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Further results concerning the Nörlund summability of orthogonal series

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1. The aim of the present paper is to investigate the Nörlund summability of orthogonal series. It contains also a result (Lemma 1) obtained by H. Zaremba, which establishes a solution of a problem raised by the author of this paper. Since this result is strictly connected with a new approach to certain classes of Nörlund means, I thought it proper to include it in this paper.

Let $\{p_n\}$ denote an arbitrary sequence of real numbers such that

$$p_0+p_1+\ldots+p_n = P_n \neq 0 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

A sequence $\{s_n\}$ is said to be *limitable by the method* (N, p_n) *to the value* s if $t_n \to s$, where

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k . \tag{T}$$

The transform (T) will be called the *n-th* (N, p_n)*-mean of the sequence* $\{s_n\}$ and s the *generalized limit* of this sequence [1]. We then write

$$(N, p_n)-\lim s_n = s .$$

By the theorem of Toeplitz the nth (N, p_n)-mean is *regular* if and only if

$$\sum_{v=0}^{n} |p_v| = O(P_n), \tag{a}$$

$$\lim_{n\to\infty} \frac{p_{n-v}}{P_n} = 0, \quad v \geqslant 0 . \tag{b}$$

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[1] Cesàro's method of summability (C, a), $a > 0$, is a special case of the (N, p_n)-means if we write

$$p_n = \binom{n+a-1}{a-1} = \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(a)} .$$

The last condition may be replaced by

$$\text{(c)} \qquad \lim_{n\to\infty} \frac{p_n}{P_n} = 0\,,$$

for we have (with M = a constant)

$$|P_n| \leqslant \sum_{v=0}^{n} |p_v| < M|P_n|\,, \qquad n \geqslant m \geqslant 0\,,$$

$$0 \leqslant \left|\frac{p_{n-v}}{P_n}\right| = \left|\frac{P_{n-v}}{P_n}\right| \cdot \left|\frac{p_{n-v}}{P_{n-v}}\right| < M \left|\frac{p_{n-v}}{P_{n-v}}\right|, \qquad n \geqslant v \geqslant 0\,,$$

where M is an absolute constant (see [10], p. 27).

Thus, condition (c) being satisfied, condition (b) is also satisfied. Conversely, condition (b) implies condition (c), with $v = 0$.

In the previous paper (see [6], pp. 231-232) we have introduced the following two classes of Nörlund means:

The sequence $\{p_n\}$ is said to *belong to the class* M^a, or shortly $\{p_n\} \in M^a$, if the following conditions are satisfied:

$$\text{(i)} \qquad 0 < p_{n+1} < p_n \quad \text{or} \quad 0 < p_n < p_{n+1} \quad (n = 0, 1, 2, \ldots)\,,$$

$$\text{(ii)} \qquad p_0 + p_1 + \ldots + p_n = P_n \nearrow +\infty$$

$$\text{(iii)} \qquad \lim_{n\to\infty} \frac{(n+1)p_n}{P_n} = a\,.$$

It is easy to verify that in the case $a > 0$ conditions (i) and (iii) imply condition (ii).

The sequence $\{p_n\}$ is said to *belong to the class* BVM^a, $a > 0$, if $\{p_n\} \in M^a$ and if

$$S_n = \frac{1}{P_n} \sum_{k=0}^{n} \frac{P_k}{k+1} \qquad (n = 1, 2, \ldots)$$

is a sequence of bounded variation.

In the case of the methods (C, a), $\{p_n\} \in BVM^a$ for $0 < a < 1$.

In the note mentioned above I have dealt with (N, p_n)-summability of orthogonal series restricting myself in general to the case where $\{p_n\} \in M^a$ or $\{p_n\} \in BVM^a$, with $a > \frac{1}{2}$.

The aim of this note is to generalize some of our previous results, and to extend them partially to the case $0 < a \leqslant \frac{1}{2}$. I have succeeded in this owing to a new approach to the class M^a of (N, p_n)-means. Before we define this new class of Nörlund means, we first prove the following

LEMMA 1 [2]. *There exist certain sequences $\{p_n\} \epsilon M^a$, $a > 0$, for which the sequence $\{n(p_n - p_{n-1})/p_n\}$ may be unbounded.*

Proof. We shall distinguish three cases: 1° $a > 1$, 2° $0 < a < 1$, 3° $a = 1$. The proof in the first two cases is based upon the following sequence:

$$a_n = \begin{cases} \dfrac{1}{2^n} & \text{for} \quad n \neq 2^m, \\ \dfrac{1}{\sqrt{n}} & \text{for} \quad n = 2^m, \end{cases} \qquad (n, m = 1, 2, \ldots).$$

Of course, the series $\sum_{n=1}^{\infty} a_n$ is convergent and its partial sums form an increasing sequence $\{s_n\}$ convergent to a value $s > 1$.

1° We set $a_n = a - \lambda(s - s_n)$ $(n = 1, 2, \ldots)$, where λ denotes an arbitrary real and positive number satisfying the condition $0 < \lambda < \dfrac{2(a-1)}{2s-1}$. We find at once that $\{a_n\}$ is an increasing sequence, convergent to a. Choosing an arbitrary positive number upon p_0, we define the sequence $\{p_n\}$ and $\{P_n\}$ by the following recurrent formulae:

$$P_n/P_{n-1} = 1 + a_n/n,$$

$$p_n = P_n - P_{n-1} \quad (P_{-1} = 0), \quad (n = 1, 2, \ldots).$$

Hence we find that

$$P_n = P_0(1 + a_1/1)(1 + a_2/2) \ldots (1 + a_n/n),$$

$$p_n = p_0(1 + a_1/1)(1 + a_2/2) \ldots \big(1 + a_{n-1}/(n-1)\big) a_n/n \qquad (n = 1, 2, \ldots).$$

Basing ourselves upon the last formulae, we state that the sequence $\{p_n\}$ satisfies conditions (ii) and (iii). In order to show that it satisfies also condition (i), we observe that in view of $a_n > 1$ $(n = 1, 2, \ldots)$ we have

$$(1 - a_{n-1})n < \lambda n a_n \quad \text{for} \quad n = 2, 3, \ldots$$

Hence

$$\frac{a_{n-1}}{n-1} < \left(1 + \frac{a_{n-1}}{n-1}\right) \frac{a_n}{n}$$

and

$$P_{n-2} \frac{a_{n-1}}{n-1} < P_{n-2}\left(1 + \frac{a_{n-1}}{n-1}\right) \frac{a_n}{n} < P_{n-1} \frac{a_n}{n} \qquad (n = 2, 3, \ldots),$$

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[2] This lemma has been proved by H. Zaremba.

whence it follows that $p_{n-1} < p_n$ for $n = 2, 3, \ldots$ These inequalities are also true for $n = 1$, because $a_1 > 1$. Thus we have proved that $\{p_n\} \epsilon M^a$. It remains to show that $\{n(p_n - p_{n-1})/p_n\}$ is unbounded. For this purpose we write

$$\frac{n(p_n - p_{n-1})}{p_{n-1}} = n\left(\frac{P_{n-1}}{P_{n-2}} \cdot \frac{a_n}{a_{n-1}} \cdot \frac{n-1}{n} - 1\right) = n\left[\left(1 + \frac{a_{n-1}}{n-1}\right) \frac{a_n}{a_{n-1}} \cdot \frac{n-1}{n} - 1\right],$$

whence it follows that

$$(1) \qquad \frac{n(p_n - p_{n-1})}{p_{n-1}} = a_n - \frac{a_n}{a_{n-1}} + \frac{\lambda n a_n}{a_{n-1}} \qquad (n = 1, 2, \ldots).$$

Since the sequence $\{na_n\}$ tends to infinity for $n = 2^m$ $(m = 1, 2, \ldots)$ then according to the last equalities the sequence $\{n(p_n - p_{n-1})/p_n\}$ cannot be bounded.

2° We set

$$a_n = a + \mu(s - s_n) \qquad \text{for} \qquad n = 1, 2, \ldots,$$

where μ denotes an arbitrary real and positive number, satisfying the condition

$$0 < \mu < \frac{2(1-a)}{2s-1}.$$

We infer at once that $\{a_n\}$ is a decreasing sequence convergent to a, and that $a < a_n < 1$.

Arguing as in the first part of the proof, we get the same formulae as previously, and, moreover, we find that $\{p_n\}$ is a decreasing sequence belonging to the class M^a for $0 < a < 1$.

Basing ourselves on the equalities

$$(2) \qquad \frac{n(p_{n-1} - p_n)}{p_{n-1}} = \frac{a_n}{a_{n-1}} - a_n + \mu\frac{na_n}{a_{n-1}} \qquad (n = 2, 3, \ldots),$$

we prove that the sequence $\{n(p_{n-1} - p_n)/p_n\}$ tends to infinity for $n = 2^m$ $(m = 1, 2, \ldots)$, which ends the proof of the second part of Lemma 1.

3° We shall examine two subcases: $p_n\nearrow$, and $p_n\searrow$. Suppose that $p_n\nearrow$. We construct the following sequence:

$$\beta_n = \begin{cases} \beta & \text{for } n = 1 \text{ and } 0 < \beta < 1, \\ \dfrac{1}{\log n} & \text{for } n \neq 2^m \quad (m = 1, 2, \ldots), \\ \log n & \text{for } n = 2^m \quad (m = 1, 2, \ldots). \end{cases}$$

Obviously, we have $\overline{\lim\limits_{n\to\infty}} \beta_n = +\infty$.

