u, \bar{u}, v, \bar{v} \in B$.

Put

(14) $$L(t) = k_1(t)k_2(t), \qquad h(t) = \left\|F\Big(t, \int_0^{a(t)} f(t, s, 0)\,ds, 0\Big)\right\|, \qquad t \in I.$$

Now we can associate with the operator occurring on the right-hand side of (1) the following one:

$$\Omega\big(t, z(\cdot)\big) = L(t)\int_0^{a(t)} z(s)\,ds + l(t)z\big(\beta(t)\big).$$

Under the assumptions of Lemma 1 it is easy to prove that the operator Ω is "good" for equation (1).

In fact, we have

LEMMA 4. *If the assumptions of Lemma* 1 *are fulfilled and*

$$w_0(t) = z_0(t),$$

$$w_{n+1}(t) \stackrel{\mathrm{df}}{=} \Omega\big(t, w_n(\cdot)\big) = L(t)\int_0^{a(t)} w_n(s)\,ds + l(t)\,w_n\big(\beta(t)\big), \tag{15}$$

$$n = 0, 1, \ldots, \ t \in I,$$

then

$$0 \leqslant w_{n+1}(t) \leqslant w_n(t) \leqslant z_0(t), \qquad n = 0, 1, \ldots, \ t \in I, \tag{16}$$

and $w_n(t) \xrightarrow{u} 0$.

Proof. Relation (16) is obtained by induction. The convergence of the sequence $\{w_n\}$ is implied by (16). The limit of this sequence satisfies inequality (8) and by Lemma 1 it must be equal zero identically. The uniform convergence of $\{w_n\}$ follows from Dini's theorem.

Put

$$x_0(t) = 0,$$

$$x_{n+1}(t) = F\Big(t, \int_0^{a(t)} f\big(t, s, x_n(s)\big)\,ds,\ x_n\big(\beta(t)\big)\Big), \qquad n = 0, 1, \ldots, \qquad t \in I. \tag{17}$$

We have the following

LEMMA 5. *If the assumptions of Lemma* 1, *Assumption* H_1 *and* (14) *are fulfilled, and if the sequences* $\{w_n\}$, $\{x_n\}$ *are defined by relations* (15) *and* (17) *respectively, then*

$$\|x_n(t)\| \leqslant z_0(t), \qquad n = 0, 1, \ldots, \ t \in I, \tag{18}$$

$$\|x_{n+p}(t) - x_n(t)\| \leqslant w_n(t), \qquad n, p = 0, 1, \ldots, \ t \in I. \tag{19}$$

Proof. The induction proof is very simple.

According to (12), (13) we have

$$\|x_{n+1}(t)\| \leqslant \Big\| F\Big(t, \int_0^{a(t)} f\big(t, s, x_n(s)\big)\,ds, x_n\big(\beta(t)\big)\Big) - F\Big(t, \int_0^{a(t)} f(t, s, 0)\,ds, 0\Big)\Big\| + h(t)$$

$$\leqslant L(t)\int_0^{a(t)} \|x_n(s)\|\,ds + l(t)\,\big\|x_n\big(\beta(t)\big)\big\| + h(t)$$

and

$$\|x_{n+p+1}(t) - x_{n+1}(t)\|$$

$$\leqslant L(t)\int_0^{a(t)} \|x_{n+p}(s) - x_n(s)\|\,ds + l(t)\,\big\|x_{n+p}\big(\beta(t)\big) - x_n\big(\beta(t)\big)\big\|.$$

From these inequalities by induction we get (18) and (19) respectively.

