ANNALES
POLONICI MATHEMATICI
XLVII (1986)

Existence and uniqueness of solution of some integro-differential equation

by Antoni Leon Dawidowicz (Kraków) and Krzysztof Łoskot (Katowice)

Abstract. In this paper the existence and uniqueness of the solution of the problem

$$\frac{\partial u}{\partial t}+c(x,z)\frac{\partial u}{\partial x} = \lambda(x,u,z), \quad z(t) = \int_0^\infty u(t,x)\,dx, \quad u(0,x) = v(x),$$

are proved.

Introduction. The population dynamic can be described by the first-order partial differential equation of the form

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x} = \lambda.$$

The coefficients generally depend on the parameters of population. In the biological interpretation of this equation, $t \geqslant 0$ denotes time, $x \geqslant 0$ proliferation and $u(t,x) \geqslant 0$ is the density of distribution of individuals. As yet, this equation was considered if the coefficient c depends on t and x and the right-hand side of the equation depends on t, x and u [3]. In the presented paper, c and λ do not depend on time but depend on the global number of individuals $z(t)$ at the moment t. Under some assumptions there are proved the existence and uniqueness of solutions of the problem.

1. Formulation of theorems. Let us consider the system of equations

$$\frac{\partial u}{\partial t}+c\big(x,z(t)\big)\frac{\partial u}{\partial x} = \lambda\big(x,u,z(t)\big), \tag{1}$$

$$z(t) = \int_0^\infty u(t,x)\,dx \tag{2}$$

for $t \geqslant 0$ and $x \geqslant 0$ with the initial condition

$$u(0,x) = v(x). \tag{3}$$

In whole of the paper we assume that the coefficients c and λ satisfy the following assumptions.

ASSUMPTION C_1. The coefficient c is of the class C^1 for $x \geqslant 0$ and $z \geqslant 0$.

ASSUMPTION C_2. $c(0, z) = 0$.

ASSUMPTION C_3. $|\partial c/\partial x| \leqslant \alpha$.

ASSUMPTION Λ_1. The function λ is of the class C^1 for $x \geqslant 0$, $z \geqslant 0$, $u \geqslant 0$.

ASSUMPTION Λ_2. $\lambda(x, 0, z) = 0$.

ASSUMPTION Λ_3. $|\partial\lambda/\lambda u| \leqslant B(u, z)$ for a continuous function B.

ASSUMPTION Λ_4. $\partial\lambda/\partial u \leqslant \beta$.

THEOREM 1. *If v is bounded and continuous on $[0, \infty)$, $v(x) \geqslant 0$, and*

$$A = \int_0^\infty v(x)dx < \infty, \tag{4}$$

then there exists a function $u(t, x)$ such that $u \geqslant 0$ and u is the solution of (1), (2), (3).

Remark. We consider the solution in a generalized sense. The sense of solution will be precized in the following section.

Now we shall formulate some new assumptions.

ASSUMPTION C_4. $|\partial c/\partial z| \leqslant \gamma$,

ASSUMPTION C_5. The coefficient c is of the class C^2 and

$$|\partial^2 c/\partial x \partial z| + |\partial^2 c/\partial x^2| \leqslant \mu(z).$$

ASSUMPTION Λ_5. $|\partial\lambda/\partial x| + |\partial\lambda/\partial z| \leqslant \nu(z, u)u$.

In Assumptions C_5 and Λ_5,

$$\mu \text{ and } \nu \text{ are continuous.} \tag{5}$$

THEOREM 2. *Under the assumption of Theorem* 1 *and Assumptions* C_4, C_5 *and* Λ_5, *the solution of problem* (1), (2), (3) *is exactly one.*

2. The method of characteristics and construction of operator. Let $C_+(\Delta)$ be the set of all continuous and non-negative functions on $\Delta = [0, T]$ or $[0, \infty)$. At first we consider problem (1), (3) where z is an arbitrary function from $C_+(\Delta)$. We define the solution of (1), (3). Denote by $\varphi(t, x) = \varphi(t, x, z)$ and $\psi(t, x, y) = \psi(t, x, y, z)$ the characteristics of (1), i.e., the solutions of

$$\xi' = c(\xi, z(t)), \quad \xi(0) = x \tag{6}$$

and

$$\eta' = \lambda(\xi, \eta, z(t)), \quad \eta(0) = y, \tag{7}$$

respectively, for $t \in \Delta$.

