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On traces of solutions of a semi-linear partial differential equation of elliptic type

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Dedicated to the memory of Jacek Szarski

Abstract. In this paper we investigate the traces of generalized solutions of a semi-linear elliptic equation. In particular, we obtain a sufficient condition for a solution belonging to $W^{1,p}_{\text{loc}}$ to have an L^p-trace on the boundary.

In the theory of partial differential equations the problem of the behaviour of the given solution near the boundary arises in a natural way. One such problem is always that of determining if the given solution has a trace on the boundary. Several function spaces arise as the spaces of traces of solutions of partial differential equations. The purpose of this paper is to obtain conditions giving L^p-traces on the boundary of solutions of a semi-linear elliptic equation. The main result is Theorem 6.

The plan of the paper is as follows. Section 1 is devoted to preliminaries. Section 2 deals with the problem of traces for solutions in $W^{1,p}_{\text{loc}}$, $p > 1$. In Section 3 we discuss traces of the first derivatives of solutions. Section 4 extends these results to the solutions of weakly coupled elliptic systems The arguments which we give here are based partially on the references [6] and [7].

1. Consider the semi-linear elliptic equation of the form

$$\sum_{i,j=1}^{n} D_j\big(a_{ij}(x)D_i u\big) - b(x, u, Du) = 0 \tag{1}$$

in a bounded domain $Q \subset \boldsymbol{R}_n$ with the boundary ∂Q of the class C^2, $Du = (D_1 u, \ldots, D_n u)$, $D_i u = \partial u/\partial x_i$.

We make the following assumptions:

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(A) There is a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \leqslant \gamma|\xi|^2$$

for all $\xi \in \boldsymbol{R}_n$ and $x \in Q$.

(B) The coefficients a_{ij} belong to $C^1(\bar{Q})$.

(C) The function $b(x, u, s)$ is defined for $(x, u, s) \in Q \times \boldsymbol{R}_{n+1}$, $s = (s_1, \ldots, s_n)$, and satisfies the Carathéodory conditions; that is,

(i) for a.e. $x \in Q$, $b(x, \cdot, \cdot)$ is a continuous function on $\boldsymbol{R}_{n+1}$,

(ii) for every fixed $(u, s) \in \boldsymbol{R}_{n+1}$, $b(\cdot, u, s,)$ is a measurable function on Q.

Moreover, we assume that

$$|b(x, u, s)| \leqslant f(x) + L(|u| + |s|),$$

for all $(x, u, s) \in Q \times \boldsymbol{R}_{n+1}$, where L is a positive constant and f is a non-negative measurable function of $x \in Q$ such that

$$\int_Q f(x)^p r(x)^\theta dx < \infty,$$

$1 < p < \infty$, $p \leqslant \theta < 2p-1$, $r(x) = \operatorname{dist}(x, \partial Q)$.

Remark 1. Under assumption (C), $b(x, u(x), s(x))$ is a measurable function of $x \in Q$, where $(u(x), s(x))$ is a measurable function and

$$b(x, \cdot, \cdot)\colon L^1_{\mathrm{loc}}(Q)^{n+1} \to L^1_{\mathrm{loc}}(Q)$$

is continuous (see [5]).

In the sequel we use the notion of a generalized solution involving the Sobolev spaces $W^{1,p}_{\mathrm{loc}}(Q)$ and $W^{1,p}(Q)$.

We denote by $W^{k,p}(Q)$ the Sobolev space of real functions u such that u and its distributional derivatives up to order k belong to $L^p(Q)$. This space is provided with the standard norm

$$\|u\|^p_{W^{k,p}} = \sum_{|a| \leqslant k} \|D^a u\|^p_{L^p},$$

where $a = (a_1, \ldots, a_n)$, the a_j are positive integers,

$$|a| = a_1 + \ldots + a_n, \quad \text{and} \quad D^a u = \frac{\partial^{|a|} u}{\partial x_1^{a_1} \ldots \partial x_n^{a_n}}.$$

The space of functions f such that $f \in W^{k,p}(Q')$ for every domain Q' satisfying $\bar{Q}' \subset Q$ is denoted by $W^{k,p}_{\mathrm{loc}}(Q)$. For related material on Sobolev spaces see [3].

A function $u(x)$ is said to be a *weak solution* of equation (1) in Q if $u \in W^{1,p}_{\mathrm{loc}}(Q)$ and u satisfies

$$\int_Q \Big\{ \sum_{i,j=1}^{n} a_{ij}(x) D_i u D_j v + b(x, u, Du) v \Big\}\, dx = 0 \tag{2}$$

for every $v \in W^{1,p'}(Q)$ with compact support in Q, where $1/p + 1/p' = 1$.

We now discuss a number of preliminary results that will be needed in the following sections.

We say that u belongs to $A(Q)$ provided that there is a measurable function $\tilde{u}$ on Q such that $u = \tilde{u}$ a.e. on Q and for almost every line τ parallel to any coordinate axis x_i, $i = 1, \ldots, n$, $\tilde{u}$ is absolutely continuous on each compact subinterval $\tau \cap Q$.

When $u \in A(Q)$ we identify u with $\tilde{u}$ so that $D_i u$ exists a.e. on Q.

The following characterization of $W^{1,p}(Q)$ is due to Gagliardo, Morrey and Calkin.

LEMMA 1. *Let $1 \leqslant p < \infty$. A function u defined on Q is in $W^{1,p}(Q)$ if and only if $u \in A(Q)$ and*

(i) $D_i u \in L^p(Q)$, $i = 1, \ldots, n$,

(ii) $u \in L^p(Q)$.

Moreover, $D_i u$ coincides a.e. in Q with the corresponding distributional derivative.

If Q is a bounded domain satisfying the cone property, then condition (ii) is superfluous. This result is essentially contained in Sections 1 and 2 of Gagliardo [1].

From Lemma 1 it follows that $u \in W^{1,p}_{\mathrm{loc}}(Q)$ if and only if $u \in A(Q)$ and $D_i u \in L_{\mathrm{loc}}(Q)$ for $i = 1, \ldots, n$.

We will also need the following well-known change of variables theorem which is an immediate consequence of the previous result.

LEMMA 2. *Let $g\colon H \to G$ be a one-to-one mapping of the domain $H \subset \boldsymbol{R}_n$ to the domain $G \subset \boldsymbol{R}_n$ such that g and g^{-1} are locally Lipschitz. Let $v(y) = u[g(y)]$; then $v \in W^{1,p}_{\mathrm{loc}}(H)$ if and only if $u \in W^{1,p}_{\mathrm{loc}}(G)$. In this case*

$$Dv(y) = Du[g(y)]\, Dg(y).$$

Here $D_i u[g(y)] D_j g_i(y)$ is interpreted as zero whenever $D_j g_i(y) = 0$, where $g = (g_1, \ldots, g_n)$.

Another result we use frequently is the following.

LEMMA 3. *Let (X, χ, μ) be a measure space, $f \in L^1([a, b] \times X)$ and for μ a.e. $x \in X$ let f be absolutely continuous on $[a, b]$. If $\partial f/\partial \delta \in L^1([a, b] \times X)$, then there is an absolutely continuous function $F(\delta)$ on $[a, b]$ such that*

$$F(\delta) = \int_X f(\delta, x)\, d\mu(x),$$

and

$$F'(\delta) = \int_X \frac{\partial f(\delta, x)}{\partial \delta} \, d\mu(x)$$

for a.e. $\delta \in [a, b]$.

This result is used in Gagliardo [1].

If follows from the regularity of the boundary ∂Q that there is a number $\delta_0 > 0$ such that for $\delta \in (0, \delta_0]$ the domain $Q_\delta = Q \cap \{x;\ \min_{y \in \partial Q} |x-y| > \delta\}$, with the boundary ∂Q_δ, possesses the following property: to each $x_0 \in \partial Q$ there is a unique point $x_\delta(x_0) = x_0 - \delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The inverse mapping $x_0 \to x_\delta(x_0)$ is given by the formula $x_0 = x_\delta + \delta\nu_\delta(x_\delta)$, where $\nu_\delta(x_\delta)$ is the outward normal to ∂Q_δ at x_δ.