Let p_0 be an arbitrary positive number, and let

$$\frac{n(p_n - p_{n-1})}{p_{n-1}} = \beta_n \quad \text{for} \quad n = 1, 2, \ldots$$

Hence we find that

$$p_n = p_0(1+\beta_1/1)(1+\beta_2/2)\ldots(1+\beta_n/n) \quad \text{for} \quad n = 1, 2, \ldots,$$

whence we conclude that $\{n(p_n - p_{n-1})/p_{n-1}\}$ is an unbounded sequence and, moreover, that

$$0 < p_{n-1} < p_n \quad \text{for} \quad n = 1, 2, \ldots \text{ (condition (i))}.$$

Since

$$\sum_{n=1}^{\infty} \frac{\beta_n}{n} = +\infty,$$

then the infinite product

$$p_0 \prod_{n=1}^{\infty} (1+\beta_n/n)$$

diverges to infinity. We then have $\lim_{n\to\infty} p_n = +\infty$, and $P_n \nearrow +\infty$ (condition (ii)).

It remains to show that the sequence $\{p_n\}$ satisfies condition (iii) for $a = 1$. We write

$$\begin{aligned} a_n &= \frac{np_n}{P_{n-1}} = \frac{p_0 + (p_1 - p_0) + (2p_2 - p_1) + \ldots + [np_n - (n-1)p_{n-1}]}{P_{n-1}} \\ &= \frac{p_0 + (p_1 - p_0) + p_1 + 2(p_2 - p_1) + \ldots + p_{n-1} + n(p_n - p_{n-1})}{P_{n-1}} \\ &= 1 + \frac{(p_1 - p_0) + (p_2 - p_1)2 + \ldots + (p_n - p_{n-1})n}{P_{n-1}} \\ &= 1 + \frac{p_0(p_1 - p_0)/p_0 + p_1(p_2 - p_1)2/p_1 + \ldots + p_{n-1}(p_n - p_{n-1})n/p_{n-1}}{P_{n-1}} \\ &= 1 + \frac{p_0\beta_1 + p_1\beta_2 + \ldots + p_{n-1}\beta_n}{P_{n-1}}. \end{aligned}$$

Hence we have

$$a_n = 1 + \frac{p_0\beta_1 + p_1\beta_2 + \ldots + p_{n-1}\beta_n}{P_{n-1}}.$$

In order to show that $\lim_{n\to\infty} a_n = 1$, it suffices to prove that the sequence $\dfrac{p_0\beta_1 + p_1\beta_2 + \ldots + p_{n-1}\beta_n}{P_{n-1}}$ tends to 0. We observe that

$$\frac{p_0\beta_1+p_1\beta_2+\ldots+p_{n-1}\beta_n}{P_{n-1}}$$

$$=\begin{cases}\dfrac{p_0\beta_1+p_1 0+p_2\beta_3+p_3 0+\ldots+p_{n-1}0}{P_{n-1}}+\dfrac{p_1\beta_2+p_3\beta_4+\ldots+p_{2^m-1}\beta_{2^m}}{P_{n-1}} \\ \qquad\qquad \text{for } n=2^m \ (m=1,2,\ldots), \\ \dfrac{p_0\beta_1+p_1 0+p_2\beta_3+p_3 0+\ldots+p_{n-1}0}{P_{n-1}}+\dfrac{p_1\beta_2+p_3\beta_4+\ldots+p_{2^m-1}\beta_{2^m}}{P_{n-1}} \\ \qquad\qquad \text{for } 2^m<n<2^{m+1} \ (m=1,2,\ldots).\end{cases}$$

The first terms of the above two sums always tend to 0, as the weighed means of a null-convergent sequence (considering that $P_{n-1}\to\to+\infty$). The second components also tend to 0, as is evident from the following estimations (considering that $\beta_n<1$ for $2^{m-1}<n<2^m$, $m=1,2,\ldots$)

$$\frac{p_{2^m-1}\log 2^m}{P_{2^m-1}-P_{2^{m-1}-1}}$$

$$=\frac{mp_{2^m-1}\log 2}{p_{2^{m-1}}+p_{2^{m-1}+1}+\ldots+p_{2^m-1}}<\frac{mp_{2^m-1}\log 2}{2^{m-1}p_{2^m-1}}$$

$$=\frac{m(1+\beta_{2^{m-1}+1}/2^{m-1}+1)(1+\beta_{2^{m-1}+2}/2^{m-1}+2)\ldots(1+\beta_{2^m-1}/2^m-1)\log 2}{2^{m-1}}$$

$$<\frac{m(2^{m-1}+2)(2^{m-1}+3)\ldots 2^m\log 2}{2^{m-1}(2^{m-1}+1)(2^{m-1}+2)\ldots(2^m-1)}=\frac{m2^m\log 2}{2^{m-1}(2^{m-1}+1)}=\frac{4m\log 2}{2^m+2}.$$

We then have

$$\lim_{m\to\infty}\frac{p_{2^m-1}\log 2^m}{P_{2^m-1}-P_{2^{m-1}-1}}=\lim_{m\to\infty}\frac{4m\log 2}{2^m+2}=0.$$

Let us notice that

$$0<\frac{p_1\beta_2+p_2\beta_4+\ldots+p_{2^m-1}\beta_{2^m}}{P_{n-1}}<\frac{p_1\beta_2+p_2\beta_4+\ldots+p_{2^m-1}\beta_{2^m}}{P_{2^m-1}},$$

for $2^m\leqslant n<2^{m+1}$ $(m=1,2,\ldots)$. Applying to the last expression the Stolz lemma (see e.g. [6], p. 232) we conclude, in view of the above estimations, that the sequence in question is a null-convergent sequence. Thus we have proved that $\lim_{n\to\infty}a_n=1$ (condition (iii)).

The proof of the second subcase of 3° is based upon the same sequence $\{\beta_n\}$, which we have used in the statement of the first subcase of 3°. We construct the sequence $\{p_n\}$ choosing an arbitrary positive number upon p_0 and for $\{p_n\}$ the numbers satisfying the equalities

$$\frac{n(p_n - p_{n-1})}{p_{n-1}} = -\beta_n \qquad (n = 1, 2, \ldots).$$

Hence we find that

$$p_n = p_0(1-\beta_1/1)(1-\beta_2/2)(1-\beta_3/3)\ldots(1-\beta_n/n).$$

We infer at once that $\{p_n\}$ is a positive and decreasing sequence, and that the sequence $\{n(p_n - p_{n-1})/p_{n-1}\}$ is unbounded.

In fact, we have

$$\overline{\lim_{n\to\infty}} \frac{n(p_{n-1}-p_n)}{p_{n-1}} = -\underline{\lim_{n\to\infty}}(-\beta_n) = \overline{\lim_{n\to\infty}}\beta_n = \lim_{m\to\infty}\log 2^m = +\infty.$$

After analogous transformations to those made before, we get the formula

$$a_n = 1 - \frac{p_0\beta_1 + p_1\beta_2 + \ldots + p_{n-1}\beta_n}{P_{n-1}}.$$

In order to show that $\lim\limits_{n\to\infty} a_n = 1$, it suffices to prove that

$$\lim_{n\to\infty} \frac{p_0\beta_1 + p_1\beta_2 + \ldots + p_{n-1}\beta_n}{P_{n-1}} = 0.$$

For this purpose we first prove that $P_n \nearrow +\infty$. In fact, considering that $\beta_n < 1$ for $2^m \leqslant n < 2^{m+1}$ $(m = 1, 2, \ldots)$, we can write

$$p_n > p_0(1-\beta)\left(1-\frac{1}{2}\right)\ldots\left(1-\frac{1}{n}\right)\frac{(1-\log 2/2)\ldots(1-\log 2^m/2^m)}{(1-1/2)\ldots(1-1/2^m)}$$

$$> p_0(1-\beta)\,\frac{1}{2}\cdot\frac{2}{3}\ldots\frac{n-1}{n}(1-\log 2/2)\ldots(1-\log 2^m/2^m),$$

$$> p_0(1-\beta)\frac{1}{n}\prod_{m=1}^{\infty}(1-\log 2^m/2^m) = p_0(1-\beta)a/n.$$

Therefore

$$P_n = p_0 + p_1 + \ldots + p_n \geqslant p_0 + p_0(1-\beta)\frac{a}{1}\ldots + p_0(1-\beta)\frac{a}{n}$$

$$> p_0(1-\beta)a\sum_{k=1}^{n}\frac{1}{k} \to \infty.$$

Arguing as before, we find that

$$\lim_{n\to\infty} \frac{p_1\beta_2+p_2\beta_4+\ldots+p_{2m-1}\beta_{2m}}{P_{n-1}} = 0\,,$$

whence, with regard to the formula for a_n, it follows that $\lim\limits_{n\to\infty} a_n = 1$. This completes the proof of Lemma 1.

Remark. The sequence $\{p_n\}$ considered above tends monotonically to 0, for p_n is the nth partial product of the infinite product $p_0 \prod\limits_{n=1}^{\infty} (1-\beta_n/n)$ divergent to zero.

In order to formulate the next lemma, we introduce the following class of (N, p_n)-means:

A sequence $\{p_n\}$ is said to *belong to the class* M^* if

(A) $$p_n > 0 \qquad (n = 0, 1, \ldots)\,,$$

(B) $$\{p_n\} \text{ is convex or concave,}$$

(C) $$0 < \varliminf_{n\to\infty} (np_n/P_n) \leqslant \varlimsup_{n\to\infty} (np_n/P_n) < +\infty \qquad (P_n = p_0+p_1+\ldots+p_n)\,.$$

LEMMA 2. *If $\{p_n\} \in M^*$, then the sequence $\{n(p_n-p_{n-1})/p_n\}$ is bounded.*

Proof. Suppose that $\{p_n\} \in M^*$. We shall examine three cases: 1) $\{p_n\}$ is convex and bounded, 2) $\{p_n\}$ is convex and unbounded, 3) $\{p_n\}$ is concave. Passing to the examination of the first case, we notice that $\{p_n\}$ is non-increasing (see [11], p. 93). Let us assume that the sequence $\{n(p_n-p_{n-1})/p_n\}$ is unbounded. Then there exist such a strictly increasing and lacunary sequence $\{k_n\}$ of indices that for an arbitrary positive number M the following inequalities are satisfied:

(3) $$\frac{k_n(p_{k_n-1}-p_{k_n})}{p_{k_n}} > M \quad \text{for} \quad n = 1, 2, \ldots,$$

where $k_n/k_{n-1} \geqslant q > 1$ for $n = 1, 2, \ldots$; q = constant.