DEFINITION 1. The function u: $\Delta \times [0, \infty) \to \boldsymbol{R}$ is a *solution of* (1), (3) if for every $t \in \Delta$, $x \geqslant 0$

$$u(t, \varphi(t, x)) = \psi(t, x, v(x)). \tag{8}$$

Remark. For a given z and for a given v there exists exactly one non-negative solution of (1), (3). (It follows from the classical theory of first-order partial differential equations [1], [4].)

In the following section we shall prove the proposition

PROPOSITION 1. *Under assumptions* C_1–C_3 *and* Λ_1–Λ_4, *if* $z \in C_+(\Delta)$, v *satisfies* (4), *and* u *is the solution of* (1), (3), *then for* $t \geq 0$

$$\int_0^\infty u(t, x)\,dx < \infty, \tag{9}$$

and the function $\Delta \ni t \mapsto \int_0^\infty u(t, x)\,dx$ *is continuous.*

In fact, u depends up on z. (We omit this dependence in notation.)

For fixed $v \geq 0$ define Θz by the formula

$$\Theta z(t) = \int_0^\infty u(t, x)\,dx. \tag{10}$$

From Proposition 1 follows that Θ: $C_+(\Delta) \to C_+(\Delta)$.

DEFINITION 2. The function u: $\Delta \times [0; \infty) \to \boldsymbol{R}$ is the *solution of* (1), (2), (3) if u is the solution of (1), (3) for z satisfying the condition

$$\Theta z = z. \tag{11}$$

Remark. To prove the existence or uniqueness of the solution of (1), (2), (3) it is sufficient to prove the existence or uniqueness (respectively) of the fixed point of operator Θ.

3. Proof of Theorem 1. We start from the proof of Proposition 1. In the proof we shall use the following lemmas.

LEMMA 1. *The* C^1*-function* φ *is defined on* $\Delta \times \boldsymbol{R}_+$ *and the* C^1*-function* ψ *is defined on* $\Delta \times \boldsymbol{R}_+ \times \boldsymbol{R}_+$. *Moreover, for fixed* t *the function* $x \mapsto \varphi(t, x)$ *is a bijection of* $\boldsymbol{R}_+$ *onto* $\boldsymbol{R}_+$.

The lemma is a simple consequence of our assumptions.

Let

$$S(t, x, z) = S(t, x) = \frac{\partial}{\partial x}\varphi(t, x). \tag{12}$$

It is obvious that S satisfies the condition

$$\frac{\partial S}{\partial t} = \frac{\partial c}{\partial x}(\varphi(t, x), z(t))\cdot S, \qquad S(0, x) = 1. \tag{13}$$

LEMMA 2. *The following inequalities are satisfied*

$$0 \leq S(t, x) \leq e^{at}, \qquad 0 \leq \varphi(t, x, y) \leq e^{\beta t} y. \tag{14}$$

The lemma follows from (7), (13), and Assumptions C_3, Λ_4.

Proof of Proposition 1. By the substitution $x = \varphi(t, \eta)$ we have

$$\int_0^\infty u(t, x)dx = \int_0^\infty u(t, \varphi(t, \eta))\frac{\partial}{\partial\eta}\varphi(t, \eta)d\eta = \int_0^\infty \psi(t, \eta, v(\eta))S(t, \eta)d\eta.$$

From Lemma 2 we have

$$0 \leqslant \psi(t, \eta, v(\eta))S(t, \eta) \leqslant e^{(\alpha+\beta)\cdot t}\cdot v(\eta) \leqslant e^{(\alpha+\beta)T}\cdot v(\eta) \tag{16}$$

if $t \leqslant T$. Hence $\int_0^\infty u(t, x)dx \leqslant Ae^{(\alpha+\beta)T} < \infty$.

Moreover, $\Theta z(t) = \int_0^\infty u(t, x)dx$ is a continuous function. This follows from (16) and the Lebesgue dominated convergence theorem. ■

Corollary. *From* (16) *it follows that*

$$\Theta z(t) \leqslant e^{(\alpha+\beta)t}\cdot A.$$

Now we shall prove the continuity of the operator Θ. We shall consider $C_+(\Delta)$ with compact convergence topology. Hence it is sufficient to prove the continuity of Θ on $C_+([0, T])$.

Proposition 2. *For every $T > 0$ the operator*

$$\Theta\colon C_+([0, T]) \to C_+([0, T])$$

is continuous.