Thus we have proved that Ω is "good" for equation (1). Summing up the results obtained we can formulate

THEOREM 1. *If Assumption* H_1 *and conditions* (4), (5) *of Lemma* 1 *are satisfied with* L *and* h *defined by* (14), *and if* $m, \overline{m} \in C(L, R_+)$, *then there exists a solution* $\overline{x} \in C(I, B)$ *of equation* (1) *with the property*

$$\|\overline{x}(t)\| \leqslant z_0(t), \qquad t \in I,$$
$$\|\overline{x}(t) - x_n(t)\| \leqslant w_n(t), \qquad t \in I, \; n = 0, 1, \ldots,$$

where x_n, w_n *are defined by* (15) *and* (17) *respectively. Moreover, the solution* $\overline{x}$ *is unique in the class* $X_0(I, B)$, *where*

$$X_0(I, B) \stackrel{\mathrm{df}}{=} \bigcup_{c \geqslant 0} [x : x \in M(I, B), \; \|x(t)\| \leqslant c z_0(t)]$$

($M(I, B)$ *is the class of all strongly measurable functions defined in* I *with range in* B).

Proof. The proof of the existence part of the theorem in view of Lemmas 4, 5 is trivial.

The uniqueness part of the theorem follows immediatelly from 3° of Lemma 1.

Indeed, if we suppose that there exists another solution $\tilde{x}$ of equation (1) belonging to $X_0(I, B)$, then we easily infer that $\|\overline{x} - \tilde{x}\| \in M_0(I, R_+)$ and

$$\|\overline{x}(t) - \tilde{x}(t)\| \leqslant L(t) \int_0^{a(t)} \|\overline{x}(s) - \tilde{x}(s)\| ds + l(t) \|\overline{x}(\beta(t)) - \tilde{x}(\beta(t))\|;$$

now by 3° of Lemma 1 we get $\|\overline{x}(t) - \tilde{x}(t)\| \equiv 0$. Thus the proof of the theorem is completed.

However, Lemma 2 implies the following

THEOREM 2. *If Assumption* H_1, *condition* (11) *with* L *and* h *defined by* (14) *are fulfilled, and if* $h(t) \leqslant H \cdot t^p$, $H \geqslant 0$, $p > 0$, $l\beta^p < 1$, *then the assertion of Theorem* 1 *holds.*

Proof of this theorem remains the same as the proof of Theorem 1.

4. Continuous dependence of the solution on the right-hand side of equation (1). Let us now consider another equation

$$y(t) = F_1\Big(t, \int_0^{a^{(1)}(t)} f_1(t, s, y(s)) ds, y(\beta^{(1)}(t))\Big) \tag{18}$$

with given F_1, f_1, $a^{(1)}$, $\beta^{(1)}$ having the properties referred to in the introduction.

Let $\bar{y}$ be a solution of equation (18). We want to evaluate the distance between $\bar{x}$ and $\bar{y}$.

Put

$$(19)\qquad v(t) \overset{\mathrm{df}}{=} \Big\| F\Big(t, \int_0^{\alpha(t)} f\big(t, s, \bar{y}(s)\big)ds, \bar{y}\big(\beta(t)\big)\Big) - F_1\Big(t, \int_0^{\alpha^{(1)}(t)} f_1\big(t, s, \bar{y}(s)\big)ds, \bar{y}\big(\beta^{(1)}(t)\big)\Big)\Big\|$$

and let $\varphi \in C(I, R_+)$ be such that

$$(20)\qquad \|\bar{x}(t) - \bar{y}(t)\| \leqslant \varphi(t), \quad t \in I.$$

Put

$$h^{(1)}(t) = \max\big(\varphi(t), v(t), h(t)\big).$$

Now we get

THEOREM 3. *If Assumptions* H_1, (14) *and conditions* (4), (5) *of Lemma* 1 *are satisfied with* h *replaced by* $h^{(1)}$, *then there exists a continuous, non-negative solution* $\bar{w}$ *of the equation*

$$(21)\qquad z(t) = L(t)\int_0^{\alpha(t)} z(s)\,ds + l(t)z\big(\beta(t)\big) + v(t)$$

such that

$$\|\bar{x}(t) - \bar{y}(t)\| \leqslant \bar{w}(t), \quad t \in I.$$

Proof. Let $z_0^{(1)}$ be the solution of equation (21) with v replaced by $h^{(1)}$.

