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Stability in generalized processes

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Abstract. General stability-like concepts with respect to sets in processes in the sense of Dafermos [4] and their generalizations are introduced by using a formal idea of Bushaw [2]. A definition is given of so-called Lyapunov functions, by an extension of a general idea presented in papers [16] and [19] by Pelczar. Theorems about the connections between stability conditions and the existence of suitable Lyapunov functions are established.

0. Introduction. The classical qualitative theory od ordinary autonomous differential equations was the origin of the theory of topological dynamical systems and their several generalizations. In particular, certain general theories of stability of the Lyapunov type, discussed with respect to very general systems and so-called semi-systems, have been established and are still investigated in various books and papers (for instance [1], [2], [3], [6]–[8], [15]–[18]; for further reference see for example [1], [18], [20], [21]), as natural extensions of the classical Lyapunov stability theory of differential equations (see the fundamental paper [14]). The first author of the present paper proposed in [18] an uniform terminology, giving a kind of classification of various general (dynamical) systems, which we recall below.

Let X be nonempty set, $(G, +)$ an abelian semi-group with the neutral element 0, and π be a mapping from $G \times X$ into X.

Definition 0.1. The triple $(X, G; \pi)$ is said to be a *pseudo-dynamical semi-system* if and only if

$$\pi(0, x) = x \quad \text{for } x \in X, \tag{0.1}$$

$$\pi\big(t, \pi(s, x)\big) = \pi(t+s, x) \quad \text{for } t, s \in G, \quad x \in X. \tag{0.2}$$

If G is a group, then the word "semi" should be removed, if X is a topological space and G is a topological abelian semi-group (or group) and simultanously π is continuous, then the term "pseudo" is dropped.

Thus in particular, $(X, G; \pi)$ is a *dynamical system* if G is a topological abelian group, X is provided with a topology, $\pi\colon G \times X \to X$ is continuous and satisfies (0.1) and (0.2). This coincides with the usual meaning of the term dynamical system, used in most papers (compare for instance [1], [20], [21]).

In order to extend and generalize the qualitative theory of ordinary nonautonomous differential equations by using an idea similar to that used for general dynamical systems, one has to add another parameter. In this way we approach the theory of so-called processes and their generalizations.

Let X be a nonempty set, called *space* in the sequel, $(G, +)$ an abelian semi-group with the neutral element 0, H a sub-semi-group of G ($0 \in H$) and μ a mapping from the Cartesian product $G \times X \times H$ into the space X.

DEFINITION 0.2. The quadruple $(X, G, H; \mu)$ is said to be a *pseudo-process* if and only if

$$\mu(t, x, 0) = x \quad \text{for } (t, x) \in G \times X, \tag{0.3}$$

$$\mu(t, x, s+r) = \mu(t+s, \mu(t, x, s), r) \quad \text{for } t \in G,\ s, r \in H,\ x \in X. \tag{0.4}$$

This definition, proposed first in [16] (see also [18]), is a minor modification of the original definition of Dafermos ([4], [5]). We use here the name "pseudo-process" in order to underline that we do not assume any continuity condition with respect to μ, while Dafermos in [4] required some regularity assumption (and of course, some topological structure was introduced in X in [4]; G and H were simply the real line and half-line).

It is clear that every pseudo-dynamical system is a pseudo-process. Every dynamical system is a regular process (μ is continuous).

Consider two examples:

EXAMPLE 0.1. Let $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ be continuous and such that for every $(t^0, y^0) \in \boldsymbol{R}^2$ there exists exactly one solution $y(\cdot\,; t^0, y^0)$ of the initial value Cauchy problem

$$y' = f(t, y), \quad y(t^0) = y^0 \tag{0.5}$$

defined in $[t^0, \infty)$ (understood, as usual, as a function continuous on $[t^0, \infty)$, differentiable on (t^0, ∞) and having the right-hand derivative at t^0, fulfilling (0.5) for $t \geqslant t^0$).

Putting

$$\mu(t, x, s) := y(t+s; x, s) \quad \text{for } t, s, x \in \boldsymbol{R},\ s \geqslant 0, \tag{0.6}$$

we obtain a pseudo-process $(\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}_*; \mu)$ with

$$\boldsymbol{R}_* := \{s \in \boldsymbol{R} \colon s \geqslant 0\} \tag{0.7}$$

playing the role of the semi-group H of the group (formally: semi-group) $G = \boldsymbol{R}$.

Remark 0.1. It is clear that we can consider $f\colon \boldsymbol{R}_* \times \boldsymbol{R} \to \boldsymbol{R}$ and assume that (0.5) has exactly one solution for $t^0 \geqslant 0$. Then the pseudo-process induced by (0.5) and by (0.6) will correspond to the case $G = H = \boldsymbol{R}_*$.

EXAMPLE 0.2. Let r be a fixed positive real number. Denote by $\mathscr{C}$ the space of real continuous functions defined on the interval $[-r, 0]$, provided with the usual maximum norm. If a real continuous function y is defined on $[a-r, \infty)$ with some $a > 0$, then by y_t we denote the element of $\mathscr{C}$ defined by

$$y_t(s) := y(t+s), \quad s \in [-r, 0]. \tag{0.8}$$

Suppose that q: $\boldsymbol{R} \times \mathscr{C} \to \boldsymbol{R}$ is continuous and such that for every $t^0 \in \boldsymbol{R}$ and every $x^0 \in \mathscr{C}$ the functional differential initial problem

$$x'(t) = q(t, x_t), \tag{0.9}$$

$$x_{t^0} = x^0, \tag{0.10}$$

has exactly one solution $x(t^0, x^0; \cdot)$ in the interval $[t^0-r, \infty)$.

Recall (see for instance [11]–[13]) that a function x: $[t^0-r, \infty) \to \boldsymbol{R}$ is said to be a solution of (0.9)–(0.10) if and only if it is continuous at every point of its domain, is differentiable in (t^0, ∞) and satisfies (0.9)–(0.10). For details concerning such functional differential equations we refer the reader to the books [9], [10] and papers [11]–[13]; further references can be found in [9]–[13]. Note that some authors require, moreover, that any solution have the right-hand derivative at the point t^0; this condition is satisfied automatically in a large number of particular cases.

Let us now put: $G = H = \boldsymbol{R}_*$, $X = \mathscr{C}$ and

$$\mu(t, \tilde{x}, s) := x_{t+s}(t, \tilde{x}) \quad \text{for } t, s \in \boldsymbol{R}_*, \ \tilde{x} \in \mathscr{C}, \tag{0.11}$$

where

$$x_u(t, \tilde{x}) \quad \text{stands for } x_u(t, \tilde{x}; \cdot)$$

with (see (0.8))

$$x_u(t, \tilde{x}; s) = x(t, \tilde{x}; s+u) \quad \text{for } s \in [-r, 0].$$

It is easy to show by a direct calculation that formula (0.11) gives a pseudo-process.

It is clear that our previous Example 0.1 (modified as in Remark 0.1) can be considered as a "limit case" of Example 0.2, with $r = 0$.

R e m a r k 0.2. It is known that every nonautonomous system of n ordinary differential equations can be replaced by a suitable system of $(n+1)$ autonomous equations. This idea permits us to construct a pseudo-dynamical (or dynamical) semi-system induced by nonautonomous differential equations (for details see for instance [21]), as well as to replace (in a general situation) any given pseudo-process $(X, G, H; \mu)$ by the pseudo-dynamical semi-system $(G \times X, H; \pi)$, where

$$\pi\colon H \times (G \times X) \ni (s, (t, y)) \mapsto (t+s, \mu(t, y, s)) \in G \times X. \tag{0.12}$$

The above method was employed for example in [16]. For our purpose, however, this method does not seem to be so useful as direct investigations with respect to the given pseudo-process.

The paper deals with general stability-like concepts. Stability questions in various versions have been discussed in a large number of papers. Recently one can observe some tendency towards relatively uniform formulations and presentations. One type of such uniform presentation is proposed by Bushaw in [2], [3] and then developed and used in order to obtain some general results by Dana [6], Habets and Peiffer [7], [8] and Trzepizur [22].