Analysing the proof of the author's lemma (see [6], Lemma 6, p. 235-236), we find that it remains valid if we assume that $\{p_n\} \in M^*$. Hence it follows that $\{P_{k_n}\}$, where $P_n = p_0+p_1+\ldots+p_n$, is a strictly increasing and lacunary sequence. Then there exists a constant $q_1 > 1$ such that $P_{k_n}/P_{k_{n-1}} \geqslant q_1 > 1$ for $n = 1, 2, \ldots$ We distinguish two subcases: 1) the series $\sum\limits_{v=1}^{\infty} p_{k_v}$ is divergent, 2) the series $\sum\limits_{v=1}^{\infty} p_{k_v}$ is convergent.

We write

$$p_{k_{n-1}}-p_{k_n}=(p_{k_{n-1}}-p_{k_{n-1}+1})+(p_{k_{n-1}+1}-p_{k_{n-1}+2})+\ldots+(p_{k_n-1}-p_{k_n})$$
$$>(k_n-k_{n-1})(p_{k_n-1}-p_{k_n})>0\,.$$

Hence and from inequalities (3) it follows that

$$p_{k_N}=\sum_{N+1}^{n}(p_{k_{v-1}}-p_{k_v})+p_{k_n}>\sum_{N+1}^{n}(k_v-k_{v-1})(p_{k_v-1}-p_{k_v})$$
$$>M\sum_{N+1}^{n}p_{k_v}(1-k_{v-1}/k_v)>M\left(1-\frac{1}{q}\right)\sum_{N+1}^{n}p_{k_v}\,.$$

Then we have

$$p_{k_N}>M\left(1-\frac{1}{q}\right)\sum_{N+1}^{n}p_{k_v}\,,$$

which in subcase 1) contradicts the boundedness of the sequence $\{p_n\}$. In subcase 2) there exists for the number M defined above such a positive integer N that the following inequalities are satisfied:

$$\text{(4)}\qquad \begin{gathered} P_{k_N}\geqslant 1\,,\qquad p_{k_N}/P_{k_N}\leqslant M\left(1-\frac{1}{q}\right)\frac{q_1-1}{2}\,,\\ p_{k_N+1}+p_{k_N+2}+\ldots+p_{k_{N+1}-1}<\frac{q_1-1}{2}\,. \end{gathered}$$

We write

$$p_{k_N}>M\left(1-\frac{1}{q}\right)\sum_{N+1}^{n}p_{k_v}$$
$$>M\left(1-\frac{1}{q}\right)[P_{k_{N+1}}-P_{k_N}-(p_{k_N+1}+p_{k_N+2}+\ldots+p_{k_{N+1}-1})]\,.$$

Dividing the last inequalities by P_{k_N}, and taking into account the first and the third inequalities of (4), we find that

$$p_{k_N}/P_{k_N}>M\left(1-\frac{1}{q}\right)\frac{q_1-1}{2}\,,$$

contrary to the second inequality of (4).

Passing to the next case, let us assume that $\{p_n\}$ is convex and unbounded. By hypothesis, $\{p_n-p_{n+1}\}$ is non-increasing, and this implies the existence of the limit $\lim\limits_{n\to\infty}(p_n-p_{n+1})$ (finite or infinite). It cannot be positive, for then $\{p_n\}$ would be strictly decreasing for sufficiently large values n, which would imply the existence of the finite limit $\lim\limits_{n\to\infty}p_n$

($\{p_n\}$ being still positive) contrary to hypothesis. Thus $\{p_n\}$ is non-decreasing. Suppose the sequence $\{n(p_n-p_{n-1})/p_n\}$ to be unbounded. Then there exists such a strictly increasing and lacunary sequence $\{k_n\}$ of indices that the following inequalities are satisfied:

$$\text{(a)} \qquad k_{n+1}/k_n \geqslant q > 1 \quad \text{for} \quad n = 0, 1, 2, \ldots, \quad q = \text{a constant},$$

$$\text{(b)} \qquad \frac{k_n(p_{k_n+1}-p_{k_n})}{p_{k_n}} > M \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

where M denotes an arbitrarily large positive number such that $M \geqslant 2M_1 q/(q-1)^2$, with M_1 satisfying the condition: $(n+1)p_n/P_n < M_1$ for $n = 0, 1, 2, \ldots$

We write

$$p_{k_n} - p_{k_{n-1}} = (p_{k_n} - p_{k_n-1}) + (p_{k_n-1} - p_{k_n-2}) + \ldots + (p_{k_{n-1}+1} - p_{k_{n-1}}).$$

Taking into consideration the convexity of the sequence $\{p_n\}$, we find that

$$\text{(5)} \qquad p_{k_n} - p_{k_{n-1}} > (k_n - k_{n-1})(p_{k_{n-1}+1} - p_{k_{n-1}}) > 0,$$

whence with regard to (b) it follows that

$$p_{k_n} = \sum_{v=1}^{n} (p_{k_v} - p_{k_{v-1}}) + p_{k_0} > M(q-1)\sum_{v=1}^{n} p_{k_{v-1}} > M(q-1)p_{k_{n-1}}.$$

Then we should have

$$M_1 P_{k_n}/(k_n+1) > \frac{(k_n+1)p_{k_n}}{P_{k_n}} \cdot \frac{P_{k_n}}{k_n+1} > M(q-1)\frac{(k_{n-1}+1)p_{k_{n-1}}}{P_{k_{n-1}}} \cdot \frac{P_{k_{n-1}}}{k_{n-1}+1}$$
$$> M(q-1)P_{k_{n-1}}/k_{n-1}+1.$$

Since $\{p_n\}$ is non-decreasing, $\{P_n/(n+1)\}$, is also non-decreasing. In virtue of the last inequalities, we then obtain

$$M_1 P_{k_n}/k_n > M(q-1)P_{k_0}/(k_0+1) \geqslant M(q-1)P_{k_0}/2k_0 \qquad (n = 0, 1, 2, \ldots).$$

Summing up the last inequalities from $n = 0$ to $n = +\infty$, and considering that by condition (a) $k_0/k_1 \leqslant 1/q$, $k_0/k_2 \leqslant 1/q^2, \ldots, k_0/k_n \leqslant 1/q^n, \ldots$, we find that

$$\frac{M_1 q}{q-1} = M_1\left(1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^n} + \ldots\right) \geqslant M_1 k_0 \sum_{n=0}^{\infty} \frac{1}{k_n}$$
$$> M\frac{q-1}{2}P_{k_0}\sum_{k=0}^{\infty}\frac{1}{P_{k_n}} > M\frac{q-1}{2}\,[3]$$

[3] The series appearing here are convergent, which is evident from the lacunarity of the sequences $\{k_n\}$ and $\{P_{k_n}\}$.

or ultimately

$$M < \frac{2M_1 q}{(q-1)^2},$$

contrary to the definition of the number M.

According to the examination of the last case, we assume that $\{p_n\}$ is concave. Then $\{p_n\}$ is non-decreasing. In fact, the sequence $\{p_n - p_{n+1}\}$ is non-positive, for otherwise there would exist such a positive integer k that $p_{n-1} - p_n > 0$ for $n = k, k+1, \ldots$ This would imply the existence of the finite limit $\lim_{n\to\infty} p_n$ and, of course, the convergence of the series $\sum_{n=1}^{\infty}(p_{n-1} - p_n)$. However, this is impossible as $\{p_{n-1} - p_n\}$ is a positive and non-decreasing sequence.

Suppose that $\{n(p_n - p_{n-1})/p_n\}$ is unbounded. Then there exists a strictly increasing and lacunary sequence $\{k_n\}$ of indices such that for an arbitrarily large positive number $M \geqslant q/(q-1)$ the following inequalities are satisfied:

$$\frac{k_n(p_{k_n} - p_{k_{n-1}})}{p_{k_n}} > M \quad \text{for} \quad n = 0, 1, 2, \ldots$$

(where $k_n/k_{n-1} \geqslant q > 1$ for $n = 1, 2, \ldots$, $q =$ a constant). Hence, with respect to the concavity of the sequence $\{p_n\}$, we find that

$$p_{k_n} = \sum_{v=1}^{n}(p_{k_v} - p_{k_{v-1}}) + p_{k_0} > M\left(1 - \frac{1}{q}\right)\sum_{v=1}^{n} p_{k_v} > M\left(1 - \frac{1}{q}\right)p_{k_n}.$$

Then we have $M(1 - 1/q) < 1$, which contradicts the definition of the number M and completes at the same time the proof of Lemma 2.

Remark. Lemma 2 is also true if we assume only that $\{p_n\}$ is convex and bounded and that the method (N, p_n) is positive and regular. H. Zaremba has noticed that it is also valid if we assume only that $p_0 > 0$ and $p_n \geqslant 0$ for $n = 1, 2, \ldots$, provided that $\{p_n\}$ is concave.

In order to formulate the next lemma we introduce the following classes of (N, p_n)-means:

If for some sequence $\{p_n\}$ conditions (i) and (ii) (see p. 238) are satisfied and, moreover, if

$$\lim_{n\to\infty}\frac{n\Delta p_{n-1}}{p_n} = 1 - a, \quad \text{where} \quad a \geqslant 0,\ \Delta p_{n-1} = p_{n-1} - p_n,$$

then we shall say that the sequence $\{p_n\}$ *belongs to the class* $\bar{M}^a$.

If $\{p_n\}$ is convex or concave, conditions (i), (ii) are satisfied and, moreover, if

$$\lim_{n\to\infty}\frac{n\Delta^2 p_{n-2}}{\Delta p_{n-1}} = 2-\alpha, \quad \text{where} \quad \alpha \geqslant 0,\ \Delta^2 p_{n-2} = \Delta p_{n-2}-\Delta p_{n-1},$$

then we shall say that the sequence $\{p_n\}$ *belongs to the class* $\overline{\overline{M}}{}^{\alpha}$ (4).