Proof. Let us consider $H\colon [0, T]\times \boldsymbol{R}_+ \times C_+([0, T]) \to \boldsymbol{R}$ defined by the formula

$$H(t, x, z) = \psi(t, x, v(x), z)S(t, x, z).$$

From the continuous dependence of the solution on the right-hand side the function H is continuous. Moreover, from (16),

$$H(t, x, z) \leqslant e^{(\alpha+\beta)T}\cdot v(x). \tag{17}$$

Hence the function

$$[0, T]\times C_+([0, T]) \ni (t, z) \mapsto \int_0^\infty H(t, x, z)dx$$

is continuous, which implies the continuity of Θ. ■

Proposition 3. *Let X be the set of all $z \in C_+(\Delta)$ satisfying the condition*

$$z(t) \leqslant Ae^{(\alpha+\beta)\cdot t}. \tag{18}$$

Then $\Theta(X)$ is relatively compact.

Proof. Since v is bounded, from (15) it follows that u is also bounded for $t \leqslant T$. Since z is also bounded for $t \leqslant T$, there exists a B_T such that $|\partial\lambda/\partial u| \leqslant B_T$ for $z \in X$ and u satisfying (1), (3). Hence

$$\left|\frac{\partial H}{\partial t}\right| \leqslant \left|\frac{\partial\psi(t, x, v(x), z)}{\partial t}\right| S(t, x, z) + \psi(t, x, v(x), z)\left|\frac{\partial S(t, x, z)}{\partial t}\right|$$

$$\leqslant (B_T + \varkappa)\, e^{(\alpha+\beta)T} \cdot \varrho(x).$$

Thus

$$|\Theta z(t+h) - \Theta z(t)| \leqslant A(B_T + \varkappa)\, e^{(\alpha+\beta)T} \cdot h \qquad \text{for} \qquad t, t+h \in [0, T].$$

In consequence, if $\Delta = [0, T]$ the proof is complete. If $\Delta = [0, \infty)$, then the set $K \subset C(\Delta)$ is relatively compact if and only if for every $T > 0$ the set of restrictions $\{z| [0, T]: z \in K\}$ is relatively compact. This known theorem completes the proof. ■

To prove Theorem 1 it is sufficient to notice that the set $\tilde{K}$ of all functions from $C_+(\Delta)$ bounded by $Ae^{(\alpha+\beta)t}$ and satisfying the Lipschitz condition with the constant $N(T) = A(B_T + \varkappa)\exp(\alpha+\beta)T$ is convex and compact. From the generalized Schauder fixed-point theorem [2] we obtain the theorem.

4. Proof of Theorem 2. To prove Theorem 2 we must claim the following proposition:

PROPOSITION 4. *Under the assumptions of Theorem 2 for $z_1, z_2 \in \tilde{K}$ the following inequality is satisfied:*

$$\|\Theta z_1 - \Theta z_2\|_T \leqslant M(T)\|z_1 - z_2\|_T, \tag{19}$$

where $\tilde{K}$ is defined in the previous section, $\| \; \|_T$ denotes the norm in $C([0, T])$ and

$$\lim_{T\to 0} M(T) = 0. \tag{20}$$

To prove this proposition we shall at first prove some lemmas.

LEMMA 3. *Under the assumptions of Theorem 2, ψ satisfies the inequality*

$$\int_0^\infty |\psi(t, x, v(x), z_1) - \psi(t, x, v(x), z_2)|\, dx \leqslant M_1(T)\|z_1 - z_2\|_T \tag{21}$$

for $t \in [0, T]$ and $z_1, z_2 \in \tilde{K}$. Moreover,

$$\lim_{T\to 0} M_1(T) = 0. \tag{22}$$

Proof. Let $w(t, x) = \psi(t, x, v(x), z_1) - \psi(t, x, v(z), z_2)$. Obviously, $w(0, x) = 0$. We shall estimate $\dfrac{\partial w}{\partial t}(t, x)$. At first we notice that for $z_1, z_2 \in \tilde{K}$

we have

(23) $$z_i(t) \leqslant Ae^{(\alpha+\beta)T} \quad \text{for } i = 1, 2,$$

(24) $$\psi(t, x, v(x), z_i) \leqslant \sup_{\xi \geqslant 0} v(\xi) e^{\beta T}$$

and, consequently, there exists a compact set F such that $(z_i(t), \psi(t, x, v(x), z_i)) \in F$. There exists a finite number