We shall apply here the idea of D. Bushaw, introducing below some formal notation, and prove theorems covering several special cases of the results. Our main purpose is to state and prove theorems on connections between some stability-like conditions for given sets and the existence of so-called Lyapunov functions.

1. W-stability of sets. Before we state our formal definitions we shall fix the following notation: for a nonempty set Y we denote by $\mathscr{P}(Y)$ the family of nonempty subsets of Y. We shall consider subfamilies of $\mathscr{P}(X)$ and mappings ranged in $\mathscr{P}(\mathscr{P}(X))$.

1(i) Let $(X, G, H; \mu)$ be a pseudo-process and let a nonempty subset M of X be given (and fixed in the sequel). Assume that there is a mapping

$$\mathscr{B}\colon\ M \ni \psi \mapsto \mathscr{B}(\psi) \in \mathscr{P}(\mathscr{P}(X)) \tag{1.1}$$

and also a family $\mathscr{G}$ of nonempty subsets of X is given.

We introduce the following formal convention: instead of

$$\forall_{\psi\in M},\ \exists_{\psi\in M},\ \forall_{\Gamma\in\mathscr{G}},\ \exists_{\Gamma\in\mathscr{G}},\ \forall_{\Delta\in\mathscr{B}(\psi)},\ \exists_{\Delta\in\mathscr{B}(\psi)},$$

$$\forall_{\sigma\in H},\ \exists_{\sigma\in H},\ \forall_{\tau\in G},\ \exists_{\tau\in G},\ \forall_{\varphi\in\Delta},\ \exists_{\varphi\in\Delta},$$

we shall write

$$P,\ p,\ G,\ g,\ D,\ d,\ S,\ s,\ T,\ t,\ F,\ f,$$

respectively.

By $(-)$ we denote the condition

$$\mu(\tau, \varphi, \sigma+\varrho) \in \Gamma \quad \text{for } \varrho \in H. \tag{1.2}$$

DEFINITION 1.1. Any sequence (ordered subset) of six symbols taken from the set

$$\{P,\ p,\ G,\ g,\ D,\ d,\ S,\ s,\ T,\ t,\ F,\ f\} \tag{1.3}$$

such that

(a) any capital letter excludes the corresponding small letter,

(b) a capital or small letter p stands before a capital or small letter d,

(c) a capital or small letter d stands before a capital or small letter f, is called a *word of the first type*, or – if no misunderstanding is possible – shortly a *word.*

DEFINITION 1.2. Let W be a word of the first type. We say that the set M is *W-stable* if and only if the sentence

$$W(-) \tag{1.4}$$

is true.

Remark 1.1. If a word W has a capital letter S at the last place, then (1.4) is equivalent to

$$W' \ (\mu(\tau, \varphi, \varrho) \in \Gamma \text{ for } \varrho \in H), \tag{1.4'}$$

where W' is the sequence of five letters obtained from the sequence forming W by droping the last letter S.

1(ii) The definition of W-stability covers various particular definitions of stability-like notions considered in the theory of differential equations and functional differential equations as well as in the theory of general dynamical systems and semi-systems.

Let us discuss some of them, especially those related to functional differential equations. So, in the first nine examples, let $(\mathscr{C}, \boldsymbol{R}_*, \boldsymbol{R}_*; \mu)$ be as in Section 0 (Example 0.2) and, for a given $c \geqslant 0$, let

$$M_c := \{\varphi \in \mathscr{C}: \ \|\varphi\| \leqslant c\} \tag{1.5}$$

and

$$\mathscr{G} = \{\Gamma_{c,\varepsilon}\}_{\varepsilon > 0}, \tag{1.6}$$

where

$$\Gamma_{c,\varepsilon} := \{\varphi \in \mathscr{C}: \ \|\varphi\| < c + \varepsilon\} \tag{1.7}$$

and finally

$$\mathscr{B}(\psi) = \{\Delta_\delta^\psi\}_{\delta > 0} \quad \text{for } \psi \in M_c \tag{1.8}$$

with

$$\Delta_\delta^\psi := \{\varphi \in \mathscr{C}: \ \|\varphi - \psi\| < \delta\}. \tag{1.9}$$

We shall consider also

$$\mathscr{B}^0(\psi) = \{{}^0\Delta_\delta^\psi\}_{\delta > 0}, \tag{1.10}$$

where

$${}^0\Delta_\delta^\psi := \{\varphi \in \mathscr{C}: \ \|\varphi\| < \delta\}. \tag{1.11}$$

Observe that ${}^0\Delta_\delta^\psi = \Delta_\delta^0$, where 0 is understood as the constant (zero) function belonging to $\mathscr{C}$.

EXAMPLE 1.1. Let us consider $(\mathscr{C}, \boldsymbol{R}_*, \boldsymbol{R}_*; \mu)$, M_c, $\mathscr{G}$, $\mathscr{B}(\psi)$ as above (see (1.5)–(1.9)) and take the word

$$W = GTPdFS.$$

According to Remark 1.1 we can remove S, and so, in place of $W(-)$ defining the W-stability of M_c, we can consider the equivalent condition

$$GTPdF(\mu(\tau, \varphi, \varrho) \in \Gamma_{c,\varepsilon} \text{ for } \varrho \geqslant 0). \tag{1.12}$$

By virtue of the fact that $\Gamma_{c,\varepsilon}$ is uniquely determined by ε and Δ_δ^ψ by δ (and so we can write $\forall_{\varepsilon>0}$ instead of $\forall_{\Gamma\in\mathscr{G}}$ and $\exists_{\delta>0}$ instead of $\exists_{\Delta\in\mathscr{B}(\psi)}$) we can write condition (1.12) in the form

$$\big(\forall_{\varepsilon>0}\forall_{\tau\geqslant 0}\forall_{\psi\in M_c}\exists_{\delta>0}\forall_{\varphi:\|\varphi-\psi\|<\delta}\big)\big(\|x_{\tau+\varrho}(\tau, \varphi)\| < c+\varepsilon \text{ for } \varrho \geqslant 0\big). \tag{1.13}$$

If, in particular, $c = 0$ and so $M_c = M_0 = \{0\} = \{\text{the constant function } 0\}$, then the letter P is superfluous; more precisely, in this case, condition (1.13) is equivalent to the condition

$$\big(\forall_{\varepsilon>0}\forall_{\tau\geqslant 0}\exists_{\delta>0}\forall_{\varphi:\|\varphi\|<\delta}\big)\big(\|x_s(\tau, \varphi)\| < \varepsilon \text{ for } s \geqslant \tau\big). \tag{1.13'}$$

If the right-hand side of (0.9) vanishes at every point of the form $(t, 0) \in \boldsymbol{R}_* \times \mathscr{C}$, then $x \equiv 0$ is a solution of (0.9)–(0.10) (with $\varphi \equiv 0$) and then, instead of stability questions for M, one discusses stability problems for the zero solution in the classical way. In particular, the zero solution is said to be *stable* (see [11]) if and only if (1.13′) holds true. So it seems to be resonable to call the set M_c (for any $c \geqslant 0$) *stable* if and only if (1.12) holds true.

EXAMPLE 1.2. Consider again the same pseudo-process and the same M_c, $\mathscr{G}$ and $\mathscr{B}(\cdot)$, as in Example 1.1. Let us now discuss the word

$$W_1 = GPdTFS.$$

The W_1-stability of M_c is now equivalent to

$$GPdTF(\mu(\tau, \varphi, \varrho) \in \Gamma_{c,\varepsilon} \text{ for } \varrho \geqslant 0), \tag{1.14}$$

We say that M_c is *uniformly stable* if and only if (1.14) is satisfied. Condition (1.14) reduces to the uniform stability of the solution $x \equiv 0$ in the sense of Definition II in [11] if $c = 0$ and $q(t, 0) = 0$ for $t \geqslant 0$. In this case we replace $\mathscr{B}(\cdot)$ by $\mathscr{B}^0(\cdot)$.