From these definitions and from the well-known theorems (see e.g. [4], p. 84 (44), 4, p. 116 (69), 19, p. 134 (80)) it follows that if $\{p_n\} \in \overline{M}^{\alpha}$, then $\{p_n\} \in M^{\alpha}$, and if $\{p_n\} \in \overline{\overline{M}}{}^{\alpha}$, then $\{p_n\} \in \overline{M}^{\alpha}$ and $\{p_n\} \in M^{\alpha}$.

LEMMA 3. *If* $\{P_n\} \in \overline{M}^{\alpha}$, $\alpha > \frac{1}{2}$, $\{q_n\} \in \overline{M}^{\beta}$, $\beta > \frac{1}{2}$, *then*

$$\lim_{n\to\infty}\frac{(n+1)r_n}{R_n} = \alpha+\beta-1, \tag{6}$$

where $p_0 = P_0,\ p_n = P_n - P_{n-1}\ (n = 1, 2, \ldots),\ r_n = \sum_{k=0}^{n} p_{n-k}q_k,\ R_n = \sum_{k=0}^{n} r_k$.

Proof. We write

$$\begin{aligned} r_0 &= p_0 q_0, \\ r_1 &= p_0 q_1 + p_1 q_0, \\ &\cdots\cdots\cdots\cdots \\ r_n &= p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0. \end{aligned} \tag{7}$$

Summing up both sides of equations (7) we find that

$$R_n = \sum_{k=0}^{n} p_k Q_{n-k} \quad \text{or} \quad R_n = \sum_{k=0}^{n} q_{n-k} P_k. \tag{8}$$

Under the assumptions of Lemma 3 it follows from the last formula that

$$R_n > 0 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Differentiating both sides of the equality

$$\Big(\sum_{n=0}^{\infty} p_n x^{n+1}\Big)\Big(\sum_{n=0}^{\infty} q_n x^{n+1}\Big) = \sum_{n=0}^{\infty} r_n x^{n+1} \quad (|x| < 1),$$

and multiplying it subsequently by x, we compare the coefficients of x^{n+1}. Dividing the equality obtained by R_n, we find the formula

$$\frac{1}{R_n}\sum_{k=0}^{n} q_{n-k}P_k\frac{(k+1)p_k}{P_k} + \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k}Q_k\frac{(k+1)q_k}{Q_k} = \frac{(n+1)r_n}{R_n}. \tag{9}$$

(4) It is easy to verify that (C, α)-means belong to the classes: $\overline{M}^{\alpha}$ and $\overline{\overline{M}}{}^{\alpha}$ for $\alpha > 0$.

We now remark that the method $\|a_{nk}\|$, where

$$a_{nk} = \begin{cases} \dfrac{q_{n-k}P_k}{R_n} & \text{for} \quad k = 0, 1, 2, \dots, n, \\ 0 & \text{otherwise} \end{cases}$$

is regular if and only if

$$\lim_{n\to\infty} a_{nk} = 0 \quad \text{for} \quad k = 0, 1, 2, \dots, n. \tag{10}$$

Let $\{P_n\} \in \bar{M}^\alpha$, $\frac{1}{2} < \alpha < 1$, $\{q_n\} \in \bar{M}^\beta$, $\beta > \frac{1}{2}$. We remark that in this case $0 < P_n \searrow$ and $p_0 > 0$, $p_n < 0$ for $n = 1, 2, \dots$ Moreover, the sequence $\{P_n\}$ is a null-convergent sequence. In fact, there would be $\lim\limits_{n\to\infty} P_n = g \neq 0$; then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P_k = g \neq 0 \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=0}^{n} P_k = +\infty.$$

Hence, according to the Stolz lemma, we should have

$$\lim_{n\to\infty} \frac{(n+1)P_n}{\sum\limits_{k=0}^{n} P_k} = \lim_{n\to\infty} \frac{np_n}{P_n} + 1 = \alpha < 1,$$

contrary to

$$\lim_{n\to\infty} \frac{(n+1)P_n}{\sum\limits_{k=0}^{n} P_k} = \lim_{n\to\infty} \frac{P_n}{\dfrac{1}{n+1} \sum\limits_{k=0}^{n} P_k} = 1.$$

Hence, and from some theorems of the author (see [6], Th. 4, p. 247 and Th. 8, p. 251), it follows that

$$\text{(a)} \qquad \lim_{n\to\infty} \frac{n}{Q_n^2} \sum_{k=0}^{n} q_k^2 = \frac{1}{2\beta - 1},$$

$$\text{(11)} \quad \text{(b)} \qquad nP_n^2 \nearrow +\infty \quad \text{as} \quad n \nearrow +\infty,$$

$$\text{(c)} \qquad \lim_{n\to\infty} \frac{1}{nP_n^2} \sum_{k=0}^{n} P_k^2 = \frac{1}{2\alpha - 1}.$$

Basing ourselves upon relation $(11)_{(b)}$, we may estimate the elements of the method $\|a_{nk}\|$. We shall examine two cases: $0 < q_n \searrow$ and $0 < q_n \nearrow$. Since $0 < P_n \searrow$, then $R_n > P_n Q_n$. By a lemma (see [6], Lemma 3, p. 232) we infer that $P_n Q_n \to +\infty$ as $n \to \infty$, whence it follows that $R_n \to +\infty$ as $n \to +\infty$. If $0 < q_n \searrow$, then

$$a_{nk} < \frac{q_{n-k} P_k}{P_n Q_n} < \frac{q_0 M}{P_n Q_n} \quad \text{for} \quad k = 0, 1, 2, \ldots, n,$$

and if $0 < q_n \nearrow$, then

$$a_{nk} < \frac{q_{n-k} P_k}{P_n Q_n} < \frac{(n+1) q_n}{Q_n} \cdot \frac{M^2}{n P_n^2} \quad \text{for} \quad k = 0, 1, \ldots, n,$$

where M denotes a positive number such that $P_n < M$ $(n = 0, 1, \ldots)$. Hence it follows that $a_{nk} \to 0$ for $k = 0, 1, 2, \ldots, n$, in both cases considered. Consequently, the method $\|a_{nk}\|$ satisfies condition (10) of regularity.

Let us denote by A_n and B_n, respectively, the first and the second expression on the right side of formula (9).

The regularity of the method $\|a_{nk}\|$ implies the formula

$$\lim_{n \to \infty} A_n = a - 1. \tag{12}$$

Passing to the estimation of the expression B_n, we notice that the method $\|b_{nk}\|$, where

$$b_{nk} = \begin{cases} \dfrac{p_{n-k} Q_k}{R_n} & \text{for} \quad k = 0, 1, 2, \ldots, n, \\ 0 & \text{otherwise}, \end{cases}$$

is in general not regular.

In order to prove this we remark that from $(11)_{(a)}$, $(11)_{(c)}$ and from the evident inequality

$$\frac{P_n Q_n}{R_n} \geqslant \sqrt{\frac{Q_n^2}{n \sum_{k=0}^{n} q_k^2} \cdot \frac{n P_n^2}{\sum_{k=0}^{n} P_k^2}}$$

it follows that

$$\varliminf_{n \to \infty} \frac{P_n Q_n}{R_n} \geqslant \sqrt{(2a-1)(2\beta-1)}.$$

Suppose the method $\|b_{nk}\|$ to be regular; thus by the Toeplitz's theorem there would exist a finite and positive number M such that

$$\sup_n \sum_{k=0}^{n} |b_{nk}| < M.$$

Let us choose for an arbitrarily given small number $\varepsilon > 0$ such a positive integer N that for $n > N$

$$\frac{P_n Q_n}{R_n} > \sqrt{(2\alpha-1)(2\beta-1)} - \varepsilon .$$

Now taking $n > N$, we can write

$$\sum_{k=0}^{n} |b_{nk}| = \frac{1}{R_n}\sum_{k=0}^{n} |p_{n-k}| Q_k = \frac{1}{R_n}\Big(\sum_{k=0}^{N} |p_k| Q_{n-k} - \sum_{k=N+1}^{n} p_k Q_{n-k}\Big)$$

$$= \frac{1}{R_n}\sum_{k=0}^{N} (|p_k| + p_k) Q_{n-k} - 1 = \frac{2p_0}{P_n}\cdot\frac{P_n Q_n}{R_n} - 1$$

$$> \frac{2p_0}{P_n}\Big(\sqrt{(2\alpha-1)(2\beta-1)} - \varepsilon\Big) - 1 .$$

Since $P_n \to 0$, the last expression tends to infinity, which proves that the method $\|b_{nk}\|$ is not regular.

This fact enables us to make a certain transformation of expression B_n, which will show the convergence of the sequence under consideration. We introduce the following notations:

$$\beta_n = \frac{(n+1)q_n}{Q_n}; \qquad \bar{\beta}_n = \frac{n(q_{n-1} - q_n)}{q_n}; \qquad \Delta\beta_{n-1} = \beta_{n-1} - \beta_n .$$

We infer at once that

$$\Delta\beta_{n-1} = \frac{q_n}{Q_{n-1}}(\beta_n - \bar{\beta}_n - 1) . \tag{13}$$

Now we can write (with $P_{-1} = p_{-1} = 0$)

$$B_n = \frac{1}{R_n}\sum_{k=0}^{n} p_k Q_{n-k}\beta_{n-k} = \frac{1}{R_n}\sum_{k=0}^{n} (P_k - P_{k-1}) Q_{n-k}\beta_{n-k}$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} P_k Q_{n-k}\beta_{n-k} - \frac{1}{R_n}\sum_{k=0}^{n} P_k Q_{n-k-1}\Delta\beta_{n-k-1}$$

$$= \frac{1}{R_n}\sum_{k=0}^{n} P_k q_{n-k}\beta_{n-k} - \frac{1}{R_n}\sum_{k=0}^{n} P_k q_{n-k}(\beta_{n-k} - \bar{\beta}_{n-k} - 1)$$

$$= 1 + \frac{1}{R_n}\sum_{k=0}^{n} P_k q_{n-k}\bar{\beta}_{n-k} .$$

Hence we have

$$(14) \qquad B_n = 1 + \frac{1}{R_n} \sum_{k=0}^{n} q_k P_{n-k} \bar{\beta}_k .$$

We observe that the method $\|a^*_{nk}\|$, where

$$a^*_{nk} = \begin{cases} \dfrac{q_k P_{n-k}}{R_n} & \text{for} \quad k = 0, 1, 2, \ldots, n, \\ 0 & \text{otherwise} \end{cases}$$

is regular, because $\|a_{nk}\|$ is so.