Put

$$w_0(t) = z_0^{(1)}(t)$$

and

$$w_{n+1}(t) = L(t)\int_0^{\alpha(t)} w_n(s)\,ds + l(t)w_n\big(\beta(t)\big) + v(t).$$

By induction we get

$$0 \leqslant w_{n+1}(t) \leqslant w_n(t) \leqslant z_0^{(1)}(t), \quad t \in I,\ n = 0, 1, \ldots$$

From this we see that the sequence $\{w_n\}$ is convergent to $\bar{w}$, $\bar{w} \leqslant z_0^{(1)}$, which satisfies equation (21). However, in view of Lemma 1, there exists only one solution of this equation (see Remark 1) in the class of measurable functions satisfying the condition $0 \leqslant z(t) \leqslant z_0^{(1)}(t)$, $t \in I$.

Further we get easily

$$(22)\qquad \|\bar{x}(t) - \bar{y}(t)\| \leqslant w_n(t), \quad n = 0, 1, \ldots, \ t \in I.$$

Indeed, we have

$$\|\bar{x}(t)-\bar{y}(t)\| = \Big\|F\Big(t, \int_0^{\alpha(t)} f\big(t, s, \bar{x}(s)\big)ds, \bar{x}\big(\beta(t)\big)\Big) -$$

$$-F_1\Big(t, \int_0^{\alpha^{(1)}(t)} f_1\big(t, s, \bar{y}(s)\big)ds, \bar{y}\big(\beta^{(1)}(t)\big)\Big)\Big\|$$

$$\leqslant \Big\|F\Big(t, \int_0^{\alpha(t)} f\big(t, s, \bar{x}(s)\big)ds, \bar{x}\big(\beta(t)\big)\Big) - F\Big(t, \int_0^{\alpha(t)} f\big(t, s, \bar{y}(s)\big)ds, \bar{y}\big(\beta(t)\big)\Big)\Big\| + v(t),$$

$$\leqslant L(t)\int_0^{\alpha(t)} \|\bar{x}(s)-\bar{y}(s)\|ds + l(t)\big\|\bar{x}\big(\beta(t)\big)-\bar{y}\big(\beta(t)\big)\big\| + v(t)$$

$$\leqslant L(t)\int_0^{\alpha(t)} z_0^{(1)}(s)ds + l(t)z_0^{(1)}\big(\beta(t)\big) + v(t) = w_1(t),$$

since inequality (20) implies the following one

$$\|\bar{x}(t)-\bar{y}(t)\| \leqslant z_0^{(1)}(t), \quad t \in I.$$

Now relation (22) results by induction.

Leting $n \to \infty$ in (22) we get the assertion of the theorem.

Remark 2. If we take the sequence of equations of type (18)

$$y(t) = F_i\Big(t, \int_0^{\alpha^{(i)}(t)} f_i\big(t, s, y(s)\big)ds, \; y\big(\beta^{(i)}(t)\big)\Big), \quad i = 1, 2, \ldots, \tag{23}$$

such that the assumptions of Theorem 3 are fulfilled for

$$h^*(t) = \sup_i \max\big(\varphi_i(t), v_i(t), h(t)\big)$$

and $v_i(t) \searrow 0$ as $i \to \infty$ (or $v_i(t) \xrightarrow{u} 0$), then

$$\|\bar{x}(t)-\bar{y}_i(t)\| \leqslant w_i(t)$$

and $w_i(t) \searrow 0$ as $i \to \infty$ (or $w_i(t) \xrightarrow{u} 0$).

We note that the index i denotes the suitable functions which are associated with equation (23).

II.1. Assumptions. We are now going to consider the existence problem for equation (1) in a finite dimensional Banach space B under slightly weaker assumptions. We now replace the Lipschitz condition for the functions F and f by the following assumptions:

ASSUMPTION H_2. $1°$ *$B = R^n$ and there exist functions $l, L_0, L_1, H_0, H_1 \in C(I, R_+)$ such that*

$$\|F(t,u,v)-F(t,u,\bar{v})\| \leqslant l(t)\|v-\bar{v}\|,$$
$$\|F(t,u,v)\| \leqslant L_0(t)\|u\|+l(t)\|v\|+H_0(t),$$
$$\|f(t,s,u)\| \leqslant L_1(t)\|u\|+H_1(t)$$

for $t \in I$, $u, v, \bar{u}, \bar{v} \in R^n$;