EXAMPLE 1.3. Consider the same pseudo-process as in Example 1.1 and the same family $\mathscr{G}$ given by (1.6)–(1.7), but for the special case $c = 0$ (and so $M_c = \{0\}$); take $\mathscr{B}^0(\psi)$ defined by (1.10) in place of $\mathscr{B}(\psi)$. Let us discuss

$$W_2(-), \tag{1.15}$$

where

$$W_2 = TPdFGs.$$

It is clear that (1.15) is equivalent to

$$(\forall_{\tau \geq 0} \forall_{\psi \in M_0} \exists_{\delta > 0} \forall_{\varphi: \|\varphi\| < \delta} \forall_{\varepsilon > 0} \exists_{\delta > 0}) (\|x_{\tau+\varrho+\sigma}(\tau, \varphi)\| < \varepsilon \text{ for } \varrho \geq 0).$$

Since M_0 has exactly one element we can omit $\forall_{\psi \in M_0}$, and so – replacing $\varrho+\sigma$ by ϱ' – we get

$$(1.16) \quad (\forall_{\tau \geq 0} \exists_{\delta > 0} \forall_{\varphi: \|\varphi\| < \delta} \forall_{\varepsilon > 0} \exists_{\sigma > 0})(\|x_{\tau+\varrho'}(\tau, \varphi)\| < \varepsilon \text{ for } \varrho' \geq \sigma),$$

which is equivalent to

$$(1.17) \quad (\forall_{\tau \geq 0} \exists_{\delta > 0}) \left(\text{if } \|\varphi\| < \delta, \text{ then } \lim_{\varrho \to \infty} \|x_{\tau+\varrho}(\tau, \varphi)\| = 0\right).$$

The authors of paper [11] call the zero solution quasi-asymptotically stable if (1.17) is satisfied.

EXAMPLE 1.4. The solution $x \equiv 0$ of (0.9)–(0.10) is said to be *quasi-uniformly-asymptotically stable* (see [11], Definition IV) if and only if

$$(1.18) \quad (\exists_{\delta > 0} \forall_{\tau \geq 0}) \left(\text{if } \|\varphi\| < \delta, \text{ then } \lim_{\varrho \to \infty} \|x_{\tau+\varrho}(\tau, \varphi)\| = 0\right).$$

It is not difficult to show that (1.18) can be written in our notation as $W_3(-)$ for $M_0 = \{0\}$, where

$$(1.19) \quad W_3 = PdTFGs.$$

EXAMPLE 1.5. The solution of (0.9)–(0.10) with $q(t, 0) = 0$ and $\varphi \equiv 0$, being the zero solution, is said to be *asymptotically stable* (see [11], Definition V) if and only if it is stable (compare Example 1.1) and quasi-asymptotically stable. So $x \equiv 0$ is in our situation asymptotically stable if and only if the conditions $W(-)$ and $W_3(-)$ are satisfied.

EXAMPLE 1.6. Let us consider once again problem (0.9)–(0.10) with $q(t, 0) = 0$ and $\varphi \equiv 0$. The zero solution is said to be *equiasymptotically stable* if and only if (see [11], Definition VI)

$$(1.20) \quad (\forall_{\tau \geq 0} \exists_{\delta > 0} \forall_{\varepsilon > 0} \exists_{\sigma > 0} \forall_{\varphi: \|\varphi\| < \delta})(\|x_{\tau+\varrho'}(\tau, \varphi)\| < \varepsilon \text{ for } \varrho' \geq \sigma).$$

As in Examples 1.1–1.3, we can easily show that (1.20) is in our notation equivalent to

$$(1.21) \quad TPdGsF(-)$$

for $M_c = M_0$.

EXAMPLE 1.7. Assuming that $q(t, 0) = 0$ and $\varphi \equiv 0$, we say that $x \equiv 0$ is *uniform-equiasymptotically stable* (Definition VII in [11]) if and only if

$$(1.22) \quad (\exists_{\delta > 0} \forall_{\varepsilon > 0} \exists_{\sigma > 0} \forall_{\tau \geq 0} \forall_{\varphi: \|\varphi\| < \delta})(\|x_{\tau+\varrho'}(\tau, \varphi)\| < \varepsilon \text{ for } \varrho' \geq \sigma).$$

Condition (1.22) is equivalent to

$$(1.23) \quad PdGsTF(-).$$

EXAMPLE 1.8. Take $M_c = M_0 = \{0\}$ once again and assume the $q(t, 0) = 0$ and $\varphi \equiv 0$. Take $W_b = TPdFgs$. The W_b-stability of M_0 is equivalent to

$$(1.24)\qquad (\forall_{\tau\geqslant 0}\exists_{\delta>0}\forall_{\varphi:\|\varphi\|<\delta}\exists_{\varepsilon>0}\exists_{\sigma>0})(\|x_{\tau+\varrho}(\tau, \varphi)\| < \varepsilon \text{ for } \varrho \geqslant \sigma)$$

(since P can be omitted because M_0 has only one element). This means that every solution of (0.9)–(0.10) starting from any initial function φ which is sufficiently close to zero is bounded. We can say that problem (0.9)–(0.10) has solutions bounded near zero.

There are some other conditions giving a reasonable boundedness property of solutions of problem (0.9)–(0.10), such as for instance those given by the words $TPdFgS$, $TPdgFs$, $TPdgFS$ etc. We shall consider again certain conditions with small g with respect to a slightly more general case in Example 1.1.

EXAMPLE 1.9. Let us consider the pseudo-process, the set M_c, the same family $\mathcal{G}$ and mapping $\mathcal{B}$ as in Example 1.1. Take the word $tpDfgs$. The "stability" condition defined by it means in particular that there is a solution of (0.9)–(0.10), belonging to M_c and being an accumulation point of bounded solutions.

EXAMPLE 1.10. Consider $(\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}_*: \mu)$ defined in Example 0.1 and assume that $f(t, 0) = 0$ and $y^0 = 0$. Take $M = \boldsymbol{R} \times \{0\}$, $\mathcal{B}((x, 0)) = \{B((x, 0), a)\}_{a>0}$, where $B((x, 0), a) := \{(u, v): (x-u)^2 + v^2 < a^2\}$ and finally $\mathcal{G} = \{\Gamma_\varepsilon\}$ with $\Gamma_\varepsilon = \{(x, y):$ if x is an integer, then $|y| < \varepsilon\}$.

Now the condition $W_0(-)$ with $W_0 = PTdFgs$ means that every solution of (0.5) starting from a point sufficiently close to the x-axis is "almost oscillating" for large arguments. The above word W_0 gives a condition of "stability" which seems indeed far removed from classical ones.

EXAMPLE 1.11. We shall now consider a more general situation, showing that classical stability conditions and their direct generalizations are covered by a suitable W-stability.

Let $(X, G, H; \mu)$ be a pseudo-process (arbitrarily fixed). Assume that $M \in \mathcal{P}(X)$, $\mathcal{G} \in \mathcal{P}(\mathcal{P}(X))$, $\mathcal{B}$: $M \to \mathcal{P}(\mathcal{P}(X))$ and if $\psi \in M$, then $\psi \in B$ for every $B \in \mathcal{B}(\psi)$. In this case the condition

$$GPdFTS(-) \Leftrightarrow GPdFT(\mu(\tau, \varphi, \varrho) \in \Gamma \text{ for } \varrho \in H)$$

is equivalent to the condition $S''(\mathcal{G}, \mathcal{B})$ from [16] (Definition 8, p. 544).

EXAMPLE 1.12. Let $(X, G; \pi)$ be a pseudo-dynamical semi-system, and let $M \in \mathcal{P}(X)$, $\mathcal{G} \in \mathcal{P}(\mathcal{P}(X))$ and a mapping $\mathcal{B}$: $M \to \mathcal{P}(\mathcal{P}(X))$ be given. Assume that $\mathcal{B}$ satisfies the condition supposed in Example 1.11. Recall that M is said to be *$S(\mathcal{G}, \mathcal{B})$-stable* (see [16]–[19]) if and only if for every $\Gamma \in \mathcal{G}$ and every $x \in M$, there is a $B \in \mathcal{B}(x)$ such that $\pi(\tau, \bar{x}) \in \Gamma$ for $\tau \in G$ and $\bar{x} \in B$. We can consider $(X, G; \pi)$ as a pseudo-process $(X, G, G; \mu)$, putting $\mu(\tau, x, \varrho) := \pi(\varrho, x)$. Then the $S(\mathcal{G}, \mathcal{B})$-stability of the set M in $(X, G; \pi)$ is equivalent to the W-stability of M in $(X, G, G; \mu)$ with $W = GPdFTS$; this is again equivalent (as has been pointed out in Example 1.11) to the condition

$$GPdFT(\mu(\tau, \varphi, \varrho) = \pi(\varrho, \varphi) \in \Gamma \text{ for } \varrho \in H = G).$$

This observation permits us to conclude that also conditions of type S' from [16] (Definition 7, p. 544) are covered by suitable conditions of the type of W-stability for corresponding product spaces constructed by using the idea presented in [16]. More precisely, we employ the following method: a given pseudo-process is replaced by a suitable pseudo-dynamical semi-system, as is done in [16], and then we treat the transferred stability conditions with respect to the "higher dimensional" pseudo-process obtained from the pseudo-dynamical semi-system by using the formula $\mu(\tau, x, \varrho) = \pi(\varrho, x)$ applied above.