Hence and from (14) we get the formula

$$(15) \qquad \lim_{n\to\infty} B_n = \beta .$$

Finally, from (9), (12) and (15) it follows that

$$(16) \qquad \lim_{n\to\infty} \frac{(n+1) r_n}{R_n} = a + \beta - 1 ,$$

and this is the required result.

LEMMA 4. *If* $\{p_n\} \in M^*$, *then*

$$\sup_{k;n\geqslant k} \frac{|p_{n-k} P_n - p_n P_{n-k}|}{k p_n p_{n-k}} < +\infty .$$

Proof. We shall distinguish two cases: 1) $\{p_n\}$ is convex and bounded, 2) $\{p_n\}$ is concave or convex and unbounded.

Passing to the first case, we notice that $\{p_n\}$ is then non-increasing. Therefore

$$0 < P_n p_{n-k} - P_{n-k} p_n = k p_n p_{n-k} \left(\frac{P_n}{k p_n} - \frac{P_{n-k}}{k p_{n-k}} \right)$$
$$< k p_n p_{n-k} P_{2k} / k p_{2k} = O(k p_n p_{n-k})$$

for $n = k, k+1, k+2, \ldots, 2k$.

For the remaining values of n (required by Lemma 4), we write

$$0 < P_n p_{n-k} - P_{n-k} p_n = P_{n-k}(p_{n-k} - p_n) + (P_n - P_{n-k}) p_{n-k}$$
$$= P_{n-k}[(p_{n-k} - p_{n-k+1}) + (p_{n-k+1} - p_{n-k+2}) + \ldots + (p_{n-1} - p_n)] +$$
$$+ p_{n-k}(p_{n-k+1} + \ldots + p_n) .$$

The proof runs further after the following estimate:

$$\frac{P_n p_{n-k} - P_{n-k} p_n}{p_n p_{n-k}} = \frac{P_{n-k}}{(n-k+1)p_{n-k}} \left[\frac{(n-k+1)(p_{n-k}-p_{n-k+1})}{p_{n-k}} \cdot \frac{p_{n-k}}{p_n} + \right.$$

$$+ \frac{(n-k+2)(p_{n-k+1}-p_{n-k+2})}{p_{n-k+1}} \cdot \frac{p_{n-k+1}}{p_n} \cdot \frac{n-k+1}{n-k+2} +$$

$$\left. + \ldots + \frac{(n+1)(p_{n-1}-p_n)}{p_n} \cdot \frac{n-k+1}{n+1} \right] + \left(1 + \frac{p_{n-1}}{p_n} + \ldots + \frac{p_{n-k+1}}{p_n}\right) = O(k)$$

for $n = 2k+1, 2k+2, 2k+3, \ldots$, which completes the proof in the first case.

Passing to the second case, we remark that $\{p_n\}$ is then non-decreasing. Hence

$$\frac{|P_n p_{n-k} - P_{n-k} p_n|}{p_n p_{n-k}} \leqslant \frac{P_{n-k}}{p_n p_{n-k}} (p_n - p_{n-k}) + \frac{p_{n-k+1} + \ldots + p_n}{p_n}$$

$$= \frac{P_{n-k}}{(n-k+1)p_{n-k}} \left[\frac{(n+1)(p_n - p_{n-1})}{p_n} \cdot \frac{n-k+1}{n+1} + \right.$$

$$\left. + \frac{n(p_{n-1}-p_{n-2})}{p_{n-1}} \cdot \frac{n-k+1}{n} \cdot \frac{p_{n-1}}{p_n} + \ldots + \frac{(n-k+1)(p_{n-k+1}-p_{n-k})}{p_{n-k+1}} \cdot \frac{p_{n-k+1}}{p_n} \right] +$$

$$+ \left(1 + \frac{p_{n-1}}{p_n} + \frac{p_{n-2}}{p_n} + \ldots + \frac{p_{n-k+1}}{p_n}\right) = O(k)$$

for $n = k, k+1, k+2, \ldots$, which completes the proof of Lemma 4.

Remark. Lemma 4 holds if we replace class M^* by the class $\bar{M}^a$, with $a > 0$.

2. Let ON $\{\varphi_n(x)\}$ denote an orthonormal system, defined in the interval $\langle 0, 1 \rangle$. Moreover, let

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{17}$$

denote an orthogonal series, whose coefficients satisfy the condition $\{c_n\} \in l^2$, i.e.

$$\sum_{n=0}^{\infty} c_n^2 < +\infty. \tag{18}$$

In order to formulate (in a more concise form) further theorems we introduce the following definition: The orthogonal series (17), whose coefficients satisfy condition (18), is said to be an *orthonormal series*.

LEMMA 5. *Suppose that $\{p_n\} \in \bar{M}^{a}$, $a > \frac{1}{2}$, and let $t_n(x)$ denote the n-th (N, p_n)-mean of orthonormal series* (17). *Then the series*

$$\sum_{n=1}^{\infty} n\big(t_n(x) - t_{n-1}(x)\big)^2$$

is convergent a.e. (almost everywhere).

Proof. Assuming $p_{-1} = P_{-1} = 0$, we can write

$$t_n(x) - t_{n-1}(x) = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n} c_k \varphi_k(x) \sum_{v=k}^{n} (p_{n-v} P_{n-1} - p_{n-v-1} P_n)$$

$$= \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n} c_k \varphi_k(x) [(P_n - p_n) P_{n-k} - P_{n-k-1} P_n]$$

or omitting the argument x (for the sake of brevity)

$$t_n - t_{n-1} = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n} c_k \varphi_k (P_n p_{n-k} - p_n P_{n-k}) .$$

Hence we find that

$$\sum_{n=1}^{\infty} n \int_0^1 (t_n - t_{n-1})^2 dx = \sum_{k=1}^{\infty} c_k^2 \sum_{n=k}^{\infty} \frac{n}{P_n^2 P_{n-1}^2} (p_{n-k} P_n - p_n P_{n-k})^2 .$$

Decomposing the inner sum of the last expression in two sums from $n = k$ to $n = 2k$ and from $n = 2k+1$ to $n = +\infty$, and then applying Lemma 4 and the author's lemma (see [6], Lemma 4, p. 233), we find that

$$\sum_{n=1}^{\infty} n \int_0^1 (t_n - t_{n-1})^2 dx = O(1) \Big[\sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=k}^{2k} \frac{n p_n^2 p_{n-k}^2}{P_n^2 P_{n-1}^2} + \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{n=2k+1}^{\infty} \frac{p_n p_{n-k}^2}{P_n P_{n-1}^2} \Big]$$

$$= O(1) \Big[\sum_{k=1}^{\infty} c_k^2 \frac{k}{P_k^2} \sum_{v=0}^{k} p_v^2 + \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{v=k+1}^{\infty} \frac{1}{v^3} \Big] < +\infty .$$

Thus the series

$$\sum_{n=1}^{\infty} n\big(t_n(x) - t_{n-1}(x)\big)^2$$

converges a.e., and this is the required result.

In order to formulate the next theorems we introduce the following definition:

A series $\sum u_n$ is said to be *strongly* (N, p_n)*-summable to* s, with $\{p_n\} \in M^a$, $a > 0$, if

$$\frac{1}{n+1}\sum_{k=0}^{n}(t_k - s)^2 \to 0 \quad \text{as} \quad n \to +\infty,$$

where

$$t_k = \frac{1}{p_k}\sum_{v=0}^{k}(p_{k-v} - p_{k-v-1})s_v,$$

and

$$s_v = u_0 + u_1 + \ldots + u_v \quad (v = 0, 1, 2, \ldots), \quad p_{-1} = 0\ (^5).$$

THEOREM 1. *Suppose that* $\{p_n\} \in \overline{\overline{M}}^a$, $a > \frac{1}{2}$, $\{q_n\} \in \overline{\overline{M}}^\beta$, $\frac{1}{2} < \beta < 1$, $a + \beta \neq 2$. *Then the strong* (N, p_n)*-summability of the series* $\sum u_n$ *implies its* (N, r_n)*-summability, with*

$$r_n = \sum_{k=0}^{n}(p_k - p_{k-1})q_{n-k} \quad \textit{and} \quad \{r_n\} \in M^{a+\beta-1},$$

for sufficiently large n $(^6)$.

Proof. Assuming that the series $\sum u_n$ is strongly (N, p_n)-summable to s, let us denote by $\bar{s}_n$, $\bar{T}_n$, and $\bar{t}_n$, respectively, the nth partial sum, the nth (N, r_n)-mean and the nth $(N, p_n - p_{n-1})$-mean of the series: $(u_0 - s) + u_1 + u_2 + \ldots + u_n + \ldots$ (see [1], pp. 73-74). Then the strong (N, p_n)-summability of the series $\sum u_n$ means the same as

$$\sum_{k=0}^{n}\bar{t}_k^2 = o(n) \quad \text{as} \quad n \to +\infty. \tag{19}$$

Since $r_n = \sum\limits_{k=0}^{n}(p_k - p_{k-1})q_{n-k}$, then with respect to

$$\sum_{n=0}^{\infty}\bar{s}_n x^n \sum_{n=0}^{\infty}(p_n - p_{n-1})x^n \sum_{n=0}^{\infty}q_n x^n$$

$$= \sum_{n=0}^{\infty}x^n \sum_{k=0}^{n}(p_{n-k} - p_{n-k-1})\bar{s}_k \sum_{n=0}^{\infty}q_n x^n = \sum_{n=0}^{\infty}x^n \sum_{k=0}^{n}q_{n-k}p_k\bar{t}_k$$

$(^5)$ The definition of strong (C, a)-summability coincides with the definition of strong (N, p_n)-summability if we take $p_n = \binom{n+a-1}{n}$, with $a > 0$.