Let x_δ denote an arbitrary point of ∂Q_δ. For fixed $\delta \in (0, \delta_0]$ let

$$A_\varepsilon = \partial Q_\delta \cap \{x;\ |x - x_\delta| < \varepsilon\}, \qquad B_\varepsilon = \{x;\ x = \tilde{x}_\delta + \delta\nu_\delta(\tilde{x}_\delta), \tilde{x}_\delta \in A_\varepsilon\},$$

and

$$\frac{dS_\delta}{dS_0}(x_\delta) = \lim_{\varepsilon \to 0} \frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denotes the $n-1$ dimensional Hausdorff measure of a set A. Mihailov [7] proved that there is a positive number γ_0 such that

$$\gamma_0^{-2} \leqslant \frac{dS_\delta}{dS_0} \leqslant \gamma_0^2 \tag{3}$$

and

$$\lim_{\delta \to 0} \frac{dS_\delta}{dS_0}(x_\delta) = 1 \tag{4}$$

uniformly with respect to $x_\delta \in \partial Q_\delta$.

According to Lemma 1 in [2], p. 382, the distance $r(x)$ belongs to $C^2(\bar{Q} - Q_{\delta_0})$ if δ_0 is sufficiently small. Denote by $\varrho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties: $\varrho(x) = r(x)$ for $x \in \bar{Q} - Q_{\delta_0}$, $\varrho \in C^2(\bar{Q})$, $\varrho(x) \geqslant 3\delta_0/4$ in Q_{δ_0}, $\gamma_1^{-1} r(x) \leqslant \varrho(x) \leqslant \gamma_1 r(x)$ in Q for some positive constant γ_1, $\partial Q_\delta = \{x;\ \varrho(x) = \delta\}$ for $\delta \in (0, \delta_0]$ and finally $\partial Q = \{x;\ \varrho(x) = 0\}$.

We will use the surface integrals $M_1(\delta) = \int_{\partial Q} |u(x_\delta(x))|^p dS_x$ and $M(\delta) = \int_{\partial Q_\delta} |u(x)|^p dS_x$, where $u \in W^{1,p}_{\mathrm{loc}}(Q)$ and the values of $u(x_\delta(x))$ on ∂Q and $u(x)$ on ∂Q_δ are understood in the sense of traces (see [3], chapter 6). The following lemma shows that $M_1(\delta)$ and $M(\delta)$ are absolutely continuous on $[\delta_1, \delta_0]$ for every $0 < \delta_1 < \delta_0$.

LEMMA 4. *Let Φ: $R \to R$ be locally absolutely continuous, $u \in W^{1,1}_{\mathrm{loc}}(Q)$ and $|D\Phi(u)| \in L^2_{\mathrm{loc}}(Q)$; then $\int_{\partial Q} \Phi(u[x_\delta(x)])^2 dS_x$ is absolutely continuous on $[\delta_1, \delta_0]$ for every $0 < \delta_1 < \delta_0$ and $\Phi(u) \in W^{1,2}_{\mathrm{loc}}(Q)$.*

Here $D_i\Phi(u) = \Phi'(u)D_i u$, where the right-hand side is interpreted as zero whenever $D_i u = 0$.

Proof. The second part is immediate from Lemma 1 and Theorem 4.3 of [4]. The first part now follows from Lemmas 2 and 3, and Theorem 4.3 of [4], as follows. It suffices to note that $\Phi^2(u[x_\delta(x)])$ and $\frac{\partial}{\partial\delta}\Phi^2(u[x_\delta(x)])$ are in $L^1([\delta_1, \delta_0]\times\partial Q)$ and $\Phi^2(u[x_\delta(x)])$ is absolutely continuous on $[\delta_1, \delta_0]$ for dS_0 a.e. $x \in \partial Q$.

An essential tool used in this paper is the following well-known generalization of the classical Green's formula:

$$\int_Q gD_i f dx = -\int_{\partial Q} gfD_i\varrho dS_x - \int_Q fD_i g dx$$

for any $f \in W^{1,p}(Q)$ and $g \in W^{1,p'}(Q)$, $1/p+1/p' = 1$, $1 \leqslant p < \infty$. Here the values of f and g on ∂Q are understood in the sense of traces. This formula follows immediately from the formula with $f, g \in C^1(\bar{Q})$ and Theorem 6.4.1 of [3].

2. Our first objective in the study of the traces of solutions of (1) is the derivation of a criterion for the continuity of $M_1(\delta)$ and $M(\delta)$ on $[0, \delta_0]$ which plays an essential part in the ensuing treatment of these traces. To obtain this criterion we need the following two lemmas.

For $v \in L^1_{\mathrm{loc}}(Q)$ we may define the surface integral

$$\tilde{M}(\delta) = \int_{\partial Q_\delta} |v(x)| dS_x \quad \text{for almost every } \delta \in (0, \delta_0].$$

LEMMA 5. *Suppose that $\tilde{M}(\delta)$ is a bounded function on $(0, \delta_0]$. Then for every $0 < a < 1$ there is a positive constant C such that*

$$\int_{Q_\delta} \frac{|v(x)|}{\big(\varrho(x)-\delta\big)^a} dx \leqslant C \quad \textit{for every } \delta \in \left(0, \frac{\delta_0}{2}\right].$$

Proof. For $\delta \in (0, \delta_0/2]$,

$$\begin{aligned}\int_{Q_\delta} \frac{|v(x)|}{\big(\varrho(x)-\delta\big)^a} dx &\leqslant \int_{Q_{\delta_0}} \frac{|v(x)|}{(\varrho-\delta)^a} dx + \int_{Q_\delta - Q_{\delta_0}} \frac{|v(x)|}{(\varrho-\delta)^a} dx \\ &\leqslant \left(\frac{4}{\delta_0}\right)^a \int_{Q_{\delta_0}} |v(x)| dx + \int_\delta^{\delta_0} \frac{dt}{(t-\delta)^a} \int_{\partial Q_t} |v(x)| dS_x \\ &\leqslant \left(\frac{4}{\delta_0}\right)^a \int_{Q_{\delta_0}} |v(x)| dx + \frac{\delta_0^{-a+1}}{1-a} \sup_{0<\delta<\delta_0} \tilde{M}(\delta).\end{aligned}$$

LEMMA 6. *Let Φ: $R \to R$ be locally absolutely continuous and $u \in W^{1,1}_{\mathrm{loc}}(Q)$ satisfy*

$$\int_Q |D\Phi(u)|^2 r(x)\,dx < \infty;$$

then for every $a \in (0,1)$ there is a positive constant C such that

$$\int_{Q_\delta} \frac{\Phi(u)^2}{\big(\varrho(x)-\delta\big)^a}\,dx \leqslant C \quad \textit{for every } \delta \in (0, \delta_0/2].$$

Proof. Let $\delta \in (0, \delta_0/2]$. By assumption $D_i\Phi(u) \in L^2_{\mathrm{loc}}(Q)$ and thus $\Phi(u) \in W^{1,2}_{\mathrm{loc}}(Q)$, by Lemma 1. Now

$$\int_{Q_\delta} \frac{\Phi(u)^2}{(\varrho-\delta)^a}\,dx = \int_{Q_\delta - Q_{\delta_0}} \frac{\Phi(u)^2}{(\varrho-\delta)^a}\,dx + \int_{Q_{\delta_0}} \frac{\Phi(u)^2}{(\varrho-\delta)^a}\,dx$$

and

$$\int_{Q_{\delta_0}} \frac{\Phi(u)^2}{(\varrho-\delta)^a}\,dx \leqslant \left(\frac{4}{\delta_0}\right)^a \int_{Q_{\delta_0}} \Phi(u)^2\,dx.$$

Moreover,

$$\int_{Q_\delta - Q_{\delta_0}} \frac{\Phi(u)^2}{(\varrho-\delta)^a}\,dx = \int_\delta^{\delta_0} \frac{dt}{(t-\delta)^a} \int_{\partial Q} \Phi\big(u[x_t(x_0)]\big)^2 \frac{dS_t}{dS_0}\,dS_0$$

$$\leqslant \gamma_0^2 \int_\delta^{\delta_0} \frac{dt}{(t-\delta)^a} \int_{\partial Q} \Phi\big(u[x(x_0)]\big)\,dS_0.$$