(25) $$v_0 = \sup\{v(z, u): (z, u) \in F\}.$$

Now we estimate $\frac{\partial}{\partial t} w(t, x)$,

(26) $$\frac{\partial}{\partial t} w(t, x) = D_1 + D_2 + D_3,$$

where

$$D_1 = \lambda(\varphi(z_1), \psi(z_1), z_1) - \lambda(\varphi(z_2), \psi(z_1), z_1),$$
$$D_2 = \lambda(\varphi(z_2), \psi(z_1), z_1) - \lambda(\varphi(z_2), \psi(z_2), z_1),$$
$$D_3 = \lambda(\varphi(z_2), \psi(z_2), z_1) - \lambda(\varphi(z_2), \psi(z_2), z_2).$$

(In the last formulae $\varphi(z_i) = \varphi(t, x, z_i)$, $\psi(z_i) = \psi(t, x, v(x), z_i)$.) From Assumption Λ_5 and (25)

$$|D_1| \leqslant v_0 |\varphi(t, x, z_1) - \varphi(t, x, z_2)| \psi(t, x, v(x), z_1).$$

But

$$\frac{\partial}{\partial t}[\varphi(t, x, z_1) - \varphi(t, x, z_2)] = c(\varphi(t, x, z_1), z_1) - c(\varphi(t, x, z_2), z_2).$$

From Assumptions C_3, C_4 and the Gronwall inequality [5]

(27) $$|\varphi(t, x, z_1) - \varphi(t, x, z_2)| \leqslant \bar{M}(T),$$

where $\lim_{T \to 0} \bar{M}(T) = 0$, and in consequence

$$|D_1| \leqslant v_0 \bar{M}(T) v(x) e^{(\alpha+\beta)T}, \qquad |D_2| \leqslant B_T w(t, x),$$

where B_T is defined in Proposition 3,

$$|D_3| v_0 |z_1(t) - z_2(t)| \psi(t, x, v(x), z_2) \leqslant v_0 \|z_1 - z_2\|_T e^{\beta T} v(x).$$

Therefore,

$$\left|\frac{\partial}{\partial t} w(t, x)\right| \leqslant B_T |w(t, x)| + M'(T) \|z_1 - z_2\|_T v(x),$$

where $\overline{\lim_{T\to 0}} M'(T) < \infty$. From the Gronwall inequality [5]

$$|w(t, x)| \leqslant M'(T)v(x)\|z_1 - z_2\|_T B_T^{-1}(e^{B_T T} - 1).$$

By integration of the last formula we obtain

$$\int_0^{\infty} w(t, x)\,dx \leqslant M'(T)B_T^{-1}(\exp B_T T - 1)A\|z_1 - z_2\|_T.$$

Let $M_1 = M'(T)B_T^{-1}(\exp B_T T - 1)$. We obtain (21). Since we may define $B_T = B_{T_0}$ for $T < T_0$ and some arbitrary T_0, formula (22) is also obvious. ■

The following lemma permits to estimate $S(t, x, z_1) - S(t, x, z_2)$.

LEMMA 4. *Under assumptions of Theorem 2, for $t \leqslant T$ and $z_1, z_2 \in \dot{R}$,*

(28) $$|S(t, x, z_1) - S(t, x, z_2)| \leqslant M_2(T)\|z_1 - z_2\|_T.$$

Moreover,

(29) $$\lim_{T\to 0} M_2(T) = 0.$$

Proof. There exists

(30) $$\mu_0 = \sup\{\mu(z)\colon z \leqslant A\exp(\alpha + \beta)T\}.$$

We shall estimate $\sigma(t, x) = S(t, x, z_1) - S(t, x, z_2)$. From (13)

(31) $$\sigma(0, x) = 0$$

and

(32) $$\partial\sigma/\partial t = E_1 + E_2 + E_3,$$

where

$$E_1 = \left[\frac{\partial c}{\partial x}(\varphi(t, x, z_1), z_1(t)) - \frac{\partial c}{\partial x}(\varphi(t, x, z_2), z_1(t))\right]S(t, x, z_1),$$

$$E_2 = \left[\frac{\partial c}{\partial x}(\varphi(t, x, z_2), z_1(t)) - \frac{\partial c}{\partial x}(\varphi(t, x, z_2), z_2(t))\right]S(t, x, z_2),$$

$$E_3 = \frac{\partial c}{\partial x}(\varphi(t, x, z_2), z_2(t))\sigma.$$

From (27) $|E_1| \leqslant \mu_0 \bar{M}(T)e^{\alpha T}$.