Similarly we can prove that W-stability conditions contain as special cases the weak semi-stability, semi-stability, etc. from [17].

EXAMPLE 1.13. Let (X, r) be a metric space. Assume that $(X, \boldsymbol{R}, \boldsymbol{R}_*; \mu)$ is a pseudo-process, $M \in \mathscr{P}(X)$. Put for $\lambda > 0$

$$B(M, \lambda) := \{x \in X \colon r(x, M) = \inf\{r(x, y) \colon y \in M\} < \lambda\}$$

and for $\varphi \in M$

$$B(\varphi, \lambda) := \{\varphi \colon r(\varphi, \psi) < \lambda\}.$$

For $\psi \in M$ we now put

$$\mathscr{B}(\psi) := \{B(\psi, \lambda) \cap N\}_{\lambda > 0},$$

where N is a given (and fixed) nonempty subset of X.

Finally we put

$$\mathscr{G} := \{B(M, \lambda)\}_{\lambda > 0}.$$

Generalizing Examples 1.1 and 1.2, we shall say that M is *stable* (*uniformly stable*) if and only if M is W-stable with $W = GTPdFS$ ($W = GPdTFS$, respectively).

EXAMPLE 1.14. Assume that X, M, $\mathscr{G}$ are as in Example 1.13 but put $\mathscr{B}(\psi) := \mathscr{G}$. Generalizing Examples 1.3 and 1.4, we shall say that M is *quasi-asymptotically stable* (*quasi-uniformly asymptotically stable*) if and only if M is W-stable with $W = TPdFGs$ ($W = PdTFGs$, respectively).

EXAMPLE 1.15. Let $(X, \|\cdot\|)$ be a vector normed space and let $(X, \boldsymbol{R}, \boldsymbol{R}_*; \mu)$ be a pseudo-process. Assume that $M = \{\varphi^0\}$ and put

$$\mathscr{B}(\varphi^0) := \{B(\varphi^0, \lambda)\}_{\lambda > 0} = \{\{\varphi \colon \|\varphi - \varphi^0\| < \lambda\}\}_{\lambda > 0}, \quad \mathscr{G} := \{B(0, \lambda)\}_{\lambda > 0}.$$

The words $W_{\mathrm{I}} = gPdFts$, $W_{\mathrm{II}} = gPdFTs$, $W_{\mathrm{III}} = PdgFTs$ and similar ones (all with the small letter g) give conditions which are some boundedness properties for the trajectories of the pseudo-process discussed here. The words W_{I} and W_{II} give, in particular, conditions of a uniform type. The first word gives example $W_{\mathrm{I}}(-)$ which is the condition

$$\exists_{\varepsilon>0}\exists_{\delta>0}\ (\text{for every } \varphi \text{ such that } \|\varphi-\varphi^0\|<\delta \text{ there are } \tau\in\boldsymbol{R} \text{ and } \sigma\in\boldsymbol{R}_* \text{ for which } \|\mu(\tau,\varphi,\varrho)\|<\varepsilon \text{ for } \varrho\geqslant\sigma).$$

2. Some conditions equivalent to W-stability. Let $(X, G, H; \mu)$ be a pseudo-process. Assume that $M\in\mathscr{P}(X)$ and there are a mapping (1.1) and a subfamily $\mathscr{G}$ of $\mathscr{P}(X)$. We extend the convention adopted in Section 1.

Assume that there is a nonempty subfamily $\mathscr{L}$ of $\mathscr{P}(G\times X)$. Instead of

$$\forall_{\Lambda\in\mathscr{L}} \quad \text{and} \quad \exists_{\Lambda\in\mathscr{L}}$$

we shall write

$$L \text{ and } l, \text{ respectively.}$$

DEFINITION 2.1. Every sequence of six letters chosen out of the elements of the set

$$\{P,\ p,\ L,\ l,\ D,\ d,\ S,\ s,\ T,\ t,\ F,\ f\} \tag{2.1}$$

satisfying conditions (a)–(c) of Definition 1.1 is said to be a *word of the second type*.

If W is a word of the first type, then by $\hat{W}$ we shall denote the word of the second type obtained from W by substituting L for G or l for g.

By $(\hat{-})$ we denote the condition

$$(\tau+\sigma,\ \mu(\tau,\varphi,\sigma))\in\Lambda. \tag{2.2}$$

Let us adopt the following notation: if $\tau\in G$, then

$$\Lambda_\tau := \{\varphi\in X\colon (\tau,\varphi)\in\Lambda\}. \tag{2.3}$$

Finally, we introduce a convention concerning sequences of three symbols taken from the set $\{G, g, L, l, T\}$, proposing the following

DEFINITION 2.2. If W is a word of the first type, then by Z_W we denote the sequence of three elements taken from the above set given by the formula

$$Z_W := \begin{cases} GlT & \text{if } G \text{ appears in } W,\\ LgT & \text{if } g \text{ appears in } W.\end{cases}$$

DEFINITION 2.3. We say that the pair $(M, \mathscr{L})$ satisfies the condition $C(\hat{W})$ if and only if

(i) $\hat{W}(\hat{-})$ holds true,

(ii) $Z_W(\Lambda_\tau\subset\Gamma)$ holds true,

(iii) $(\tau,\varphi)\in\Lambda \Rightarrow (\tau+\varrho,\ \mu(\tau,\varphi,\varrho))\in\Lambda$ for $\varrho\in H$.

We shall now state and prove our first theorem, which will be used in the sequel as an auxiliary result.

THEOREM 1.1. *The set M is W-stable if and only if there exists a subfamily $\mathscr{L}$ of $\mathscr{P}(G \times X)$ such that $(M, \mathscr{L})$ satisfies the condition $C(\hat{W})$.*

Proof. Assume that M is W-stable and try to construct a nonempty family $\mathscr{L} \subset \mathscr{P}(G \times X)$ satisfying (i)–(iii).

Let us put

$$E(\Gamma) := \{(\tau, \varphi): \mu(\tau, \varphi, \varrho) \in \Gamma \text{ for } \varrho \in H\} \quad \text{for } \Gamma \in \mathscr{G}. \tag{2.4}$$

We claim that

$$\mathscr{L} = \{E(\Gamma): \Gamma \in \mathscr{G}\}$$

has the required properties.

We start with condition (iii). Let $(\tau, \varphi) \in E(\Gamma)$. So $\mu(\tau, \varphi, \varrho) \in \Gamma$ for $\varrho \in H$. Thus, in particular, for every fixed $\varrho \in H$ we have

$$\mu(\tau, \varphi, \varrho + \sigma) \in \Gamma \quad \text{for } \sigma \in H. \tag{2.5}$$

Since

$$\mu(\tau, \varphi, \varrho + \sigma) = \mu(\tau + \varrho, \mu(\tau, \varphi, \varrho), \sigma),$$

we get from (2.5)

$$\mu(\tau + \varrho, \mu(\tau, \varphi, \varrho), \sigma) \in \Gamma \quad \text{for } \sigma \in H, \tag{2.6}$$

which – by virtue of definition (2.4) of $E(\Gamma)$ – gives directly

$$(\tau + \varrho, \mu(\tau, \varphi, \varrho)) \in E(\Gamma). \tag{2.7}$$

Condition (iii) is satisfied.