As $\int_{\partial Q} \Phi\big(u[x_t(x)]\big)^2 dS_x$ is absolutely continuous on $[\delta, \delta_0]$, integrating by parts

$$\int_{Q_\delta - Q_{\delta_0}} \frac{\Phi(u)^2}{(\varrho-\delta)^a}\,dx \leqslant \gamma_0^2 \frac{\delta_0^{1-a}}{1-a} \int_{\partial Q} \Phi\big(u[x_{\delta_0}(x)]\big)^2 dS_0 +$$

$$+\gamma_0^2 \frac{2}{1-a} \int_\delta^{\delta_0} (t-\delta)^{1-a}\,dt \int_{\partial Q} \big|\Phi\big(u[x_t(x_0)]\big)\big|\big|D\Phi\big(u[x_t(x_0)]\big)\big| \left|\frac{\partial}{\partial t} x_t(x_0)\right| dS_0$$

$$\leqslant \gamma_0^4 \frac{\delta_0^{1-a}}{1-a} \int_{\partial Q_{\delta_0}} \Phi(u)^2\,dS + \gamma_0^4 \frac{2}{1-a} \int_{Q_\delta - Q_{\delta_0}} \Phi(u)^2 |D\Phi(u)| (\varrho-\delta)^{1-a}\,dx$$

$$\leqslant \frac{\gamma_0^4 \delta_0^{1-a}}{1-a} \int_{\partial Q_{\delta_0}} \Phi(u)^2\,dS + \frac{2\beta\gamma_0^4}{1-a} \int_{Q_\delta - Q_{\delta_0}} \frac{\Phi(u)^2}{(\varrho-\delta)^a}\,dx +$$

$$+ \frac{2\gamma_0^4 \delta_0^{1-a}}{\beta(1-a)} \int_{Q_\delta - Q_{\delta_0}} |D\Phi(u)|^2 (\varrho-\delta)\,dx,$$

where we have used Young's inequality in the final step. Now choosing $2\gamma_0^4\beta/(1-a) = \frac{1}{2}$ the result follows.

Remark 2. Under the assumptions of Lemma 6, for all $0 < 2\delta \leqslant \eta \leqslant \delta_0/2$

$$\int\limits_{Q_\delta - Q_\eta} \frac{\Phi(u)^2}{(\varrho-\delta)^a}\,dx \leqslant \frac{2\gamma_0^4\eta^{1-a}}{1-a}\int\limits_{\partial Q_\eta}\Phi(u)^2 dS + \frac{16\gamma_0^8\eta^{1-a}}{(1-a)^2}\int\limits_{Q_\delta}|D\Phi(u)|^2(\varrho-\delta)\,dx.$$

THEOREM 1. *Let u be a solution of* (1) *belonging to $W^{1,p}_{\mathrm{loc}}(Q)$ for fixed $p \geqslant 2$; then the following conditions are equivalent*:

(I) *$M(\delta)$ is a bounded function on* $(0, \delta_0]$;

(II) $\int\limits_Q |Du(x)|^2|u(x)|^{p-2}r(x)\,dx < \infty$;

(III) *$M_1(\delta)$ is continuous on* $[0, \delta_0]$.

Proof. We show that

$$(\mathrm{I}) \Rightarrow (\mathrm{II}) \Rightarrow (\mathrm{I}) \quad \text{and} \quad (\mathrm{II}) \Rightarrow (\mathrm{III}) \Rightarrow (\mathrm{I}).$$

Let

$$\Psi(x) = \begin{cases} u|u|^{p-2}(\varrho-\delta) & \text{for } x \in Q_\delta, \\ 0 & \text{for } x \in Q - Q_\delta. \end{cases}$$

Using Hölder's inequality and Lemma 1 it is easy to prove that Ψ is an admissible test function in (2). Substituting Ψ in (2), we obtain

$$(5)\quad \int\limits_{Q_\delta}\Big\{\sum_{i,j=1}^n [a_{ij}D_iuD_j(u|u|^{p-2})(\varrho-\delta) + a_{ij}D_iuu|u|^{p-2}D_j\varrho] + b(x,u,Du)u|u|^{p-2}(\varrho-\delta)\Big\}dx = 0.$$

Since $a_{ij}|u|^pD_j\varrho \in W^{1,1}(Q_\delta)$, by the Green formula we have

$$(6)\quad \Big|\int\limits_{Q_\delta}\sum_{i,j=1}^n a_{ij}D_iuu|u|^{p-2}D_j\varrho\,dx\Big|$$
$$= \Big|\frac{1}{p}\int\limits_{Q_\delta}\sum_{i,j=1}^n D_i(a_{ij}|u|^pD_j\varrho)\,dx - \frac{1}{p}\int\limits_{Q_\delta}\sum_{i,j=1}^n D_i(a_{ij}D_j\varrho)|u|^p\,dx\Big|$$
$$= \Big|\frac{1}{p}\int\limits_{\partial Q_\delta}\sum_{i,j=1}^n a_{ij}|u|^pD_i\varrho D_j\varrho\,dS_x + \frac{1}{p}\int\limits_{Q_\delta}\sum_{i,j=1}^n D_i(a_{ij}D_j\varrho)|u|^p\,dx\Big|$$
$$\leqslant \frac{\gamma}{p}M(\delta) + \frac{Cd^a}{p}\int\limits_{Q_\delta}\frac{|u|^p}{(\varrho-\delta)^a}\,dx,$$

where

$$C = \max_Q \Big|\sum_{i,j=1}^{n} D_i(a_{ij}D_j\varrho)\Big|, \quad a = \frac{\theta-1}{p-1} - 1, \quad d = \text{diameter}\, Q.$$

Using Assumption (C) and Young's inequality we have the estimates

$$\text{(7)} \quad \Big|\int_{Q_\delta} b(x,u,Du)u|u|^{p-2}(\varrho-\delta)\,dx\Big|$$
$$\leqslant \int_{Q_\delta} \{f(x)|u|^{p-1}(\varrho-\delta) + L(|u|^p + |Du|\,|u|^{p-1})(\varrho-\delta)\}\,dx$$
$$\leqslant \int_{Q_\delta} \Big\{L\Big[|u|^p(\varrho-\delta) + s|Du|^2|u|^{p-2}(\varrho-\delta) + \frac{1}{s}\,|u|^p(\varrho-\delta)\Big] +$$
$$+ f(x)^p(\varrho-\delta)^\theta + \frac{|u|^p}{(\varrho-\delta)^a}\Big\}\,dx.$$

Thus combining (5), (6) and (7) we obtain

$$\text{(8)} \quad \int_{Q_\delta} |Du|^2|u|^{p-2}(\varrho-\delta)\,dx \leqslant \gamma \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}D_iuD_ju\,|u|^{p-2}(\varrho-\delta)\,dx$$
$$\leqslant \frac{\gamma^2}{p(p-1)}\, M(\delta) + \int_{Q_\delta} \Big\{\frac{K|u|^p}{(\varrho-\delta)^a} + \frac{\gamma}{p-1} f(x)^p(\varrho-\delta)^\theta\Big\}\,dx +$$
$$+ \frac{\gamma Ls}{p-1} \int_{Q_\delta} |Du|^2|u|^{p-2}(\varrho-\delta)\,dx,$$

where

$$K = \frac{\gamma}{p-1}\Big(\frac{Cd^a}{p} + 1 + L + \frac{Ld^{1+a}}{s} + Ld^{1+a}\Big).$$

Choosing s such that $\dfrac{\gamma Ls}{p-1} = \frac{1}{2}$ we obtain

$$\tfrac{1}{2}\int_{Q_\delta} |Du|^2|u|^{p-2}(\varrho-\delta)\,dx \leqslant \frac{\gamma^2}{p(p-1)}\, M(\delta) + \frac{\gamma}{p-1}\int_{Q_\delta} f(x)^p(\varrho-\delta)^\theta dx +$$
$$+ K\int_{Q_\delta} \frac{|u|^p}{(\varrho-\delta)^a}\,dx.$$

Implication (I) $\Rightarrow$ (II) now follows from Lemma 5, with $v = |u|^p$, and the Monotone Convergence Theorem.