$(^6)$ That is, if the sequence $\{r_n\}$ satisfies condition (i) for sufficiently large n. In this sense this relation is understood in the next theorems.

and

$$\sum_{n=0}^{\infty} \bar{s}_n x^n \sum_{n=0}^{\infty} (p_n - p_{n-1}) x^n \sum_{n=0}^{\infty} q_n x^n = \sum_{n=0}^{\infty} \bar{s}_n x_n \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} (p_{n-k} - p_{n-k-1}) q_k$$
$$= \sum_{n=0}^{\infty} \bar{s}_n x^n \sum_{n=0}^{\infty} r_n x^n = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} r_{n-k} \bar{s}_k ,$$

we get the formula

$$\bar{T}_n = \frac{1}{R_n} \sum_{k=0}^{n} r_{n-k} \bar{s}_k = \frac{1}{R_n} \sum_{k=0}^{n} q_{n-k} p_k \bar{t}_k , \tag{20}$$

where $R_n = \sum_{k=0}^{n} q_{n-k} p_k$.

Applying the Cauchy inequality to formula (20), we obtain the inequality

$$|\bar{T}_n| \leqslant \frac{1}{R_n} \sqrt{\sum_{k=0}^{n} p_k^2 q_{n-k}^2 \sum_{k=0}^{n} \bar{t}_k^2} .$$

In order to estimate the last expression we shall distinguish two cases: $0 < p_n \searrow$ and $0 < p_n \nearrow$. Since, by hypothesis, $0 < q_n \searrow$, then in the first case (for $n \geqslant 2$)

$$\frac{\sum_{k=0}^{n} p_k^2 q_{n-k}^2}{\left(\sum_{k=0}^{n} p_k q_{n-k}\right)^2} = \frac{\sum_{k=0}^{m} p_k^2 q_{n-k}^2 + \sum_{k=m+1}^{n} p_k^2 q_{n-k}^2}{\left(\sum_{k=0}^{m} p_k q_{n-k} + \sum_{k=m+1}^{n} p_k q_{n-k}\right)^2}$$
$$< \frac{q_m^2 \sum_{k=0}^{n} p_k^2 + p_m^2 \sum_{k=0}^{n} q_k^2}{m^2 p_m^2 q_n^2 + m^2 q_m^2 p_n^2} < \frac{16 \sum_{k=0}^{n} p_k^2}{(n^2 p_n^2/P_n^2) P_n^2} + \frac{16 \sum_{k=0}^{n} q_k^2}{(n^2 q_n^2/Q_n^2) Q_n^2}$$
$$= O\left(\frac{1}{n}\right) \left(\frac{n}{P_n^2} \sum_{k=0}^{n} p_k^2 + \frac{n}{Q_n^2} \sum_{k=0}^{n} q_k^2\right) = O\left(\frac{1}{n}\right),$$

where $m = [n/2]$ (⁷).

It remains to examine the second case. We have for $n > N$

$$\frac{1}{R_n^2} \sum_{k=0}^{n} p_k^2 q_{n-k}^2 < \frac{p_n^2}{q_n^2 P_n^2} \sum_{k=0}^{n} q_k^2 = \frac{1}{n} \cdot \frac{n^2 p_n^2}{P_n^2} \cdot \frac{1}{(n^2 q_n^2/Q_n^2)} \cdot \frac{n}{Q_n^2} \sum_{k=0}^{n} q_k^2 = O\left(\frac{1}{n}\right).$$

(⁷) By $[x]$ we denote the integral part of x.

Thus, according to formula (19) and the last inequality for $|\bar{T}_n|$, we find in both cases the relation

$$\bar{T}_n = o(1) \quad \text{as} \quad n \to \infty .$$

Now we shall show that $\{r_n\} \in M^{\alpha+\beta-1}$ for sufficiently large n. Since $R_n > 0$ and in view of Lemma 3 $\lim\limits_{n\to\infty} (n+1)r_n/R_n = \alpha+\beta-1 > 0$, we have $r_n > 0$ ([8]) for sufficiently large n. Hence it follows that $R_n \nearrow +\infty$ as $n \to \infty$.

In order to prove that $\{r_n\}$ is a strictly monotone sequence, we write

$$(21) \qquad r_{n-1} - r_n = \sum_{k=0}^{n} (p_k - p_{k-1})(q_{n-k-1} - q_{n-k}) \qquad (p_{-1} = q_{-1} = 0),$$

and we introduce the following notations:

$$\alpha_n = \frac{(n+1)p_n}{P_n}, \quad \bar{\alpha}_n = \frac{(n+1)\Delta p_{n-1}}{p_n}, \quad \bar{\bar{\alpha}}_n = \frac{(n+1)\Delta^2 p_{n-2}}{\Delta p_{n-1}},$$

and analogously

$$\beta_n = \frac{(n+1)q_n}{Q_n}, \quad \bar{\beta}_n = \frac{(n+1)\Delta q_{n-1}}{q_n}, \quad \bar{\bar{\beta}}_n = \frac{(n+1)\Delta^2 q_{n-2}}{\Delta q_{n-1}}.$$

Multiplying both sides of equality (21) by $(n+2)$, we obtain the formula

$$(n+2)(r_{n-1} - r_n) = \sum_{k=0}^{n} (p_{n-k} - p_{n-k-1}) q_k \bar{\beta}_k + \sum_{k=0}^{n} (q_{n-k} - q_{n-k-1}) p_k \bar{\alpha}_k .$$

Multiplying again both sides of the last equality by $(n+2)$, we find that

$$\begin{aligned}
&(n+2)^2(r_{n-1} - r_n) \\
&= \sum_{k=0}^{n} (n-k-1)(p_{n-k} - p_{n-k-1})\bar{\beta}_k q_k + \sum_{k=0}^{n} (n-k+1)(q_{n-k} - q_{n-k-1})\bar{\alpha}_k p_k + \\
&\quad + \sum_{k=0}^{n} (k+1)(p_{n-k} - p_{n-k-1})\bar{\beta}_k q_k + \sum_{k=0}^{n} (k+1)(q_{n-k} - q_{n-k-1})\bar{\alpha}_k p_k \\
&= \sum_{k=0}^{n} \alpha_k \bar{\alpha}_k (q_{n-k} - q_{n-k-1}) P_k + \sum_{k=0}^{n} \beta_k \bar{\beta}_k (p_{n-k} - p_{n-k-1}) Q_k - 2\sum_{k=0}^{n} \bar{\alpha}_k \bar{\beta}_{n-k} p_k q_{n-k} .
\end{aligned}$$

([8]) If $0 < q_n \searrow$, then $\{q_n\}$ tends to zero. The sequence $\{r_n\}$ is in the case of decreasing $\{p_n\}$ also a null-convergent sequence, which is evident from the following theorem: Let both $\{x_n\}$ and $\{y_n\}$ be null-convergent sequences, the second of which satisfies, moreover, the condition that the sums $y_0 + y_1 + y_2 + \ldots + y_n$ are for every n less than a certain constant K. Then the numbers $z_n = x_0 y_n + x_1 y_{n-1} + \ldots + x_n y_0$ also form a null-convergent sequence (see e.g. [4], p. 85, (44) 9b).

Finally, we obtain the formula

$$(22)\qquad (n+2)^2(r_{n-1}-r_n) = \sum_{k=0}^{n} a_k\bar{a}_k(q_{n-k}-q_{n-k-1})P_k + \\ + \sum_{k=0}^{n} \beta_k\bar{\beta}_k(p_{n-k}-p_{n-k-1})Q_k - 2\sum_{k=0}^{n} \bar{a}_k\bar{\beta}_{n-k}p_kq_{n-k}\,.$$

The last formula may also be written in the form

$$(n+2)^2(r_{n-1}-r_n) = \sum_{k=0}^{n} a_k\bar{a}_kq_{n-k}p_k + \sum_{k=0}^{n} \beta_k\bar{\beta}_kp_{n-k}q_k + \\ + \sum_{k=0}^{n} p_kq_{n-k}\left[-\frac{k^2}{(k+1)^2}\bar{\bar{a}}_k\bar{a}_k + \frac{2k+1}{k+1}\bar{a}_k - a_k\bar{a}_k\right] + \\ + \sum_{k=0}^{n} q_kp_{n-k}\left[-\frac{k^2}{(k+1)^2}\bar{\bar{\beta}}_k\bar{\beta}_k + \frac{2k+1}{k+1}\bar{\beta}_k - \beta_k\bar{\beta}_k\right] - 2\sum_{k=0}^{n} \bar{a}_k\bar{\beta}_{n-k}p_kq_{n-k}\,.$$

Applying the Cauchy inequality to the last expression, and remarking that the method $\|a_{nk}\|$, where

$$a_{nk} = \begin{cases} \dfrac{p_kq_{n-k}}{R_n} & \text{for} \quad k = 0,1,2,\ldots,n\,, \\ 0 & \text{otherwise} \end{cases}$$

is regular, we find that

$$\lim_{n\to\infty}\frac{(n+2)^2(r_{n-1}-r_n)}{R_n} \geqslant (a+\beta-1)(2-a-\beta) > 0, \quad \text{if} \quad a+\beta<2$$

or

$$\lim_{n\to\infty}\frac{(n+2)^2(r_n-r_{n-1})}{R_n} \geqslant (a+\beta-1)(a+\beta-2) > 0, \quad \text{if} \quad a+\beta>2\,.$$

Consequently, $0<r_n\searrow$ or $0<r_n\nearrow$ for sufficiently large n, which completes the proof of Theorem.