From (14), (30) and Assumption C_5

$$|E_2| \leqslant \mu_0|z_1(t) - z_2(t)|e^{\alpha t} \leqslant \mu_0\|z_1 - z_2\|_T e^{\alpha T}.$$

From (32) and Assumption C_3

(33) $$|\partial\sigma/\partial t| \leqslant M''(T)\|z_1 - z_2\|_T + \alpha|\sigma|,$$

where $\overline{\lim_{T\to 0}} M''(T) < \infty$. Hence, using the Gronwall inequality, from (31) and (33) we obtain

$$|\sigma(t)| \leqslant M''(T)\alpha^{-1}(e^{\alpha T}-1)\|z_1 - z_2\|_T.$$

Denoting $M_2(T) = \alpha^{-1}(e^{\alpha T}-1)M''(T)$ proves Lemma 4. ■

Now, we shall prove Proposition 4. For $t \leqslant T$, $z_1, z_2 \in \tilde{K}$

$$|\Theta z_1(t) - \Theta z_2(t)|$$
$$= \left|\int_0^\infty [\psi(t, x, v(x), z_1)S(t, x, z_1) - \psi(t, x, v(x), z_2)S(t, x, z_2)]\,dx\right|$$

but this is not greater than

$$\int_0^\infty |\psi(t, x, v(x), z_1) - \psi(t, x, v(x), z_2)|\,S(t, x, z_1)\,dx +$$
$$+ \int_0^\infty \psi(t, x, v(x), z_2)|\sigma(t, x)|\,dx$$
$$\leqslant M_1(T)e^{\alpha T}\|z_1 - z_2\|_T + Ae^{\beta T}M_2(T)\|z_1 - z_2\|_T.$$

Denoting $M(T) = M_1(T)e^{\alpha T} + AM_2(T)e^{\beta T}$, we obtain Proposition 4. ■

To prove Theorem 2 it is sufficient to notice that for sufficiently small T the operator Θ: $\tilde{K}_T \to \tilde{K}_T$ fulfils the assumption of the Banach fixed-point theorem ($\tilde{K}_T = \{z|\,[0, T]$: $z \in \tilde{K}\}$). Hence the operator Θ has exactly one fixed point in $\tilde{K}_T$. Since $\Theta(C_+(\Delta)) \subset \tilde{K}$, Θ has no fixed point out of $\tilde{K}$ and Θ has exactly one fixed point in $C_+([0, T])$. To prove that Θ has exactly one fixed point in $C_+(\boldsymbol{R}_+)$ we notice that problem (1), (2), (3) is time-independent, i.e., Theorem 2 remains true in $\Delta = [t_0, T]$ with initial condition

$$u(t_0, x) = \bar{v}(x). \tag{34}$$

From this follows that the set of all $t_0 \in \boldsymbol{R}_+$ for which (1), (2), (3) has exactly one solution in $[0, t_0] \times \boldsymbol{R}_+$ is open in $\boldsymbol{R}_+$. Obviously, it is also closed. This completes the proof. ■

Remark. It is obvious that the presented results remain true if they are considered for $x \leqslant a$. In this case the proof is simpler and some assumptions about bounding may be omitted.

Remark. Some assumptions about bounding are essential for the existence of the solution integrable on $\boldsymbol{R}_+$ defined for $t \geqslant 0$. The authors suppose that some assumptions in the theorem on uniqueness may be omitted. (In this situation the proof must be different from the presented one.)

References

[1] R. Courant, *Partial differential equations*, New York, London 1962.
[2] J. Dugundji, A. Granas, *Fixed point theory*, PWN, Warszawa 1982.
[3] M. E. Gurtin, R. McCamy, *Non-linear age-dependent Population Dynamics*, Arch. Rat. Mech. Anal. 54 (1974), 281–300.
[4] E. Kamke, *Differentialgleichungen II*, Akad. Verlagsellschaft, Leipzig 1965.
[5] J. Szarski, *Differential inequalities*, PWN, Warszawa 1965.

INSTYTUT MATEMATYKI UNIWERSYTETU JAGIELLOŃSKIEGO, KRAKÓW
INSTYTUT MATEMATYKI UNIWERSYTETU ŚLĄSKIEGO, KATOWICE
INSTITUTE OF MATHEMATICS, JAGIELLONIAN UNIVERSITY, CRACOW
INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY, KATOWICE

Reçu par la Rédaction le 1984.06.14