To prove (II) $\Rightarrow$ (I) note that Lemma 6 implies

$$\int_{Q_\delta} \frac{|u|^p}{(\varrho-\delta)^a}\,dx \leqslant C, \quad 0 < a < 1,$$

for $\delta \in (0, \delta_0/2]$, where C is independent of δ. By Lemma 4 it suffices to prove continuity of $M(\delta)$ at $\delta = 0$. From the first part of the proof

$$\int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij}|u|^p D_i\varrho D_j\varrho\, dS_x$$
$$= \int_{Q_\delta} \Big\{ \sum_{i,j=1}^{n} [D_i(a_{ij}D_j\varrho)|u|^p - p(p-1)a_{ij}D_iuD_ju|u|^{p-2}(\varrho-\delta)] - $$
$$-pb(x, u, Du)u|u|^{p-2}(\varrho-\delta)\Big\} dx.$$

Thus $\lim\limits_{\delta\to 0} \int_{\partial Q_\delta} \sum\limits_{i,j=1}^{n} a_{ij}|u|^p D_iuD_ju\, dS_x$ exists by the Dominated and Monotone Convergence Theorems. Since

$$\frac{1}{\gamma} \leqslant \sum_{i,j=1}^{n} a_{ij}D_i\varrho D_j\varrho \leqslant \gamma$$

is continuous on $\bar{Q}$ it follows that $M(\delta)$ is continuous at $\delta = 0$.

Implication (II) ⇒ (III) follows by Lemma 6 and the relationship

$$M(\delta) - M_1(\delta) = \int_{\partial Q} |u(x_\delta(x))|^p\left(\frac{dS_\delta}{dS_0} - 1\right) dS_0$$

since $dS_\delta/dS_0 \to 1$ uniformly as $\delta \to 0$ and, by the previous part of the proof, $M(\delta)$ and hence $M_1(\delta)$ is continuous.

Finally (III) ⇒ (I) follows from the proof (II) ⇒ (III).

Remark 3. Under the assumptions of Theorem 1 condition (I) can be replaced by

(I′) $\lim\limits_{\nu\to\infty} M(\delta_\nu)$ exists for some sequence $\delta_\nu > 0$ with $\lim\limits_{\nu\to\infty} \delta_\nu = 0$.

Indeed, setting $\Phi(u) = |u|^{p/2}$ in Remark 2 we have

$$K\int_{Q_\delta} \frac{|u|^p}{(\varrho-\delta)^a}\, dx \leqslant \frac{K}{\eta^a}\int_{Q_\eta} |u|^p dx + \frac{2K\gamma_0^4\eta^{1-a}}{1-a} M(\eta) +$$
$$+ \frac{4Kp^2\gamma_0^8\eta^{1-a}}{(1-a)^2}\int_{Q_\delta} |Du|^2|u|^{p-2}(\varrho-\delta)\, dx,$$

for $0 < 2\delta \leqslant \eta \leqslant \delta_0/2$. Choosing $s = (p-1)/3\gamma L$ and then

$$\eta = \left[\frac{12p^2\gamma_0 K}{(1-a)^2}\right]^{1/(a-1)}$$

we obtain from inequality (8)

$$\int_{Q_\delta} |Du|\,|u|^{p-2}(\varrho-\delta)\,dx \leqslant \frac{3\gamma^2}{p(p-1)} M(\delta) + \frac{6K\gamma_0^4\eta^{1-\alpha}}{1-\alpha} M(\eta) + \\ + \frac{3K}{\eta^\alpha}\int_{Q_\eta} |u|^p\,dx + \frac{3\gamma}{p-1}\int_{Q_\delta} f^p(\varrho-\delta)^\theta\,dx.$$

Setting $\delta = \delta_\nu < \eta/2$, the result follows from the Monotone Convergence Theorem.

Remark 4. Theorem 1 is sharp in the sense that for $\theta = 2p-1$ we have the counter example of Mihailov [7].

Remark 5. A condition such as (I) is needed even for the Laplace equation as is well known.

Remark 6. Condition (C) can be weakened by allowing

$$|b(x, u, s)| \leqslant f(x) + \frac{L}{\varrho^{1+\alpha}}\,|u| + \frac{L}{\varrho^\alpha}\,|s|,$$

where f, L and α are as above. The proof requires only minor modifications.

Under slightly stronger assumptions the conclusions of Theorem 1 hold for $1 < p < 2$. The details of proof are similar and we only sketch the proof.

THEOREM 2. *Let $u \in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1) *for fixed p, $1 < p < 2$; then conditions* (I), (II) *and* (III) *are equivalent.*

Proof. To prove (I) $\Rightarrow$ (II) introduce a test function

$$v(x) = \begin{cases} u(x)\,[u(x)^2+\varepsilon]^{(p-2)/2}\big(\varrho(x)-\delta\big) & \text{for } x\in Q_\delta,\\ 0 & \text{for } x\in Q-Q_\delta,\end{cases}$$

for $\varepsilon > 0$ and $\delta > 0$. Since $u \in W^{1,2}_{\mathrm{loc}}(Q)$ for ε and δ positive

$$\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}D_iuD_ju\,(u^2+\varepsilon)^{(p-2)/2}(\varrho-\delta)\,dx \quad \text{is finite}.$$

Substituting v into (2) we obtain

$$\int_{Q_\delta}\Big\{\sum_{i,j=1}^n \big[a_{ij}D_iuD_ju\,(u^2+\varepsilon)^{(p-4)/2}[(p-1)u^2+\varepsilon](\varrho-\delta) + \\ + a_{ij}D_iu\,(u^2+\varepsilon)^{(p-2)/2}uD_j\varrho\big] + b(x, u, Du)\,u\,(u^2+\varepsilon)^{(p-2)/2}(\varrho-\delta)\Big\}\,dx = 0.$$

Using Assumption (C) and $(p-1)u^2+\varepsilon \geqslant (p-1)(u^2+\varepsilon)$ we obtain

$$(9)\quad \int_{Q_\delta} |Du|^2(u^2+\varepsilon)^{(p-2)/2}(\varrho-\delta)\,dx$$
$$\leqslant \frac{\gamma}{(p-1)} \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i u D_j u (u^2+\varepsilon)^{(p-4)/2}[(p-1)u^2+\varepsilon](\varrho-\delta)\,dx$$
$$= -\frac{\gamma}{(p-1)} \int_{Q_\delta} \Big\{ \sum_{i,j=1}^{n} a_{ij} D_i u (u^2+\varepsilon)^{(p-2)/2} u D_j \varrho +$$
$$+ b(x, u, Du) u (u^2+\varepsilon)^{(p-2)/2}(\varrho-\delta)\Big\}\,dx.$$

On the other hand by Green's formula we have

$$(10)\quad \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i u (u^2+\varepsilon)^{(p-2)/2} u D_j \varrho\,dx = \frac{1}{p} \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i (u^2+\varepsilon)^{p/2} D_j \varrho\,dx$$
$$= -\frac{1}{p} \int_{\partial Q_\delta} (u^2+\varepsilon)^{p/2} \sum_{i,j=1}^{n} a_{ij} D_i \varrho D_j \varrho\,dx - \frac{1}{p} \int_{Q_\delta} (u^2+\varepsilon)^{p/2} D_i [a_{ij} D_j \varrho]\,dx.$$

It follows from (9), (10) and Assumption (C) that

$$\int_{Q_\delta} |Du|^2(u^2+\varepsilon)^{(p-2)/2}(\varrho-\delta)\,dx$$
$$\leqslant \frac{\gamma^2}{p(p-1)} \int_{\partial Q_\delta} (u^2+\varepsilon)^{p/2}\,dS + \frac{C\gamma}{p(p-1)} \int_{Q_\delta} (u^2+\varepsilon)^{p/2}\,dx +$$
$$+ \frac{\gamma}{p(p-1)} \int_{Q_\delta} [f + L(|u| + |Du|)]\,|u|\,(u^2+\varepsilon)^{(p-2)/2}(\varrho-\delta)\,dx,$$

with the same constant C as in the proof of Theorem 1. Using Young's inequality followed by Remark 2 with $\Phi(u) = (u^2+\varepsilon)^{p/4}$ and suitable η, $0 < 2\delta \leqslant \eta \leqslant \delta_0/2$, we obtain

$$\int_{Q_\delta} |Du|^2(u^2+\varepsilon)^{(p-2)/2}(\varrho-\delta)\,dx \leqslant C\Big\{ \int_{\partial Q_\delta} (u^2+\varepsilon)^{p/2}\,dS +$$
$$+ \int_{\partial Q_\eta} (u^2+\varepsilon)^{p/2}\,dS + \int_{Q_\eta} (u^2+\varepsilon)^{p/2}\,dx + \int_{Q_\delta} f^p (\varrho-\delta)^\theta\,dx \Big\},$$

where the constant C does not depend on δ and ε. Letting $\varepsilon \to 0$ the Monotone and Dominated Convergence theorems give

$$\int_{Q_\delta} |Du|^2 |u|^{p-2}(\varrho-\delta)\,dx \leqslant C\Big\{ M(\delta) + M(\eta) + \int_{Q_\eta} |u|^p\,dx + \int_{Q} f^p \varrho^\theta\,dx \Big\}$$

and the result follows.