Remark. If $a>1$, then $0<p_n\nearrow$, and the formula on r_n implies that $r_n>0$ for $n=0,1,2,\ldots$ If, moreover, $a>2$, then $\{p_n\}$ is convex and from the formula on r_n we deduce immediately that $0<r_n<r_{n+1}$ for $n=0,1,2,\ldots$

THEOREM 2. *If orthonormal series* (17) *is* (N,p_n)*-summable to a function* $s(x)$ *a.e., with* $\{p_n\}\in\bar{\bar{M}}^a$, $a>\frac{1}{2}$, *then it is strongly* (N,p_n)*-summable to this function a.e.*

Proof. Denoting by $T_n(x)$ and $t_n(x)$, respectively, the nth (N,p_n)-mean and the nth (N,p_n-p_{n-1})-mean of orthonormal series (17), we set $s(x) = \lim_{n\to\infty} T_n(x)$ at the point x of convergence of the sequence $\{T_n(x)\}$. Omitting for the sake of brevity the argument x, we can write

$$\sum_{k=0}^{n}(t_k-s)^2 \leqslant 2\sum_{k=0}^{n}(t_k-T_k)^2+2\sum_{k=0}^{n}(T_k-s)^2 .$$

By hypothesis the last expression is of the order $o(n)$ a.e.

Obviously, our statement will be proved if we show that

$$\sum_{k=0}^{n}(t_k-T_k)^2 \doteq o(n+1) \quad \text{as} \quad n\to+\infty\ (^9). \tag{23}$$

We write

$$t_k-T_k = \sum_{v=0}^{k} c_v\varphi_v \frac{p_{k-v}P_k-p_kP_{k-v}}{p_kP_k}.$$

Hence we find that

$$\sum_{k=0}^{\infty}\int_0^1 \frac{(t_k-T_k)^2}{k+1}\,dx = \sum_{k=0}^{\infty}\frac{1}{k+1}\sum_{v=0}^{k} c_v^2\left(\frac{p_{k-v}P_k-p_kP_{k-v}}{p_kP_k}\right)^2$$

$$= \sum_{v=0}^{\infty}c_v^2\sum_{k=v}^{\infty}\frac{1}{k+1}\left(\frac{p_{k-v}P_k-p_kP_{k-v}}{p_kP_k}\right)^2 .$$

Proceeding as in the proof of Lemma 5, we find that

$$\sum_{k=0}^{\infty}\int_0^1 \frac{(t_k-T_k)^2}{k+1}\,dx = O(1)\left[\sum_{k=1}^{\infty}c_k^2\frac{k}{P_k^2}\sum_{v=0}^{k}p_v^2+\sum_{k=1}^{\infty}k^2c_k^2\sum_{v=k+1}^{\infty}\frac{1}{v^3}\right]< +\infty .$$

The series $\sum_{k=0}^{\infty}(t_k-T_k)^2/(k+1)$ is then convergent a.e., whence by means of the well-known Kronecker's theorem we obtain relation (23).

In order to formulate the next theorem we introduce the following definition:

The orthonormal series (17) is said to be *very strongly* (N, p_n)*-summable to a function* $s(x)$ *a.e.* if for every increasing sequence $\{v_m\}$ of indices the relation

$$\sum_{m=1}^{n}[t_{v_m}(x)-s(x)]^2 \doteq o(n+1) \quad \text{as} \quad n\to+\infty,$$

(9) If the sequence $\{f_n(x)/g_n(x)\}$ tends to zero, resp. is bounded in the interval $\langle 0, 1\rangle$ a.e. (almost everywhere), then we shall write: $f_n(x) \doteq o(g_n(x))$, resp., $f_n(x) \doteq O(g_n(x))$.

holds, where $t_n(x)$ denotes the nth $(N, p_n - p_{n-1})$-mean of orthonormal series (17).

THEOREM 3a. *Let $\{\lambda_n\}$ be an increasing and positive sequence such that $\{n/\lambda_n\}$ is also an increasing sequence, and let $\sum_{n=1}^{\infty} 1/n\lambda_n$ be a convergent series. If orthonormal series (17) is (N, p_n)-summable on a set E a.e., with $\{p_n\} \in \overline{M}^a$, $a > \frac{1}{2}$, then the condition*

$$c_n = O\left(\frac{1}{\sqrt{n\lambda_n}}\right) \tag{24}$$

implies its very strong (N, p_n)-summability on E a.e.

Proof. As in the proof of Theorem 2 our statement will be proved if we show that

$$\sum_{m=1}^{n} [t_{v_m}(x) - T_{v_m}(x)]^2 \doteq o(n), \tag{25}$$

where $t_n(x)$ and $T_n(x)$ denote, resp., the nth $(N, p_n - p_{n-1})$-mean and the nth (N, p_n)-mean of orthonormal series (17) (see [1], 2.6.3, p. 104-105). In virtue of Lemma 4, we can write

$$\sum_{m=1}^{\infty} \frac{1}{m} \int_0^1 [t_{v_m}(x) - T_{v_m}(x)]^2 dx = O(1) \sum_{m=1}^{\infty} \frac{1}{m P_{v_m}^2} \sum_{k=1}^{v_m} p_{v_m - k}^2 k^2 c_k^2 .$$

According to condition (24) $k^2 c_k^2 = O(k/\lambda_k)$, and from the monotonicality of the sequence $\{k/\lambda_k\}$ and by $(11)_{(a)}$ it follows that for $a > \frac{1}{2}$

$$\sum_{m=1}^{\infty} \frac{1}{m} \int_0^1 [t_{v_m}(x) - T_{v_m}(x)]^2 dx = O(1) \sum_{m=1}^{\infty} \frac{1}{m\lambda_{v_m}} \cdot \frac{v_m}{P_{v_m}^2} \sum_{k=0}^{v_m} p_k^2$$

$$= O(1) \sum_{m=1}^{\infty} \frac{1}{m\lambda_m} < \infty .$$

The series

$$\sum_{m=1}^{\infty} \frac{1}{m} [t_{v_m}(x) - T_{v_m}(x)]^2$$

is then convergent a.e., whence we conclude by Kronecker's theorem the validity of relation (25).

THEOREM 3b. *Let $\{c_n\}$ denote a positive numerical sequence satisfying the following conditions:*

$$\sum_{n=0}^{\infty} c_n^2 < \infty, \tag{26}$$

$$n c_n^2 \geqslant (n+1) c_{n+1}^2 \quad (n = 1, 2, \ldots). \tag{27}$$

Further let

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{28}$$

denote an orthogonal series with coefficients satisfying the condition

$$a_n^2 = O(c_n^2) . \tag{29}$$

Suppose that the orthogonal series (28) *is under these assumptions* (N, p_n)-*summable to a function* $f(x)$ *a. e., with* $\{p_n\} \in \overline{M}^a$, $a > \frac{1}{2}$; *then it is very strongly* (N, p_n)-*summable to this function a.e.*

Proof. Let $t_n(x)$ and $T_n(x)$ have the same meaning as previously but with respect to the orthogonal series (28). By arguing as before, it remains to show only relation (25). In virtue of Lemma 4 and condition (29), we can write (omitting the argument x for the sake of brevity)

$$\sum_{m=1}^{\infty} \int_0^1 \frac{1}{m} (T_{v_m} - t_{v_m})^2 dx = O(1) \sum_{m=1}^{\infty} \frac{1}{mP_{v_m}^2} \sum_{k=1}^{v_m} k^2 a_k^2 p_{v_m-k}^2$$

$$= O(1) \Big[\sum_{m=1}^{\infty} \frac{1}{mP_{v_m}^2} \sum_{k=1}^{m-1} k^2 c_k^2 p_{v_m-k}^2 + \sum_{m=1}^{\infty} \frac{1}{mP_{v_m}^2} \sum_{k=m}^{v_m} k^2 c_k^2 p_{v_m-k}^2 \Big] .$$

Denoting by A and B, resp., the first and the second term contained in the last brackets, we shall estimate them distinguishing two cases: 1) $0 < p_n \searrow$, 2) $0 < p_n \nearrow$. Let $0 < p_n \searrow$. Changing the order of summation in the expression A and decomposing the inner sum into two sums from $m = k+1$ to $m = 2k$ and from $m = 2k+1$ to $m = \infty$, we can write

$$A = \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{m=k+1}^{\infty} \frac{p_{v_m-k}^2}{mP_{v_m}^2} < \sum_{k=1}^{\infty} k^2 c_k^2 \Big(\sum_{m=k}^{2k} \frac{p_{v_m-k}^2}{mP_{v_m}^2} + \sum_{m=2k+1}^{\infty} \frac{p_{v_m-k}^2}{mP_{v_m}^2} \Big) .$$

Considering that $v_m \geqslant m$ $(m = 1, 2, \dots)$ and applying a lemma (see [6], Lemma 4, pp. 233-234), we obtain

$$\sum_{k=1}^{\infty} k^2 c_k^2 \sum_{m=k}^{2k} \frac{p_{v_m-k}^2}{mP_{v_m}^2} \leqslant \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{m=k}^{2k} \frac{p_{m-k}^2}{mP_m^2} < \sum_{k=1}^{\infty} c_k^2 \frac{k}{P_k^2} \sum_{m=0}^{k} p_m^2$$

$$= O(1) \sum_{k=1}^{\infty} c_k^2 < \infty ,$$

and

$$\sum_{k=1}^{\infty} k^2 c_k^2 \sum_{m=2k+1}^{\infty} \frac{p_{v_m-k}^2}{m P_{v_m}^2} = O(1) \sum_{k=1}^{\infty} k c_k^2 \sum_{m=2k+1}^{\infty} \frac{1}{(v_m-k)^2}$$

$$= O(1) \sum_{k=1}^{\infty} k c_k^2 \sum_{m=k}^{\infty} \frac{1}{m^2} = O(1) \sum_{k=1}^{\infty} c_k^2 < \infty .$$

If $0 < p_n \nearrow$, then

$$A < \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{m=k}^{\infty} \frac{p_{v_m}^2}{m P_{v_m}^2} = O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{m=k}^{\infty} \frac{1}{m v_m^2}$$

$$= O(1) \sum_{k=1}^{\infty} k^2 c_k^2 \sum_{m=k}^{\infty} \frac{1}{m^3} = O(1) \sum_{k=1}^{\infty} c_k^2 < \infty .$$

Passing to the estimate of the expression B, we find after (27) and the lemma quoted above that in both cases

$$B < \sum_{m=1}^{\infty} \frac{v_m c_m^2}{P_{v_m}^2} \sum_{k=0}^{v_m} p_k^2 = O(1) \sum_{m=1}^{\infty} c_m^2 < \infty .$$

From the above estimations it follows that the series

$$\sum_{m=1}^{\infty}{}' \frac{1}{m} [T_{v_m}(x) - t_{v_m}(x)]^2$$

is convergent a.e., whence we obtain relation (25) and at the same time the required result.