In order to prove (ii) observe that for every $\Gamma \in \mathscr{G}$ and every $\tau \in G$ we have

$$E(\Gamma)_\tau \subset \Gamma. \tag{2.8}$$

Indeed, if $\varphi \in E(\Gamma)_\tau$, then $(\tau, \varphi) \in E(\Gamma)$, which means that $\mu(\tau, \varphi, \varrho) \in \Gamma$ for every $\varrho \in H$. So in particular $\varphi = \mu(\tau, \varphi, 0) \in \Gamma$; inclusion (2.8) has been proved.

Now let G be in W. So $Z_W = GlT$. Let $\Gamma \in \mathscr{G}$ be fixed. Take $\Lambda = E(\Gamma)$ and take $\tau \in G$. Inclusion (2.8) means in our case that $\Lambda_\tau \subset \Gamma$. $Z_W(\Lambda_\tau \subset \Gamma)$ is true.

If g is in W, then $Z_W = LgT$. Let $\Lambda \in \mathscr{L}$ be given. Take $\Gamma \in \mathscr{G}$ such that $E(\Gamma) = \Lambda$. Let $\tau \in G$ be arbitrarily fixed. Applying (2.8), we finish the proof of $Z_W(\Lambda_\tau \subset \Gamma)$ also in this case. Thus we have proved (ii).

In order to prove (i) it is enough to observe that for $\Lambda = E(\Gamma)$ we have the following sequence of equivalences

$$\begin{aligned}(-) &\Leftrightarrow \mu(\tau, \varphi, \sigma + \varrho) \in \Gamma \quad \text{for } \varrho \in H \\ &\Leftrightarrow \mu(\tau + \sigma, \mu(\tau, \varphi, \sigma), \varrho) \in \Gamma \quad \text{for } \varrho \in H \\ &\Leftrightarrow (\tau + \sigma, \mu(\tau, \varphi, \sigma)) \in \Lambda \Leftrightarrow (\hat{-}).\end{aligned}$$

This proves that for given families $\mathscr{G}$ and $\mathscr{L} = \{E(\Gamma)\colon \Gamma \in \mathscr{G}\}$ we have the equivalence

$$W(-) \Leftrightarrow \hat{W}(\hat{-}). \tag{2.9}$$

Thus $\mathscr{L}$ satisfies all the conditions required in $C(\hat{W})$.

Let us assume now that there is a nonempty subfamily $\mathscr{L}$ of $\mathscr{P}(G \times X)$ such that the pair $(M, \mathscr{L})$ satisfies the condition $C(\hat{W})$.

Assume that G appears in W and so condition (ii) means that

$$GlT(\Lambda_\tau \subset \Gamma). \tag{2.10}$$

Observe that if for some $\Lambda \in \mathscr{L}$ and $\Gamma \in \mathscr{G}$ we have

$$\Lambda_\tau \subset \Gamma \quad \text{for } \tau \in G \tag{2.11}$$

and (for some τ, σ, φ)

$$\big(\tau+\sigma,\ \mu(\tau, \varphi, \sigma)\big) \in \Lambda, \tag{2.12}$$

then (because of (iii))

$$\big(\tau+\sigma+\varrho,\ \mu(\tau+\sigma,\ \mu(\tau, \varphi, \sigma), \varrho)\big) \in \Lambda \quad \text{for } \varrho \in H, \tag{2.13}$$

which means that

$$\big(\tau+\sigma+\varrho,\ \mu(\tau, \varphi, \sigma+\varrho)\big) \in \Lambda \quad \text{for } \varrho \in H, \tag{2.14}$$

and so

$$\mu(\tau, \varphi, \sigma+\varrho) \in \Lambda_{\tau+\sigma+\varrho} \quad \text{for } \varrho \in H, \tag{2.15}$$

and then (see (2.11))

$$\mu(\tau, \varphi, \sigma+\varrho) \in \Gamma \quad \text{for } \varrho \in H. \tag{2.16}$$

The above reasoning proves that, assuming that (ii) and (iii) are fulfilled, we obtain the implication

$$\hat{W}(\hat{-}) \Rightarrow W(-).$$

This proves the W-stability of M under the assumption that G is in W.

The same method is applied if g is in W. The only one change is that (2.10) should be replaced by

$$LgT(\Lambda_\tau \subset \Gamma). \tag{2.17}$$

The proof of the theorem is completed.

Remark 2.1. If S appears in W in the final position, then — as we know — $W(-)$ is equivalent to $W'(\mu(\tau, \varphi, \varrho) \in \Gamma$ for $\varrho \in H)$ (see Section 1). We can extend this observation to words of the second type. If S appears in W (and so in $\hat{W}$) in the final position, then

$$\hat{W}(\hat{-}) \Leftrightarrow \hat{W}'\big((\tau, \varphi) \in \Lambda\big),$$

where $\hat{W}'$ is the sequence of five letters obtained by omitting in $\hat{W}$ the final one, namely S. Indeed, if S is the last letter in $\hat{W}$, then in the condition $(\tau+\sigma, \mu(\tau, \varphi, \sigma))\in \Lambda$ we can put $\sigma = 0$, which gives simply $(\tau, \varphi)\in \Lambda$.

3. Lyapunov functions. Let $(X, G, H; \mu)$ be a pseudo-process and let $M\in \mathscr{P}(X)$, $\mathscr{G}\in \mathscr{P}(\mathscr{P}(X))$ and $\mathscr{B}\colon M\to \mathscr{P}(\mathscr{P}(X))$ be given.

Let $(\mathfrak{A}, \leqslant)$ be a partially ordered space. We put $\mathfrak{A}_+ := \mathfrak{A}\setminus\{\min \mathfrak{A}\}$ if $\min \mathfrak{A}$ exists and $\mathfrak{A}_+ := \mathfrak{A}$ if $\min \mathfrak{A}$ does not exist. Let W be a given word of the first type.

DEFINITION 3.1. We say that a subset N of X is *invariant* if and only if

$$\varphi\in N \Rightarrow [\mu(\tau, \varphi, \varrho)\in N \text{ for every } \tau\in G,\ \varrho\in H]. \tag{3.1}$$

Remark 3.1. If $N = \emptyset$, then N is trivially invariant. In the sequel we shall consider nonempty invariant sets.

Remark 3.2. Implication (3.1) is equivalent to the following one:

$$\begin{aligned}[(\tau, \varphi, \varrho)\in G\times X\times H \text{ and } \mu(\tau, \varphi, \varrho)\in N]\\ \Rightarrow [\mu(\tau, \varphi, \varrho+\sigma)\in N \text{ for } \sigma\in H].\end{aligned} \tag{3.2}$$

Proof of the equivalence (3.1)$\Leftrightarrow$(3.2). Assume (3.1) and take (τ, φ, ϱ) such that $\mu(\tau, \varphi, \varrho)\in N$. Let σ be fixed. From (3.1) we get $\mu(\tau^0, \mu(\tau, \varphi, \varrho), \varrho^0)\in N$ for every $\tau^0\in G$ and $\varrho^0\in H$, and so, in particular for $\tau^0 = \tau+\varrho$, $\varrho^0 = \sigma$ we obtain

$$\mu(\tau+\varrho, \mu(\tau, \varphi, \varrho), \sigma) = \mu(\tau, \varphi, \varrho+\sigma)\in N.$$

Thus (3.1)$\Rightarrow$(3.2).

In order to show the inverse implication we observe that $\varphi = \mu(\tau, \varphi, 0)$, and so

$$\varphi\in N \Leftrightarrow \mu(\tau, \varphi, 0)\in N \quad \text{for } \tau\in G \Rightarrow \mu(\tau, \varphi, \varrho)\in N \quad \text{for } \tau\in G \text{ and } \varrho\in H.$$

DEFINITION 3.2. We say that a function $V\colon \mathscr{N}\to \mathfrak{A}$, where $\mathscr{N} = G\times N$ with some invariant subset N of X, $N\neq \emptyset$, is a *Lyapunov function of the type* $[W; \mathfrak{A}, \mathscr{G}, \mathscr{B}]$ for the set M if and only if the family

$$\mathscr{L} = \{\Lambda^{(\eta)}\colon \eta\in \mathfrak{A}_+\}, \tag{3.3}$$

where

$$\Lambda^{(\eta)} := \{(\tau, \varphi)\in \mathscr{N}\colon V(\tau, \varphi)\leqslant \eta\} \tag{3.4}$$

satisfies conditions (i)–(iii) of Definition 2.3, and moreover,

$$V(\tau+\varrho, \mu(\tau, \varphi, \varrho))\leqslant V(\tau, \varphi) \quad \text{for } (\tau, \varphi)\in \mathscr{N}. \tag{3.5}$$

Remark 3.3. Because of the fact that N is invariant, the set satisfies the condition

$$(\tau, \varphi)\in \mathscr{N} \Rightarrow (\tau+\varrho, \mu(\tau, \varphi, \varrho))\in \mathscr{N} \quad \text{for } \varrho\in H. \tag{3.6}$$

Conversely, if $\mathcal{N} = G \times N$ satisfies (3.6), then N is invariant.