To prove (II) $\Rightarrow$ (I) note that using (10) and letting $\varepsilon \to 0$ we arrive at

$$(11)\quad \frac{1}{p}\int_{\partial Q_\delta} |u|^p \sum_{i,j=1}^{n} a_{ij} D_i \varrho D_j \varrho \, dS_x = \int_{Q_\delta} \Big\{ (p-1)\sum_{i,j=1}^{n} a_{ij} D_i u D_j u |u|^{p-2}(\varrho-\delta) - \frac{|u|^p}{p}\sum_{i,j=1}^{n} D_i(a_{ij} D_j \varrho) + bu|u|^{p-2}(\varrho-\delta)\Big\} dx.$$

By condition (II), Lemma 6 and Young's inequality the left-hand side of (11) is dominated by an L^1 function. Thus $\lim_{\delta\to 0}\int_{\partial Q_\delta} |u|^p \sum_{i,j} a_{ij} D_i \varrho D_j \varrho \, dS_x$ exists and $M(\delta)$ is continuous at $\delta = 0$. The rest of the proof follows as in the proof of Theorem 1.

In order to establish u has a trace on ∂Q in $L^p(\partial Q)$ that is, $u(x_\delta)$ converges in $L^p(\partial Q)$ we first show that $u(x_\delta)$ converges weakly in $L^p(\partial Q)$. To do this we first need the following result.

THEOREM 3. *Let $u \in W^{1,p}_{\mathrm{loc}}(Q)$ for fixed $p \geqslant 2$ be a solution of* (1). *Assume one of conditions* (I), (II) *or* (III) *holds. There is a sequence $\delta_\nu \to 0$ as $\nu \to \infty$ and a function $\varphi \in L^p(\partial Q)$ such that*

$$\lim_{\nu\to\infty}\int_{\partial Q} u[x_{\delta_\nu}(x)] g(x)\, dS_x = \int_{\partial Q} \varphi(x) g(x)\, dS_x$$

for each $g \in L^{p'}(\partial Q)$.

LEMMA 7. *Let $u \in W^{1,p}_{\mathrm{loc}}(Q)$ be a solution of* (1) *for fixed $p \geqslant 2$; then condition* (II) *implies*

$$\int_Q |Du|^2 r(x)\, dx < \infty.$$

Proof. By Theorem 1, condition (II) implies that $\bar{M}(\delta) = \int_{\partial Q_\delta} (u^2+1)^{p/2} dS$ is bounded. Repeating the proof of (I) $\Rightarrow$ (II) of Theorem 1 with

$$v = \begin{cases} u(u^2+1)^{(p-2)/2}(\varrho-\delta) & \text{for } x \in Q_\delta, \\ 0 & \text{for } x \in Q - Q_\delta, \end{cases}$$

as a test function we obtain $\int_Q |Du|^2 (u^2+1)^{(p-2)/2} r\, dx < \infty$ and the result follows.

THEOREM 4. *Let $u \in W^{1,p}_{\mathrm{loc}}(Q)$ be a solution of* (1) *for fixed $p \geqslant 2$. If one of conditions* (I), (II) *or* (III) *holds, then the function*

$$G(\delta) = \int_{\partial Q} u[x_\delta(x)] \Psi(x)\, dS_x$$

is continuous on $[0, \delta_0]$ for any Ψ in $L^{p'}(\partial Q)$.

Proof. By Lemma 4, $G(\delta)$ is absolutely continuous on $[\delta_1, \delta_0]$ for any $\delta_1 > 0$ so it suffices to prove continuity at $\delta = 0$. Since $\Phi(x) = \sum_{i,j=1}^{n} a_{ij}(x) D_i \varrho(x) D_j \varrho(x)$ is uniformly continuous, $1/\gamma \leqslant \Phi(x) \leqslant \gamma$, $M_1(\delta)$ is bounded, and the elements of $C^1(\bar{Q})$ restricted to ∂Q are dense in $L^{p'}(\partial Q)$, it suffices to show that

$$\bar{G}(\delta) = \int_{\partial Q_\delta} u\Psi \sum_{i,j=1}^{n} a_{ij} D_i \varrho D_j \varrho \, dS_x$$

is continuous for each $\Psi \in C^1(\bar{Q})$. From (2), taking

$$v = \begin{cases} \Psi(\varrho - \delta) & \text{for } x \in Q_\delta, \\ 0 & \text{for } x \in Q - Q_\delta, \end{cases}$$

as a test function, we have

$$\int_{Q_\delta} \Big\{ \sum_{i,j=1}^{n} a_{ij} D_i u D_j \Psi(\varrho - \delta) + \sum_{i,j=1}^{n} a_{ij} D_i u \Psi D_j \varrho + b\Psi(\varrho - \delta) \Big\} dx = 0.$$

Thus

$$\bar{G}(\delta) = \int_{Q_\delta} \Big\{ \sum_{i,j=1}^{n} a_{ij} D_i u D_j \Psi(\varrho - \delta) + b\Psi(\varrho - \delta) - \sum_{i,j=1}^{n} D_i(a_{ij} D_j \varrho \Psi) u \Big\} dx.$$

The integrand on the right is dominated by

$$K\{|Du|^2 \varrho + |u|^p + f^p \varrho^\theta + 1/\varrho^a\}$$

which belongs to $L^1(Q)$ by Lemmas 5 and 7 and condition (I), where K is a positive constant independent of δ, and $a = \dfrac{\theta - 1}{p - 1} - 1$. The result follows.

Remark 7. The result holds for $1 < p < 2$ on noting that for $1 < p < 2$ the integrand on the right is dominated by

$$K\{|Du|^2 |u|^{p-2} \varrho + |u|^p + f^p \varrho^\theta + 1/\varrho^a\}$$

since

$$|Du| \varrho \leqslant \tfrac{1}{2} d |Du|^2 |u|^{p-2} \varrho + \tfrac{1}{2} |u|^{2-p}$$

and

$$|u|^{2-p} \leqslant \frac{2p - p}{p} + \frac{(2 - p)}{p} |u|^p.$$

We now prove that under appropriate assumptions on the data the solution u of (1) assumes boundary data in the sense $u(x_\delta(x))$ converges in $L^p(\partial Q, dS)$. We need the following lemmas.

For $\delta \in (0, \delta_0]$ we define the mapping $x^\delta \colon \bar{Q} \to \bar{Q}_{\delta/2}$ by

$$x^\delta(x) = \begin{cases} x & \text{for } x \in Q_\delta, \\ x_\delta + \frac{1}{2}(x - x_\delta) & \text{for } x \in Q - Q_\delta. \end{cases}$$

Thus $x^\delta(x) = x$ for each $x \in Q_\delta$ and $x^\delta(x) = x_{\delta/2}(x)$ for each $x \in \partial Q$. Moreover, $\varrho(x^\delta) \geqslant \delta/2$. Also x^δ is uniformly Lipschitz continuous and in particular

$$|x^\delta(x) - x^\delta(y)| \leqslant K|x - y|$$

for some positive constant K. Note that if $u \in W^{1,p}_{\mathrm{loc}}(Q)$, then $u(x^\delta) \in W^{1,p}(Q)$.

LEMMA 8. *Let $g \in L^1(Q)$, if $\int_{\partial Q_\delta} |g(x)|\, dS_x$ is bounded on $[0, \delta_0]$, then*

$$\frac{1}{\delta} \int_{Q - Q_\delta} |g(x)|\, dx \leqslant \sup_{[0,\delta_0]} \int_{\partial Q_t} |g(x)|\, dS_x.$$

Proof.

$$\frac{1}{\delta} \int_{Q - Q_\delta} |g(x)|\, dx \leqslant \frac{1}{\delta} \int_0^\delta dt \int_{\partial Q_t} |g(x)|\, dS_x \leqslant \sup_{[0,\delta_0]} \int_{\partial Q_t} |g(x)|\, dS_x.$$

LEMMA 9. *Let f be a non-negative function in $L^1(Q_{\delta/2} - Q_\delta)$; then*

$$\int_{Q - Q_\delta} f(x^\delta(x))\, dx \leqslant 2\gamma_0^4 \int_{Q_{\delta/2} - Q_\delta} f(x)\, dx. \tag{12}$$

If $\int_{\partial Q_t} f(x)\, dS_x$ is bounded on $[0, \delta_0]$, then

$$\int_{Q - Q_\delta} \frac{f(x^\delta(x))}{\varrho(x)^a}\, dx \leqslant \frac{\gamma_0^4 \delta^{1-a}}{1-a} \sup_{[0,\delta_0]} \int_{\partial Q_t} f(x)\, dS_x \tag{13}$$

for $0 \leqslant a < 1$.