$2°$ *the assumptions of Lemma 1 are fulfilled for h, L defined by*

$$h(t) = H_0(t)+L_0(t)\int_0^{a(t)} H_1(s)\,ds, \qquad L(t) = L_0(t)L_1(t), \qquad t \in I.$$

ASSUMPTION H_3. *There exist subadditive and non-decreasing functions $\omega_i \in C(R_+, R_+)$, $i = 1, 2, 3, 4, 5$, $\omega_i(0) = 0$, and such that*

$1°$ $\|F(t,u,v)-F(t',u',v)\| \leqslant \omega_1(|t-t'|)+\omega_2(\|u-u'\|)$ *for* $\|u\|, \|u'\| \leqslant R \overset{\mathrm{df}}{=} a\max_I L_1(t)\max_I z_0(t)+\max_I H_1(t)$,

$$\|v\| \leqslant R_1 \overset{\mathrm{df}}{=} \max_I z_0(t), \qquad t, t' \in I,$$

z_0 being defined in Lemma 1;

$2°$ $\|f(t,s,v)-f(t',s,v')\| \leqslant \omega_3(|t-t'|)+\omega_4(\|v-v'\|)$ *for $t, t' \in I$. $0 \leqslant s \leqslant \min(t,t')$, $\|v\|$, $\|v'\| \leqslant R_1$;*

$3°$ $|a(t)-a(t')| \leqslant \omega_5(|t-t'|)$, $\quad t, t' \in I$.

ASSUMPTION H_4. *The following series are convergent:*

$$1° \quad m_1(t,\delta) \overset{\mathrm{df}}{=} \sum_{i=0}^{\infty} l_i(t)\,\omega_2\big(\delta a(\beta_i(t))\big) < +\infty,$$

$$2° \quad m_2(t_1,t_2) \overset{\mathrm{df}}{=} \sum_{i=0}^{\infty} l_i(t_2)\,\omega\big(|\beta_i(t_1)-\beta_i(t_2)|\big) < +\infty \text{ for } t, t_1, t_2 \in I \text{ and } \delta \in R_+,$$

where

$$\omega(s) = \omega_1(s)+\omega_2\big(R\,\omega_5(s)\big)+\omega_2\big(a\,\omega_3(s)\big),$$

and the functions m_i, $i = 1, 2$, are continuous.

Remark 3 It is obvious that Assumption H_4 is fulfilled if for instance $l_i(t) \equiv l < 1$. The functions m_1, m_2 are continuous if the functions l, β, a are non-decreasing (in this case it is easy to find a majorizing series independent on t and to apply the Weierstrass theorem).

Let

$$W \overset{\mathrm{df}}{=} [y : y \in C(I, R^n),\ \|y(t)\| \leqslant z_0(t),\ t \in I]$$

with z_0 defined in Lemma 1.

2. Some lemmas. We have

LEMMA 6. *If Assumption* H_2 *is fulfilled, then for any* $y \in W$ *there exists the unique* $x(\cdot, y) \in W$ *which is the solution of the equation*

$$x(t) = F\Big(t, \int_0^{\alpha(t)} f\big(t, s, y(s)\big)\,ds,\ x\big(\beta(t)\big)\Big). \tag{24}$$

Proof. The proof of this lemma is standard. We define sequences:

$$x_{n+1}(t) = F\Big(t, \int_0^{\alpha(t)} f\big(t, s, y(s)\big)\,ds,\ x_n\big(\beta(t)\big)\Big), \quad x_0(t) \equiv 0,$$
$$n = 0, 1, \ldots, \ t \in I$$

and

$$z^*_{n+1}(t) = l(t) z^*_n\big(\beta(t)\big), \quad z^*_0(t) = z_0(t),$$

with z_0 defined in Lemma 1.