THEOREM 3.1. *If there exists a Lyapunov function of the type* $[W; \mathfrak{A}, \mathcal{G}, \mathcal{B}]$ *for* M, *defined in some set* $\mathcal{N}$ *of the form* $G \times N$, *the set* $N \in \mathcal{P}(X)$ *being invariant, then* M *is* W*-stable.*

Proof. By virtue of Theorem 2.1 it is enough to show that every set $\Lambda^{(\eta)}$ satisfies condition (iii) of Definition 2.3. In order to do that we apply (3.5). Let $\eta \in \mathfrak{A}_+$ be fixed. We have

$$(\tau, \varphi) \in \Lambda^{(\eta)} \Leftrightarrow V(\tau, \varphi) \leqslant \eta \Rightarrow V(\tau+\varrho, \mu(\tau, \varphi, \varrho)) \leqslant \eta \quad \text{for } \varrho \in H,$$
$$\Leftrightarrow (\tau+\varrho, \mu(\tau, \varphi, \varrho)) \in \Lambda^{(\eta)} \quad \text{for } \varrho \in H.$$

The proof is completed.

Remark 3.4. Theorem 3.1 generalizes in particular Theorem 4 from paper [16].

Now we are going to establish an inverse result. This will be effected with respect to the special case $\mathfrak{A} = \boldsymbol{R}_*$, $\mathfrak{A}_+ = \boldsymbol{R}_+ = (0, \infty)$.

THEOREM 3.2. *Suppose that* $\mathcal{G}$ *is indexed by* $\boldsymbol{R}_+$ *in such a way that* $\mathcal{G} = \{\Gamma_a\}_{a>0}$ *and* $t < s \Rightarrow \Gamma_t \subset \Gamma_s$. *Assume that there is a* $b > 0$ *and there exists an* $N \in \mathcal{P}(X)$ *satisfying condition*

$$N \text{ is invariant and } N \subset \Gamma_b. \tag{3.7}$$

If M *is* W*-stable, then there is a Lyapunov function of the type* $[W; \boldsymbol{R}_*, \mathcal{G}, \mathcal{B}]$ *for* M.

Proof. Define $\mathcal{N} := G \times N$, where N is such that (3.7) holds true. Put

$$V(\tau, \varphi) := \inf\{a \in \boldsymbol{R}_+ : \mu(\tau, \varphi, \varrho) \in \Gamma_a \text{ for } \varrho \in H\} \tag{3.8}$$

and consequently

$$\Lambda^{(a)} := \{(\tau, \varphi) \in \mathcal{N} : V(\tau, \varphi) \leqslant a\}. \tag{3.9}$$

Definition (3.8) is correct since, for every (τ, φ), we have $\mu(\tau, \varphi, \varrho) \in \Gamma_b$ for every $\varrho \in H$ (see (3.7)).

We shall show that V satisfies all conditions required from Lyapunov functions.

I. We have for $(\tau, \varphi) \in \mathcal{N}$:

$$\begin{aligned} V(\tau+\varrho, \mu(\tau, \varphi, \varrho)) &= \inf\{a \in \boldsymbol{R}_+ : \mu(\tau+\varrho, \mu(\tau, \varphi, \varrho), \sigma) \in \Gamma_a \text{ for } \sigma \in H\} \\ &= \inf\{a \in \boldsymbol{R}_+ : \mu(\tau, \varphi, \varrho+\sigma) \in \Gamma_a \text{ for } \sigma \in H\} \\ &\leqslant \inf\{a \in \boldsymbol{R}_+ : \mu(\tau, \varphi, \lambda) \in \Gamma_a \text{ for } \lambda \in H\} \\ &= V(\tau, \varphi). \end{aligned}$$

Thus we have proved that V satisfies (3.5).

II. In order to prove condition (ii) of Definition 2.3 for the family $\{\Lambda^{(a)}\}_{a>0}$ we shall first show the following

LEMMA. *If* $0 < a' < a^0 < b$, *with* $b > 0$ *satisfying* (3.7), *then for every* $\tau \in G$

$$(\Lambda^{(a')})_\tau \subset \Gamma_{a^0}. \tag{3.10}$$

Proof of lemma. We have

$$\begin{aligned}\varphi \in (\Lambda^{(a')})_\tau &\Leftrightarrow (\tau, \varphi) \in \Lambda^{(a')} \Leftrightarrow V(\tau, \varphi) \leqslant a' \\ &\Leftrightarrow \inf\{a \in \boldsymbol{R}_+ :\ \mu(\tau, \varphi, \varrho) \in \Gamma_a \text{ for } \varrho \in H\} \leqslant a' \\ &\Rightarrow \inf\{a \in \boldsymbol{R}_+ :\ \mu(\tau, \varphi, \varrho) \in \Gamma_a \text{ for } \varrho \in H\} < a^0 \\ &\Rightarrow \text{there is } c \in (0, a^0] \subset \boldsymbol{R}_+ \text{ such that } \mu(\tau, \varphi, \varrho) \in \Gamma_c \text{ for } \varrho \in H \\ &\Rightarrow \mu(\tau, \varphi, \varrho) \in \Gamma_{a^0} \text{ for } \varrho \in H \Rightarrow \mu(\tau, \varphi, 0) \in \Gamma_{a^0} \Leftrightarrow \varphi \in \Gamma_{a^0}. \blacksquare\end{aligned}$$

By using the above lemma we can now easily prove (ii) in both cases. If $Z_W = GlT$, then for every Γ_{a^0} (we can assume without loss of generality that $a^0 \leqslant b$) we take $\Lambda^{(a')}$ with any fixed $a' \in (0, a^0)$ and for every $\tau \in G$ we get (3.10). If $Z_W = LgT$, then for every fixed $\Lambda^{(c)}$ such that $c \in (0, b)$ we take $\bar{c} \in (c, b)$ and apply (3.10) for $a' = c$, $a^0 = \bar{c}$, which gives

$$(\Lambda^{(c)})_\tau \subset \Gamma_{\bar{c}}.$$

Finally, if $Z_W = LgT$ and $\Lambda^{(c)}$ is given with $c > b$, then we use the inclusion $\Gamma_b \subset \Gamma_c$ and observe that for every $\bar{c} \in \boldsymbol{R}_+$ and every $\tau \in G$ the following sequence of implications holds true:

$$\varphi \in (\Lambda^{(\bar{c})})_\tau \Rightarrow (\tau, \varphi) \in \mathcal{N} \Rightarrow \varphi \in \Gamma_b.$$

So, if $c > b$, then $(\Lambda^{(c)})_\tau \subset \Gamma_c$. Thus (ii) is satisfied for both cases.

III. In order to show that the sentence $\hat{W}(\hat{-})$ is true with respect to $\mathscr{L} = \{\Lambda^{(a)}\}_{a>0}$ it is enough to observe that for every $\tau \in G$, $\varphi \in X$ and $\sigma \in H$ we have:

$$\begin{aligned}\mu(\tau, \varphi, \sigma+\varrho) \in \Gamma_a \quad &\text{for every } \varrho \in H \Leftrightarrow (-) \\ &\Leftrightarrow \mu\big(\tau+\sigma, \mu(\tau, \varphi, \sigma), \varrho\big) \in \Gamma_a \quad \text{for } \varrho \in H \\ &\Rightarrow V\big(\tau+\sigma, \mu(\tau, \varphi, \sigma)\big) \leqslant a \Leftrightarrow \big(\tau+\sigma, \mu(\tau, \varphi, \sigma)\big) \in \Lambda^{(a)} \Leftrightarrow (\hat{-}).\end{aligned}$$

So, if $W(-)$ is true, then $\hat{W}(\hat{-})$ is also satisfied in both cases, that is for G and g appearing in W.