Proof. By change of variables we obtain

$$\int_{Q - Q_\delta} \frac{f(x^\delta(x))}{\varrho^a(x)}\, dx = \int_{x^\delta(Q - Q_\delta)} \frac{f(x)}{(2\varrho - \delta)} J_{x^\delta}^{-1}(x)\, dx$$

$$\leqslant \gamma_0^4 \int_{Q_{\delta/2} - Q_\delta} \frac{f(x)}{(2\varrho - \delta)^a}\, dx = \gamma_0^4 \int_{\delta/2}^{\delta} \frac{dt}{(2t - \delta)^a} \int_{\partial Q_t} f(x)\, dS_x.$$

Setting $a = 0$ we obtain (12). If $\int_{\partial Q_t} f(x)\, dS_x$ is bounded on $[0, \delta_0]$, then

$$\int_{Q - Q_\delta} \frac{f(x^\delta(x))}{\varrho^a(x)}\, dx \leqslant \gamma_0^4 \int_{\delta/2}^{\delta} \frac{dt}{(2t - \delta)^a} \sup_{[0,\delta_0]} \int_{\partial Q_t} f(x)\, dS_x$$

as required.

LEMMA 10. *Let $h \in L^1(Q)$; then*

$$\lim_{\delta\to 0} \int_{Q-Q_\delta} h(x^\delta(x))\,dx = 0.$$

Proof. This is a consequence of Lemma 9 and the well-known property

$$\lim_{\delta\to 0} \int_{Q_{\delta/2}-Q_\delta} h\,dx = 0.$$

LEMMA 11. *Let $g \in L^2(Q)$, $\varrho^{1/2} f \in L^2$ and suppose that $\int_{\partial Q_\delta} |g(x)|^2 dS_x$ is bounded on $[0, \delta_0]$; then*

$$\lim_{\delta\to 0} \int_{Q-Q_\delta} f(x^\delta(x))\,g(x)\,dx = 0.$$

Proof. Set $J_\delta = \int_{Q-Q_\delta} f(x^\delta(x))\,g(x)\,dx$, then

$$|J_\delta| \leqslant \int_{Q-Q_\delta} |f(x^\delta(x))|\,\varrho^{1/2}(x^\delta(x))\,\frac{|g(x)|\sqrt{2}}{\sqrt{\delta}}\,dx$$

since $\varrho(x^\delta(x)) \geqslant \delta/2$, thus by Hölder's inequality

$$J_\delta^2 \leqslant \int_{Q-Q_\delta} f(x^\delta(x))^2\,\varrho(x^\delta(x))\,dx \int_{Q-Q_\delta} \frac{2g(x)^2}{\delta}\,dx.$$

Setting $h(x) = f(x)^2 \varrho(x)$ the result follows from Lemmas 8 and 10.

LEMMA 12. *If $\varrho^{a/2} f \in L^2(Q)$, $0 \leqslant a < 1$, $g \in L^2(Q)$ and $\int_{\partial Q_\delta} g(x)^2 dS_x$ is bounded on $[0, \delta_0]$, then $\lim_{\delta\to 0} \int_{Q-Q_\delta} g(x^\delta(x)) f(x)\,dx = 0$.*

Proof. Setting

$$I_\delta = \int_{Q-Q_\delta} g(x^\delta(x)) f(x)\,dx$$

we have by Hölder's inequality and Lemma 9

$$I_\delta^2 \leqslant \int_{Q-Q_\delta} \frac{g(x^\delta(x))^2}{\varrho(x^\delta(x))^a}\,dx \int_{Q-Q_\delta} \varrho(x)^a f(x)^2\,dx$$

$$\leqslant 2\gamma_0^4 \int_{Q_{\delta/2}-Q_\delta} \frac{g(x)^2}{\varrho(x)^a}\,dx \int_{Q-Q_\delta} \varrho(x)^a f(x)^2\,dx$$

and the result follows from Lemma 5.

LEMMA 13. *If $\varrho^{1/2} f$ and $\varrho^{1/2} g$ belong to $L^2(Q)$, then*

$$\lim_{\delta\to 0} \int_{Q-Q_\delta} f(x^\delta(x))\,g(x)\,\varrho(x)\,dx = 0.$$

Proof. Let $K_\delta = \int\limits_{Q-Q_\delta} f(x^\delta(x))g(x)\varrho(x)dx$; then since $\varrho(x^\delta(x)) \geqslant \varrho(x)$

$$K_\delta^2 \leqslant \int\limits_{Q-Q_\delta} f(x^\delta(x))^2 \varrho(x^\delta(x))\,dx \int\limits_{Q-Q_\delta} g(x)^2\varrho(x)\,dx$$

and the result follows by the previous lemmas.

Let $L_1^2 = L^2(\partial Q, dS_x)$ with inner product (norm) denoted by $\langle\cdot,\cdot\rangle_1$ ($\|\cdot\|_1$) and let $L_2^2 = L^2(\partial Q, \Phi(x)dS_x)$ with inner product (norm) denoted by $\langle\cdot,\cdot\rangle_2$ ($\|\cdot\|_2$), where

$$\Phi = \sum_{i,j=1}^{n} a_{ij}D_i\varrho D_j\varrho.$$

First we prove $u(x_\delta)$ converges for the special case $p = 2$ and then, using this result we obtain the result in the case $p \geqslant 2$.

THEOREM 5. *Let $u \in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1) *such that one of conditions* (I), (II), (III) *holds for $p = 2$. Then there is $\zeta \in L^2(\partial Q)$ such that $u(x_\delta)$ converges to ζ in L_1^2 as δ converges to* 0; *that is, $\lim\limits_{\delta\to 0} u(x_\delta) = \zeta$ in L_1^2.*

Proof. As $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent it suffices to show there is $\zeta \in L_2^2$ and $\lim\limits_{\delta\to 0} u(x_\delta) = \zeta$ in L_2^2. By Theorem 4 there is $\zeta \in L_2^2$ such that $\lim\limits_{\delta\to 0} u(x) = \zeta$ weakly in L_2^2. Since L_2^2 is uniformly convex, it suffices to show that $\lim\limits_{\delta\to 0}\|u(x_\delta)\|_2 = \|\zeta\|_2$. For $\Psi \in W^{1,2}(Q)$ set

$$F(\Psi(x)) = \sum_{i,j=1}^{n} a_{ij}D_iuD_j\Psi\varrho + b\Psi\varrho - \sum_{i,j=1}^{n} D_i(a_{ij}D_j\varrho\Psi)u.$$

As in the proof of Theorem 1 we find that

$$\langle\zeta, \Psi\rangle_2 = \int\limits_Q F(\Psi(x))\,dx$$

for all $\Psi \in C^1(\bar{Q})$ and hence for all $\Psi \in W^{1,2}(Q)$. As $u(x^\delta) \in W^{1,2}(Q)$ we have

$$\langle\zeta, u(x^\delta)\rangle_2 = \int\limits_Q F(u[x^\delta(x)])\,dx = \int\limits_{Q-Q_\delta} F(u[x^\delta(x)])\,dx + \int\limits_{Q_\delta} F(u(x))\,dx,$$

since $x^\delta(x) = x$ for every $x \in Q_\delta$. We show that

$$\lim_{\delta\to 0}\int\limits_{Q-Q_\delta} F(u[x^\delta(x)])\,dx = 0$$

and that

$$\lim_{\delta\to 0}\int\limits_{Q_\delta} F(u(x))\,dx = \lim_{\delta\to 0}\|u(x^\delta)\|_2^2,$$

so that

$$\|\zeta\|_2^2 = \lim_{\delta\to 0} \langle \zeta, u(x^\delta)\rangle_2, \qquad \text{as } x^\delta(x) = x_{\delta/2}(x) \text{ on } \partial Q,$$
$$= \lim_{\delta\to 0} \|u(x^\delta)\|_2^2, \qquad \text{as required.}$$