We observe that

$$\|x_n(t)\| \leqslant z_0(t), \quad n = 0, 1, \ldots, \ t \in I$$

and

$$z^*_n(t) = l_n(t) z_0\big(\beta_n(t)\big)$$

and therefore $z_n(t) \xrightarrow{u} 0$ as $n \to \infty$, $t \in I$ (see relation (9)).

Further, by induction, we get

$$\|x_{n+p}(t) - x_n(t)\| \leqslant z^*_n(t), \quad t \in I, \ n, p = 0, 1, \ldots$$

Now the assertion of Lemma is obvious.

Consider the operator U: $W \to W$

$$U^y \stackrel{\mathrm{df}}{=} x(\cdot, y)$$

where $x(\cdot, y)$ is the solution of equation (24) for a given $y \in W$.

LEMMA 7. *If Assumptions* H_2, H_3 *and* 1° *of Assumption* H_4 *are fulfilled, then the operator* U *is continuous.*

Proof. Let $y_1, y_2 \in W$, and $x_i = x(\cdot, y_i)$, $i = 1, 2$, $u(t) = \|x_1(t) - x_2(t)\|$, $\delta = \omega_4\big(\max_I \|y_1(t) - y_2(t)\|\big)$; by our assumptions we get

$$u(t) \leqslant \omega_2\Big(\int_0^{\alpha(t)} \delta\,ds\Big) + l(t)\,u\big(\beta(t)\big)$$

and

$$u(t) \leqslant l_n(t)\,u\big(\beta_n(t)\big) + \sum_{i=0}^{n-1} l_i(t)\,\omega_2\big(\delta\alpha\big(\beta_i(t)\big)\big). \tag{25}$$

But $u(t) \leqslant 2z_0(t)$, consequently we have

$$l_n(t)u(\beta_n(t)) \leqslant 2l_n(t)z_0(\beta_n(t)) = z_n^*(t).$$

Letting $n \to \infty$ in (25) we obtain

$$u(t) \leqslant \sum_{i=0}^{\infty} l_i(t)\,\omega_2(\delta\cdot\beta_i(t)) = m_1(t, \delta).$$

By the continuity of the function m_1 we conclude the assertion of Lemma 7.

LEMMA 8. *If Assumptions* H_2, H_3 *and* H_4 *are fulfilled and*

$$l_n(t_2)z_0(\beta_n(t_1)) \to 0 \quad \text{as } n \to \infty,\ t_1, t_2 \in I, \tag{26}$$

then the set $U(W)$ *is compact.*

Proof. For $y \in W$ and $x = U^y$ we have

$$\|x(t_1, y) - x(t_2, y)\| = \Big\| F\Big(t_1, \int_0^{\alpha(t_1)} f(t_1, s, y(s))\,ds,\ x(\beta(t_1), y)\Big) -$$

$$- F\Big(t_2, \int_0^{\alpha(t_2)} f(t_2, s, y(s))\,ds,\ x(\beta(t_2), y)\Big)\Big\| \leqslant \omega_1(|t_1 - t_2|) + \omega_2(R\omega_5(|t_1 - t_2|)) +$$

$$+ \omega_2(a\omega_3(|t_1 - t_2|)) + l(t_2)\|x(\beta(t_1), y) - x(\beta(t_2), y)\|$$

$$\leqslant \omega(|t_1 - t_2|) + l(t_2)\|x(\beta(t_1), y) - x(\beta(t_2), y)\|.$$

Let $\nu(t_1, t_2) = \|x(t_1, y) - x(t_2, y)\|$; we have

$$\nu(t_1, t_2) \leqslant l(t_2)\nu(\beta(t_1), \beta(t_2)) + \omega(|t_1 - t_2|)$$

and consequently

$$\nu(t_1, t_2) \leqslant l_n(t_2)\nu(\beta_n(t_1), \beta_n(t_2)) + \sum_{i=0}^{n-1} l_i(t_2)\,\omega(|\beta_i(t_1) - \beta_i(t_2)|)$$

for $n = 0, 1, \ldots,\ t_1, t_2 \in I$.