The proof is completed.

Remark 3.5. It is possible to generalize Theorem 3.2 by replacing $\boldsymbol{R}_*$ by more general partially ordered spaces. Assume that $(\mathfrak{A}, <)$ is partially ordered space such that

(3.11) for every $\mathfrak{M} \in \mathscr{P}(\mathfrak{A})$ there is $\inf \mathfrak{M}$;

(3.12) for every $\lambda \in \mathfrak{A}_+$ there is $\lambda^0 \in \mathfrak{A}_+$ such that $\lambda^0 < \lambda$;

(3.13) for every $\mathfrak{M} \in \mathscr{P}(\mathfrak{A})$ and every $\lambda \in \mathfrak{A}_+$ such that $\inf \mathfrak{M} < \lambda$ there is $\eta \in \mathfrak{M}$ such that $\eta \leqslant \lambda$.

We easily show that

(3.14) if $\lambda, \eta \in \mathfrak{A}_+$, then $\inf\{\lambda, \eta\} \in \mathfrak{A}_+$.

Basing ourselves on the above properties, we can prove the existence of a Lyapunov function of the type $[W; \mathfrak{A}, \mathscr{G}, \mathscr{B}]$, $V: \mathscr{N} \to \mathfrak{A}$ for M, if $\mathscr{G}$ is indexed monotonically by $\mathfrak{A}_+$ and M is W-stable, by using the same method as above. We refer the reader to [18] and [19] for details concerning such partially ordered spaces $(\mathfrak{A}, \leqslant)$ and for the application of the technical calculation used there to constructions of Lyapunov functions in pseudo-dynamical semi-systems.

Remark 3.6. The assumption of the existence of an element $b > 0$ and a set N such that (3.7) holds true is satisfied in a large number of particular cases important for applications. Note for instance that many natural stability conditions imply the fact that M is invariant; if that case occurs, then for $\mathscr{G}$ such that M is contained in every set of this family (it is enough if it is contained only in one of those sets) we meet the assumption mentioned above, which is satisfied trivially with $N = M$. The condition $M \subset \Gamma$ for every $\Gamma \in \mathscr{G}$ is quite natural and frequently satisfied in classical situations. For applications we usually require that N should contain the set M together with some neighbourhood (if X is a topological space) or with a family of sets belonging to $\mathscr{B}(\psi)$, $\psi \in M$. This is ensured in many cases. We refer the reader to [19], where the main results concerning the connections between the stability and the existence of Lyapunov functions are established in such a form that those functions are defined in invariant sets containing M together with $\bigcup\{B_\psi: \psi \in M\}$, where, for every ψ, B_ψ is chosen in $\mathscr{B}(\psi)$. This is done with respect to stabilities simpler than the general W-stabilities considered above, but the same idea can be applied here. We have for instance the following

THEOREM 3.3. *Assume that* $(X, G, H; \mu)$ *is a pseudo-process,* $M \in \mathscr{P}(X)$, $\mathscr{G} \in \mathscr{P}(\mathscr{P}(X))$ *and* $\mathscr{B}: M \to \mathscr{P}(\mathscr{P}(X))$. *If* M *is W-stable with* $W = GTPdFS$, *then for every* $\Gamma \in \mathscr{G}$ *there exists a set* N *such that*

(3.15) *N is invariant, $N \subset \Gamma$, and for every $\psi \in M$ there is a $B \in \mathscr{B}(\psi)$ such that $B \subset N$.*

Proof. It is enough to observe that for every Γ the union of those B_ψ which are elements of $\mathscr{B}(\psi)$ $(\psi \in M)$ chosen according to the W-stability of M for all $\tau \in G$ is the set N satisfying all conditions (3.15).

COROLLARY. *If $W = GTPdFS$, then the assumption of the existence of Γ_b and N fulfilling* (3.7) *is superfluous. Moreover, the set N, obtained by using the method presented in the proof of Theorem* 3.3, *has the important additional property written as the third one in* (3.15). *Thus in this case Theorem* 3.2 *can be stated in a stronger form.*

4. Some sufficient conditions for W-stability. Assume that $(X, G, H; \mu)$ is a pseudo-process, $M \in \mathscr{P}(X)$, $\mathscr{G} \in \mathscr{P}(\mathscr{P}(X))$ and $\mathscr{B}$: $M \to \mathscr{P}(\mathscr{P}(X))$. Let W be a word of the first type.

THEOREM 4.1. *Assume that there is a subfamily $\mathscr{L}^0$ of $\mathscr{P}(X)$, $\mathscr{L}^0 \neq \emptyset$, such that*

(i^0) $\hat{W}^0(\mu(\tau, \varphi, \sigma) \in \Lambda^0)$,

(ii^0) $Z_W^0(\Lambda^0 \subset \Gamma)$,

(iii^0) *every set $\Lambda^0 \in \mathscr{L}^0$ is invariant,*

where $\hat{W}^0$ is obtained from $\hat{W}$ by replacing L by $L^0 \Leftrightarrow \forall_{\Lambda^0 \in \mathscr{L}^0}$ (and l by $l^0 \Leftrightarrow \exists_{\Lambda^0 \in \mathscr{L}^0}$), and $Z_W^0 = Gl^0$ if $\mathcal{G}$ is in W and $Z_W^0 = L^0 g$ if g is in W.

Then M is W-stable.

Proof. It is enough to observe that by putting $\mathscr{L} := \{G \times \Lambda^0 \colon \Lambda^0 \in \mathscr{L}^0\}$ we obtain a subfamily of $\mathscr{P}(G \times X)$ satisfying conditions (i)–(iii) of Definition 2.3, and so we can apply Theorem 1.1.

Remark 4.1. If S is in the last place in W, then (i^0) is equivalent to $\hat{W}^{0\prime}(\varphi \in \Lambda^0)$, where $\hat{W}^{0\prime}$ is obtained from $\hat{W}^0$ by omitting S and T if T is in $\hat{W}^0$ or S and t if t is in $\hat{W}^0$ (keeping all the other letters unchanged). Indeed, if S is in the last position in W (and so in $\hat{W}^0$), then in the relation $\mu(\tau, \varphi, \sigma) \in \Lambda^0$ we can put $\sigma = 0$, which gives $\varphi \in \Lambda^0$ and then S and T (or t) are superfluous.

THEOREM 4.2. *Assume that there is a nonempty and invariant subset N of X and a function $\hat{V}$: $N \to \boldsymbol{R}_*$ such that the family $\mathscr{L}^0 := \{\Lambda^{0(a)}\}_{a>0}$, where $\Lambda^{0(a)} := \{\varphi \in N\colon V(\varphi) \leqslant a\}$ satisfies conditions (i^0) and (ii^0) assumed in Theorem 4.1 and, moreover,*

$$\hat{V}(\mu(\tau, \varphi, \varrho)) \leqslant \hat{V}(\varphi) \quad \text{for } \varphi \in N,\ \tau \in G,\ \varrho \in H. \tag{4.1}$$

Then the set M is W-stable.

Proof. It is easy to see that condition (4.1) implies (iii^0) for the family $\mathscr{L}^0$, and so we can apply Theorem 4.1, which finishes the proof.

EXAMPLE 4.1. Let $(X, \boldsymbol{R}, \boldsymbol{R}_*; \mu)$, M, $\mathscr{G}$, $\mathscr{B}$ be as in Example 1.13. Let N be an invariant subset of X and let $\hat{V}$: $N \to \boldsymbol{R}_*$ be such that (4.1) holds true (with $G = \boldsymbol{R}$, $H = \boldsymbol{R}_*$) and

$$\forall_{\varepsilon>0} \exists_{\delta>0} \big(y \in N\colon\ \hat{V}(y) \leqslant \delta \Rightarrow r(M, y) < \varepsilon\big), \tag{4.2}$$

which is equivalent to

$$\forall_{\eta>0} \inf\{V(y)\colon\ y \in N \setminus B(M, \eta)\} > 0, \tag{4.2'}$$

and, moreover,

$$\forall_{\psi \in M} \forall_{\varepsilon>0} \exists_{\delta>0} \forall_{\varphi \in B(\psi,\delta) \cap N} \big(\hat{V}(\varphi) \leqslant \varepsilon\big). \tag{4.3}$$

Then M is uniformly stable in the sense of the definition introduced in Example 1.13 and also – obviously – stable. Condition (4.3) implies for $W = PGdFTS$ the condition $W^{0'}(\varphi \in \Lambda^0)$.