Setting

$$v(x) = \begin{cases} u(x)(\varrho-\delta) & \text{for } x \in Q_\delta, \\ 0 & \text{for } x \in Q - Q_\delta, \end{cases}$$

in equation (2), we have

$$\int_{Q_\delta} \Big\{ \sum_{i,j=1}^n [a_{ij} D_i u D_j u(\varrho-\delta) + a_{ij} D_i u D_j \varrho] + bu(\varrho-\delta)\Big\} dx = 0.$$

Hence

$$\begin{aligned}
&\lim_{\delta\to 0} \int_{Q_\delta} F(u)\,dx \\
&= \lim_{\delta\to 0} \int_{Q_\delta} \Big[\sum_{i,j=1}^n a_{ij} D_i u D_j u(\varrho-\delta) + bu(\varrho-\delta) - \sum_{i,j=1}^n D_i(a_{ij} D_j \varrho u)u\Big] dx \\
&= \lim_{\delta\to 0} \int_{Q_\delta} \sum_{i,j=1}^n D_i(a_{ij} D_j \varrho u^2)\,dx = \lim_{\delta\to 0} \int_{\partial Q_\delta} u^2 \Phi\, dS_x \\
&= \lim_{\delta\to 0} \int_{\partial Q} u(x_\delta)^2 \Phi\, dS_x + \lim_{\delta\to 0} \int_{\partial Q} u(x_\delta)^2 [\Phi(x_\delta) - \Phi]\, dS_x \\
&= \lim_{\delta\to 0} \int_{\partial Q} u(x_\delta)^2 \Phi\, dS_x = \lim_{\delta\to 0} \|u(x_\delta)\|_2^2.
\end{aligned}$$

Now

$$\begin{aligned}
|F(u(x^\delta(x)))| \leqslant C\{&|Du(x)||Du(x^\delta(x))|\,\varrho(x) + |u(x)|\,|u(x^\delta(x))| + \\
&+ |Du(x^\delta(x))|\,|u(x)| + |Du(x)||u(x^\delta(x))| + f(x)|u(x^\delta(x))|\,\varrho\}
\end{aligned}$$

for some positive constant C independent of δ. Noting $\varrho^{\theta/2-1}(\varrho f)$, $\varrho^{1/2}|Du| \in L^2(Q)$, $0 \leqslant \frac{1}{2}\theta - 1 < \frac{1}{2}$ and $M(\delta)$ is bounded on $[0, \delta_0]$, by earlier lemmas

$$\lim_{\delta\to 0} \int_{Q-Q_\delta} F(u(x^\delta(x)))\,dx = 0.$$

For the convergence of $u(x_\delta)$ in $L^p(\partial Q)$ in the case $p > 2$ we need the following results.

LEMMA 14. *Let $u \in W^{1,p}_{\mathrm{loc}}(Q)$ be a solution of* (1) *satisfying one of conditions* (I), (II) *or* (III) *for fixed $p > 2$; then $u(x_\delta)$ converges to ζ in $L^q(\partial Q)$ for each q, $0 < q < p$.*

Proof. First we note that $u(x_\delta)$ converges weakly to ζ in $L^p(\partial Q)$. We show that $u(x_\delta)$ converges to ζ in $L^2(\partial Q)$. Let

$$\alpha(\theta) = \begin{cases} 0 & \text{for } p \leqslant \theta < \frac{3}{2}p, \\ (\theta - 1 - p)/p & \text{for } \frac{3}{2}p \leqslant \theta < 2p-1, \end{cases}$$

and $\beta(\theta) = 2\theta/p - \alpha(\theta)$. For $p \leqslant \theta < 2p-1$, $2 \leqslant \beta < 3$ and

$$\int_Q f^2 \varrho^\beta dx \leqslant \Big(\int_Q f^p \varrho^\theta dx\Big)^{2/p} \Big(\int_Q \varrho^{-p\alpha/(p-2)} dx\Big)^{(p-2)/p} < \infty$$

by (C), as $p\alpha/(p-2) < 1$.

By Theorem 5, $u(x_\delta)$ converges to ξ in $L^2(\partial Q)$ so $\xi = \zeta$ a.e. For measurable sets $A \subset \partial Q$ and s satisfying $1/s + q/p = 1$ we have

$$\begin{aligned}\int_A |u(x_\delta) - \zeta|^q dS_x &\leqslant |A|^s \Big(\int_A |u(x_\delta) - \zeta|^p dS_x\Big)^{q/p} \\ &\leqslant |A|^s \Big\{\Big(\int_{\partial Q} |u(x_\delta)|^p dS_x\Big)^{1/p} + \Big(\int_{\partial Q} |\zeta|^p dS_x\Big)^{1/p}\Big\}^q \\ &\leqslant |A|^s \Big\{M_1(\delta) + \Big(\int_{\partial Q} |\zeta|^p dS_x\Big)^{1/p}\Big\}^q.\end{aligned}$$

Thus $u(x) - \zeta$ is equi-absolutely integrable and bounded in $L^q(\partial Q)$ and hence compact for $0 < \delta \leqslant \delta_0$. Now for any sequence $\delta_\nu \to 0$ there is a subsequence $\delta_\nu^1 \to 0$ with $u(x_{\delta_\nu^1}) - \zeta \to 0$ a.e. and the result follows.

THEOREM ON NEMYTSKY OPERATORS (see [8], p. 155). *If $f(x, u)$, defined on $Q \times R$, satisfies Carathéodory conditions (conditions* (i) *and* (ii) *of Assumption* (C)) *and*

$$|f(x, u)| \leqslant g(x) + K|u|^{s/t},$$

where $g \in L^t(Q)$, $1 \leqslant s$, $t < \infty$ and K is a positive constant, then f generates a continuous operator from $L^s(Q)$ into $L^t(Q)$ given by the formula

$$h\colon\ u(\cdot) \to f(\cdot, u(\cdot)).$$

(This operator is called the *Nemytsky operator.*)

We now establish the following L^p-convergence theorem.

THEOREM 6. *Let $u \in W^{1,p}_{\mathrm{loc}}(Q)$ be a solution of* (1) *satisfying one of conditions* (I), (II) *or* (III) *for fixed $p > 2$; then $u(x^\delta)$ converges to ζ in $L^p(\partial Q)$.*

Proof. We begin with the following remark: if $u(x^\delta)$ is bounded in $L^p(\partial Q)$ and $u(x^\delta) \to \zeta$ in $L^q(\partial Q)$ for $q < p$, then $u(x^\delta)|u(x^\delta)|^{p-2} \to \zeta|\zeta|^{p-2}$ weakly in $L^{p'}(\partial Q)$. This follows by observing

$$f(x, u(x^\delta)) = u(x^\delta)|u(x^\delta)|^{p-2}$$

is continuous from $L^q(\partial Q)$ to $L^{q/(p-1)}(\partial Q)$ by the Theorem on Nemytsky Operators. Hence $u(x^\delta)|u(x^\delta)|^{p-2} \to \zeta|\zeta|^{p-2}$ in $L^{q/(p-1)}(\partial Q)$ as $\delta \to 0$, where we take $q/(p-1) > 1$.

Also $u(x^\delta)|u(x^\delta)|^{p-2}$ is bounded in $L^{p'}(\partial Q)$ and so is weakly compact and the result follows.

The rest of the proof is similar to that of Theorem 5. For every $\Psi \in W^{1,p'}_{\mathrm{loc}}(Q)$ we have

$$\int_{\partial Q} \Psi\zeta\Phi\, dS_x = \int_Q \Big\{ \sum_{i,j=1}^n a_{ij} D_i u D_j \Psi\varrho + b\Psi\varrho - u \sum_{i,j=1}^n D_i(a_{ij}\Psi D_j\varrho)\Big\} dx$$

since $u(x^\delta) \to \zeta$ as $\delta \to 0$ weakly in $L^p(\partial Q)$.