Remark. An analogous theorem concerning the (C, a)-method, with the additional condition $n^2 c_n^2 \leqslant (n+1)^2 c_{n+1}^2$ $(n = 1, 2, \ldots)$, has been proved by G. Alexits (see [2]). K. Tandori has proved (see [9]) that this theorem is also valid without the last assumption. Recently, L. Leindler (see [5]) has solved a problem raised by K. Tandori, proving that: The strictly increasing sequence $\{v_m\}$ of indices, used in the definition of the very strong (C, a)-summability, can be replaced by an arbitrary sequence of natural numbers, all distinct, tending to infinity. This statement can be transferred without any essential difficulty to the very strong (N, p_n)-summability. Theorem 3b constitutes an analogue to that of Tandori.

3. Proceeding in the same way as in the proof of the author's theorem (see [6], Th. 3, pp. 244-245), we obtain according to Lemma 4 the following

THEOREM 4. *Let $s_n(x)$ be the n-th partial sum of orthonormal series* (17), *and let $\{n_k\}$ denote a lacunary sequence of indices satisfying the condition*

$$(26) \qquad 1 < q \leqslant n_{k+1}/n_k \leqslant r \quad \textit{for} \quad k = 0, 1, 2, \ldots,$$

where r and q are constants greater than 1.

In order that orthonormal series (17) *be (N, p_n)-summable a.e., with arbitrary $\{p_n\} \in \overline{M}^{\alpha}$, $\alpha > \frac{1}{2}$, it is necessary and sufficient that the sequence $\{s_{n_k}(x)\}$ be convergent a.e.*

THEOREM 5. *If orthonormal series* (17) *is (N, p_n)-summable, with $\{p_n\} \in M^{\alpha}$, $\alpha > 0$, to a function $f(x) \in L^2$ a.e., then it is also (N, q_n)-summable to this function a.e., with arbitrary $\{q_n\} \in \overline{M}^{\beta}$, $\beta > \frac{1}{2}$.*

Proof. Suppose that the sequences $\{s_n(x)\}$ and $\{n_k\}$ have the same meaning as previously, and let $T_n(x)$ denote the nth (N, p_n)-mean of orthonormal series (17). The sequence $\{T_n(x)\}$ is by hypothesis convergent to a function $f(x) \in L^2$ a.e. and, of course, also the sequence $\{T_{n_k}(x)\}$. Hence, by a lemma (see [6], Lemma 9, p. 242) it follows that the sequence $\{s_{n_k}(x)\}$ converges also to $f(x)$ a.e. Applying now Theorem 4, we conclude that the series under consideration is (N, q_n)-summable to this function a.e., with arbitrary $\{q_n\} \in \overline{M}^{\beta}$, $\beta > \frac{1}{2}$, and this is the required result.

THEOREM 6. *If orthonormal series* (17) *is (N, p_n)-summable to a function $f(x) \in L^2$ a.e., with arbitrary $\{p_n\} \in \overline{M}^{\alpha}$, $\alpha > \frac{1}{2}$, then it is (N, r_n)-summable to this function a.e., with $\{r_n\} \in M^{\beta}$ for sufficiently large n and arbitrary $0 < \beta < 1$.*

Proof. In virtue of Theorem 4 orthonormal series (17) is in particular (N, q_n)-summable to $f(x)$ a.e., with $\{q_n\} \in \overline{\overline{M}}^{(1+\beta)/2}$ and arbitrary $0 < \beta < 1$. Then by Theorem 2 and Theorem 1 it is (N, r_n)-summable to this function a.e., with $r_n = \sum_{k=0}^{n} q_{n-k}(q_k - q_{k-1})$ and $\{r_n\} \in M^{\beta}$ for sufficiently large n.

In order to formulate the next theorem, we extend the sequence $\{n_k\}$ satisfying the condition of lacunarity (26) to a continuous and strictly increasing function $n(x)$, assuming the value $n(k) = n_k$ at $x = k$ for $k = 0, 1, 2, \ldots$, by means of linear interpolation. We denote the inverse of the function $n(x)$ by $l(x)$. Evidently, the function $l(x)$ is continuously and strictly increasing.

THEOREM 7. *If*

$$\sum^{\infty} c_n^2 \log_+^2 [l(n)] < +\infty\ (^{10}),$$

(10) By $\log_+|f|$ we mean $\log|f|$ wherever $|f| \geqslant 1$, and 0 otherwise.

then orthonormal series (17) *is* (N, p_n)*-summable a.e., with arbitrary* $\{p_n\} \in \overline{M}^{\alpha}$, $\alpha > \frac{1}{2}$, *and* (N, r_n)*-summable a.e., with certain* $\{r_n\} \in M^{\beta}$, $0 < \beta < 1$, *for sufficiently large* n.

The proof of this theorem follows immediately from the proof of an analogous theorem (see [6], Th. 2, pp. 245-246), from Theorem 4 and Theorem 6.

THEOREM 8. *Let* $\{v(n)\}$ *be an arbitrary sequence of numbers satisfying the condition*

$$0 < v(n) \leqslant v(n+1), \qquad v(n) = o\{\log^2[l(n)]\}.$$

Moreover, let $\{p_n\} \in \overline{M}^{\alpha}$, *with* $\alpha > \frac{1}{2}$, *and* $\{r_n\} \in M^{\beta}$, *with*

$$r_n = \sum_{k=0}^{n} q_{n-k}(q_k - q_{k-1}), \qquad \text{where} \qquad \{q_n\} \in M^{(1+\beta)/2},\ 0 < \beta < 1.$$

Then there exist a system ON $\{\psi_n(x)\}$ *and a sequence of real numbers* $\{b_n\}$ *such that*

1° $\sum_{n=0}^{\infty} b_n^2 v(n) < \infty$,

2° *the series* $\sum_{n=0}^{n} b_n \psi_n(x)$ *is not summable by the methods* (N, p_n) *and* (N, r_n), *with* $\{r_n\} \in M^{\beta}$ *for sufficiently large* n, *at any point of the interval* $\langle 0, 1\rangle$.

The proof runs similarly to the proof of an analogous theorem (see [6], Th. 3, pp. 246-247), on the basis of Theorem 4.

THEOREM 9. *Suppose that* $\{p_n\} \in \overline{M}^{\alpha}$, *with* $\alpha > \frac{1}{2}$, *and* $\{r_n\} \in M^{\beta}$ *for* $0 < \beta < 1$ *and sufficiently large* n, *where*

$$r_n = \sum_{k=0}^{n} q_{n-k}(q_k - q_{k-1}) \qquad \text{and} \qquad \{q_n\} \in \overline{\overline{M}}^{(1+\beta)/2}.$$

If $t_n(x)$ *denotes the* n*-th* (N, p_n)*-mean or the* n*-th* (N, r_n)*-mean of orthonormal series* (17), *then*

$$t_n(x) \doteq o\{\log[l(n)]\} \qquad \text{as} \qquad n \to +\infty.$$

The proof is based on Theorem 7 and on the proof of an analogous theorem (see [6], Th. 4, p. 247).

THEOREM 10. *Suppose that the assumptions of Theorem* 9 *are satisfied. Moreover, let* $\{v(n)\}$ *denote an arbitrary sequence of positive numbers increasing monotonically to infinity and satisfying the condition*

$$v(n) = o\{\log^2[l(n)]\} \qquad \text{as} \qquad n \to +\infty.$$

Then there exist a system ON $\{\Phi_n(x)\}$ *and a sequence of real numbers* $\{a_n\} \in l^2$ *such that*

$$\overline{\lim_{n\to\infty}} \frac{|T_n(x)|}{v(n)} = +\infty,$$

at every point of the interval $\langle 0, 1\rangle$, *where* $T_n(x)$ *denote the n-th* (N, p_n) *resp. n-th* (N, r_n)*-mean of the orthonormal series* $\sum_{n=0}^{\infty} a_n \Phi_n(x)$.

The proof is based on Theorems 8 and 9, and on the proof of an analogous theorem (see [6], Th. 5, pp. 248-250).

Remark. The condition $\{p_n\} \in BVM^{\alpha}$, $\alpha > 0$, holds for the method (C, α) only if $0 < \alpha < 1$, while the condition $\{p_n\} \in \bar{M}^{\alpha}$, $\alpha > 0$, is satisfied for all Cesàro's methods of positive order. In this sense Lemma 5 and Theorems 2-10 generalize our previous results. Moreover, Theorems 9 and 10 establish a generalization of theorems VII and VIII of K. Tandori concerning the estimation of the (C, α)-means for all $\alpha > 0$ (see e.g. [1], pp. 112-113 and p. 173).

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