Observe that the existence of such a function $\hat{V}$ is sufficient for uniform stability; trying to find an analogous function for stability, we come a cross the same function. This shows that the sufficient conditions considered here are not necessary.

EXAMPLE 4.2. Let the pseudo-process, the set M and the family $\mathscr{G}$ be the same as before but $\mathscr{B}(\psi) = \mathscr{G}$ for $\psi \in M$. Let $\hat{V}$: $N \to \boldsymbol{R}_*$ be such that (4.1) and (4.2) are satisfied and, moreover,

$$\forall_{\tau \in \boldsymbol{R}} \exists_{\delta > 0} \forall_{\varphi \in B(M,\delta) \cap N} \forall_{\varepsilon > 0} \exists_{\sigma \geqslant 0} \big(V(\mu(\tau, \varphi, \sigma)) \leqslant \varepsilon\big) \tag{4.4}$$

or

$$\exists_{\delta > 0} \forall_{\tau \in \boldsymbol{R}} \forall_{\varphi \in B(M,\delta) \cap N} \forall_{\varepsilon > 0} \exists_{\sigma \geqslant 0} \big(V(\mu(\tau, \varphi, \sigma)) \leqslant \varepsilon\big). \tag{4.5}$$

Then M is quasi-asymptotically stable if (4.4) holds true or quasi-uniformly-asymptotically stable in the case of (4.5) (see Example 1.14).

Remark 4.2. There are several practical problems concerning classical stability conditions such that the function $\hat{V}$ of the type discussed in Examples 4.1 or 4.2 is defined a priori in a set N' which is not necessarily invariant; inequality (4.1) is supposed to be fulfilled for (τ, φ, ϱ) for which $\mu(\tau, \varphi, \varrho) \in N'$. It is possible to replace N' by a subset N which will be invariant. We omit technical details. We refer the reader to [18] for a special case (where pseudo-dynamical semi-systems are considered).

5. It has been pointed out in Section 0, that any pseudo-process $(X, G, H; \mu)$ gives a pseudo-dynamical semi-system $(G \times X, H; \pi)$ with π defined by (0.12). So it seems to be natural that some stability-like concepts considered for subsets of X with respect to $(X, G, H; \mu)$ can be replaced by suitable conditions of stability type for the corresponding subsets of $G \times X$ with respect to π given by (0.12). Theorems on equivalence for pairs of some of those conditions are given in [16]. A standard and natural way to establish theorems of that kind is to define $\tilde{M} := G \times M$ (or $\tilde{M} = A \times X$, where A is a subset of G) and to introduce subfamilies of the type

$$\Omega := \{G \times \Gamma\colon\ \Gamma \in \mathscr{G}\}$$

as well as suitable mappings

$$\tilde{\beta}\colon\ \tilde{M} \to \mathscr{P}(\mathscr{P}(G \times X)),$$

and then replace W-stabilities for M by some stability-like conditions for $\tilde{M}$ (with respect to Ω and $\tilde{\beta}$). As an example let us consider once again the stability condition $S(\Omega, \beta)$ considered in Section 1 (Example 1.12), discussing, however, its connection with a W-stability condition in the inverse direction. Consider a pseudo-process $(X, G, H; \mu)$, $M \in \mathscr{P}(X)$, $\mathscr{G} \in \mathscr{P}(\mathscr{P}(X))$ and

β: $M \to \mathscr{P}(\mathscr{P}(X))$. Let W be equal to $GPdFTS$. Put $\tilde{X} := G \times X$, $\tilde{M} := G \times M$, $\Omega := \{G \times \Gamma\colon \Gamma \in \mathscr{G}\}$, and finally $\tilde{\beta}((\tau, \psi)) := \{G \times B\colon B \in \beta(\psi)\}$. It is easy to see (compare [16], Proposition 3) that M is W-stable if and only if $\tilde{M}$ satisfies the condition $S(\Omega, \tilde{\beta})$, which means that (see Example 1.12) for every $\tilde{x} = (\tau, \psi) \in \tilde{M}$ and every $\tilde{Q} = G \times \Gamma \in \Omega$ there is a $\tilde{B} = G \times B \in \tilde{\beta}(\tilde{x})$ such that

$$\tilde{z} \in \tilde{B} \Rightarrow \pi(\sigma, \tilde{z}) \in \tilde{Q} \quad \text{for } \sigma \in H.$$

The same method can be used for many other cases; the case presented above seems to be the simplest one.

However, there are certain cases of W-stabilities with respect to which this method seems to be useless. This can be observed for instance if the capital letters D and F and the small letter t appear simultanously in the word W. In order to furnish some examples investigate

$$W_a = PDGtFS$$

and

$$W_b = PgDtFs$$

with respect to the pseudo-process $(\boldsymbol{R}, \boldsymbol{R}, \boldsymbol{R}_*; \mu)$, where μ is given by (0.6), considered in Example 0.1, with $M = \{0\} \times \boldsymbol{R}$ and $\mathscr{G} = \{B(0, \varepsilon)\colon \varepsilon > 0\} = \Delta(0)$. The condition $W_a(-)$ is equivalent here to the following one:

$$\forall_{\delta>0} \forall_{\varepsilon>0} \exists_{t^0 \in \boldsymbol{R}} \forall_{x^0} \text{ such that } |x^0| < \delta \exists_{\sigma>0} (|y(t^0+s, x^0, s)| < \varepsilon \text{ for } s > \sigma),$$

which gives the convergence (of uniform type in some sense) to zero of the solutions of the equation

$$(*) \qquad y' = f(t, y).$$

The condition $W_b(-)$ means that there is a constant C such that for every neighbourhood Δ of 0 there is an initial time t^0 such that solutions of $(*)$ passing through (t^0, x^0) with $x^0 \in \Delta$ have the absolute values bounded by C for a large time value.

The method used in [16] and recalled roughly above cannot be used here with respect to W_a and W_b stabilities (at least it cannot be used in its original form). In particular, putting $\tilde{X} = G \times X$ and $\tilde{M} = \boldsymbol{R} \times \{0\}$, we cannot suggest that Ω and $\tilde{\beta}$ should be defined as in the preceding case. Indeed, putting $\Omega = \{\boldsymbol{R} \times B(0, \varepsilon)\colon \varepsilon > 0\}$ we have to find a reasonable $\tilde{\beta}$ in such a way that the W_a-stability (W_b-stability) of M would be replaced by a suitable condition for $\tilde{M}$. Recall that $\pi(\sigma, (\tau, \varphi)) = (\tau + \sigma, \mu(\tau, \varphi, \sigma))$. So the condition

$$(5.1) \qquad \pi(\lambda, (\tau, \varphi)) \in \tilde{Q} \quad \text{for some } \lambda \in \boldsymbol{R}_*,\ \tau \in \boldsymbol{R},\ \varphi \in \tilde{B}$$

means in fact that λ can be fixed arbitrarily, while (τ, φ) is to be chosen in some domain according to some rule. This, however, is complicated since the variable φ is preceded by the universal quantifier and, moreover, φ is taken from Δ, which is also preceded by the universal quantifier, while τ is – on the contrary – preceded by the existential quantifier.

The above remarks show that the W-stabilities considered in the present paper do not seem to be reducible – in general – to stability-like conditions discussed in the theory of pseudo-dynamical systems and semi-systems. On the other hand, we have to underline that our approach to a general presentation of stability-like concepts with respect to pseudo-processes does not cover all possible ones, even for pseudo-dynamical semi-systems. In particular we have not considered stabilities of motions but only stabilities of subsets of the (phase) space X. A presentation which seems to be practically complete with respect to stability-like concepts (including motions) in the theory of pseudo-dynamical semi-systems (with $G = \boldsymbol{R}$) is given by Dana in [6].

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