But

$$\nu(t_1, t_2) \leqslant z_0(t_1) + z_0(t_2);$$

thus

$$l_n(t_2)\nu(\beta_n(t_1), \beta_n(t_2)) \leqslant l_n(t_2)z_0(\beta_n(t_1)) + l_n(t_2)z_0(\beta_n(t_2)).$$

By the assumptions of the lemma we get

$$l_n(t_2)\nu(\beta_n(t_1), \beta_n(t_2)) \to 0.$$

Finally, letting $n \to \infty$, we obtain the estimation

$$\nu(t_1, t_2) \leqslant \sum_{i=0}^{\infty} l_i(t_2)\,\omega(|\beta_i(t_1) - \beta_i(t_2)|) = m_2(t_1, t_2).$$

By the continuity of the function m_2 we arrive at the assertion of Lemma 8.

Remark 4. Relation (26) can be assumed only for $t_1 > t_2$, $t_1, t_2 \in I$; it will be satisfied if the functions l, β are non-decreasing or if there exist non-decreasing functions $\tilde{a}, \tilde{L}, \tilde{l}, \tilde{\beta}, \tilde{h}$ such that $a(t) \leqslant \tilde{a}(t)$, $l(t) \leqslant \tilde{l}(t)$, $L(t) \leqslant \tilde{L}(t)$, $\beta(t) \leqslant \tilde{\beta}(t)$, $h(t) \leqslant \tilde{h}(t)$ and (4), (5) hold with a, l, L, β, h replaced by $\tilde{a}$, $\tilde{l}$, $\tilde{L}$, $\tilde{\beta}$, $\tilde{h}$.

3. Existence theorems. Now we can formulate

THEOREM 4. *If Assumptions* H_2, H_3, H_4 *are fulfilled and relation* (26) *holds, then equation* (1) *has at least one solution* $\bar{x} \in W$.

Proof. In view of Lemmas 6, 7, 8 and the Schauder fixed-point theorem the assertion of the theorem is obvious. In fact, we see that the continuous operator U maps the bounded, closed and convex set $W \subset C(I, R^n)$ into its compact subset $U(W)$, thus it has at least one fixed-point.

Now we can indicate some more simple conditions under which the assertion of Theorem 4 holds true.

We have

THEOREM 5. *If Assumption* H_3, *condition* $1°$ *of* H_2 *and* (11) *are satisfied and if*

$$h(t) \leqslant H\cdot t^p, \quad \omega_1(s) = s^\lambda, \quad \omega_2(s) = s^q, \quad \omega_3(s) = s^r, \quad \omega_5(s) = s^\mu,$$

$$\varrho \stackrel{\mathrm{df}}{=} \min(p, q, \lambda, \mu q, rq), \quad l\beta^\varrho < 1,$$

then equation (1) *has at least one solution* $\bar{x} \in W$.

Proof. To prove this theorem it remains to observe that under assumed conditions Assumption H_4 is fulfilled.

It results from the following estimations

$$l_i(t)\,\omega_2\big(\delta a\cdot\beta_i(t)\big) \leqslant l^i\,\omega_2(\delta a\cdot\beta^i\cdot t) = (l\beta^q)^i t^q,$$

$$l_i(t_2)\,\omega\big(|\beta_i(t_1)-\beta_i(t_2)|\big) \leqslant l^i\big(\omega\big(\beta_i(t_1)\big)+\omega\big(\beta_i(t_2)\big)\big),$$

$$l^i\big(\omega(\beta^i\cdot t_1)+\omega(\beta^i\cdot t_2)\big) \leqslant 2l^i\omega(\beta^i\cdot t), \qquad t = \max(t_1, t_2)$$

and

$$l^i\omega(\beta^i t) = (l\beta^\lambda)^i t^\lambda + (l\beta^{q\mu})^i R^q t^{q\mu} + (l\beta^{qr})^i a^q t^{qr}.$$

We know (see Lemma 2) that condition $2°$ of H_2 is satisfied if $l\beta^p < 1$.

Now we find that Assumptions H_2, H_3, H_4 are fulfilled if $l\beta^\varrho < 1$. Thus the theorem is proved.

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