Set $\Psi = u(x^\delta)|u(x^\delta)|^{p-2}$ in the above and noting that $u(x^\delta) = u(x)$ on Q_δ, we obtain

$$\begin{aligned}\int_{\partial Q} |\zeta|^p \Phi\, dS_x &= \lim_{\delta\to 0} \int_{\partial Q} \zeta u(x^\delta)|u(x^\delta)|^{p-2}\Phi\, dS_x\\ &= \lim_{\delta\to 0} \int_{Q_\delta} \Big\{ \sum_{i,j=1}^n a_{ij} D_i u D_j(u|u|^{p-2})\varrho + bu|u|^{p-2}\varrho -\\ &\quad - \sum_{i,j=1}^n uD_i(a_{ij}u|u|^{p-2}D_j\varrho)\Big\} dx +\\ &\quad + \lim_{\delta\to 0} \int_{Q-Q_\delta} \Big\{ \sum_{i,j=1}^n a_{ij} D_i u D_j\big(u(x^\delta)|u(x^\delta)|^{p-2}\big)\varrho +\\ &\quad + bu(x^\delta)|u(x^\delta)|^{p-2}\varrho - \sum_{i,j=1}^n uD_i\big(a_{ij}u(x^\delta)|u(x^\delta)|^{p-2}D_j\varrho\big)\Big\} dx.\end{aligned}$$

Setting

$$v(x) = \begin{cases} u(x)|u(x)|^{p-2}\big(\varrho(x)-\delta\big) & \text{for } x \in Q_\delta,\\ 0 & \text{for } x \in Q - Q_\delta,\end{cases}$$

in (2) we obtain

$$\begin{aligned}\int_{\partial Q_\delta} |u|^p \Phi\, dS_x = \int_{Q_\delta} \Big\{ &\sum_{i,j=1}^n a_{ij} D_i u D_j(u|u|^{p-2})\varrho +\\ &+ bu|u|^{p-2}\varrho - \sum_{i,j=1}^n uD_i(a_{ij}u|u|^{p-2}D_j\varrho)\Big\} dx + E_\delta,\end{aligned}$$

where

$$E_\delta = \int_{Q_\delta} \Big\{ bu|u|^{p-2}\delta + \sum_{i,j=1}^n a_{ij} D_i u D_j(u|u|^{p-2})\delta \Big\} dx.$$

It is clear that $\lim_{\delta\to 0} E_\delta = 0$. As in the case $p = 2$ it suffices to show

$$\begin{aligned}\lim_{\delta\to 0} \int_{Q-Q_\delta} \Big\{ &\sum_{i,j=1}^n a_{ij} D_i u D_j\big(u(x^\delta)|u(x^\delta)|^{p-2}\big)\varrho +\\ &+ bu(x^\delta)|u(x^\delta)|^{p-2}\varrho - \sum_{i,j=1}^n uD_i\big(a_{ij}u(x^\delta)|u(x^\delta)|^{p-2}D_j\varrho\big)\Big\} dx = 0.\end{aligned}$$

It is easily seen that the integrand can be estimated by

$$f|u(x^\delta)|^{p-1}\varrho+K\{|u|\,|u(x^\delta)|^{p-1}+|Du|\,|u(x^\delta)|^{p-1}\varrho+ \\ +|Du|\,|(Du)(x^\delta)|\,|u(x^\delta)|^{p-2}\varrho+|u|\,|(Du)(x^\delta)|\,|u(x^\delta)|^{p-2}\},$$

where K is a suitable constant. Estimation of the integrals of the first three terms is similar to previous calculations (see the proof of Theorem 5). Now

$$abc^{p-2}\leqslant \mathrm{const}(a^p+b^2c^{p-2}+c^p).$$

Set $b=|(Du)(x^\delta)|$ and $c=|u(x^\delta)|$. The integrals of the fourth and fifth terms can be estimated by earlier lemmas setting $a=|Du|$ and $a=|u|$ respectively.

3. In this section we show that analogues of the theorems of Section 2 hold with $D_k u$ replacing u provided we suitably strengthen our assumptions. In particular we assume that (A), (B) and (C) hold, where in (C)

$$(14)\qquad |b(x,u,s)|\leqslant f(x)+L(|u|+|s|),$$

with $f\in L^2(Q)$.

Define the surface integrals

$$N(\delta)=\int_{\partial Q_\delta}|Du|^2dS_x \quad\text{and}\quad N_1(\delta)=\int_{\partial Q}\big|Du\big(x_\delta(x)\big)\big|^2dS_x.$$

We have the following regularity result.

THEOREM 7. *Let $u\in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1); *then* $u\in W^{2,2}_{\mathrm{loc}}(Q)$.

Proof. This follows immediately from [2] Theorem 8.8, where the $f(x)$ of Theorem 8.8 is given by

$$f(x)=b(x,u,Du).$$

We now obtain the analogue of Theorem 1. As details of the proof are similar to those in Theorem 1 we only sketch the proof.

THEOREM 8. *Let $u\in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1). *The following conditions are equivalent:*

(IV) *$N(\delta)$ is bounded on $(0,\delta_0]$;*

(V) $\int_Q\sum_{i,j=1}^{n}|D_{ij}u|^2r(x)\,dx<\infty$;

(VI) *$N_1(\delta)$ is a continuous function on $(0,\delta_0]$.*

Proof. By Theorem 7, $u\in W^{2,2}_{\mathrm{loc}}(Q)$ so $N(\delta)$ and $N_1(\delta)$ are continuous on $(0,\delta_0]$, by Lemma 4. Clearly $N(\delta)$ is bounded on $(0,\delta_0]$ if $N_1(\delta)$ is bounded on $(0,\delta_0]$ and the equivalence of (IV) and (VI) follows from the behaviour as $\delta\to 0$ of the expression

$$N(\delta)-N_1(\delta)=\int_{\partial Q}\big|Du\big(x_\delta(x)\big)\big|^2\left(\frac{dS_\delta}{dS_0}-1\right)dS_0.$$

To prove (IV) $\Rightarrow$ (V) let $v \in W^{2,2}(Q)$ have compact support in Q. Thus

$$\int_Q \sum_{i,j=1}^{n} a_{ij} D_i u D_j (D_k v)\, dx = -\int_Q b(x, u, Du) D_k v\, dx$$

and integrating by parts

$$(15)\qquad \int_Q \sum_{i,j=1}^{n} [(D_k a_{ij}) D_i u D_j v + a_{ij} D_i (D_k u) D_j v]\, dx = \int_Q b(x, u, Du) D_k v\, dx.$$

Now, by continuity, (15) holds for all $v \in W^{1,2}(Q)$ having compact support. Setting

$$v(x) = \begin{cases} D_k u(x)\big(\varrho(x) - \delta\big) & \text{for } x \in Q_\delta, \\ 0 & \text{for } x \notin Q_\delta, \end{cases}$$

we obtain

$$(16)\qquad \int_{Q_\delta} \sum_{i,j=1}^{n} [D_k a_{ij} D_i u D_{jk} u (\varrho - \delta) + D_k a_{ij} D_i u D_k u D_j \varrho +$$
$$+ a_{ij} D_i (D_k u) D_j (D_k u)(\varrho - \delta) + a_{ij} D_i (D_k u) D_k u D_j \varrho]\, dx$$
$$= \int_{Q_\delta} \{b(x, u, Du) D_{kk} u (\varrho - \delta) + b(x, u, Du) D_k u D_k \varrho\}\, dx.$$

Note that

$$(17)\qquad \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i (D_k u) D_k u D_j \varrho\, dx$$
$$= -\tfrac{1}{2} \int_{\partial Q_\delta} |D_k u|^2 \Phi\, dS_x - \tfrac{1}{2} \int_{Q_\delta} \sum_{i,j=1}^{n} D_i (a_{ij} D_j \varrho) |D_k u|^2\, dx.$$

Using this and Young's inequality we arrive at the inequality

$$\int_{Q_\delta} |D(D_k u)|^2 (\varrho - \delta)\, dx \leqslant C \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij} D_i (D_k u) D_j (D_k u)(\varrho - \delta)\, dx$$
$$\leqslant \tfrac{1}{2} C \gamma N(\delta) + \tfrac{1}{2} \int_{Q_\delta} |D(D_k u)|^2 (\varrho - \delta)\, dx +$$
$$+ K \int_{Q_\delta} \{|Du|^2 + f^2 + u^2\}\, dx,$$

for suitable constants C and K. Since $u \in W^{1,2}_{\text{loc}}(Q)$ and $N(\delta)$ is bounded $|Du| \in L^2(Q)$ and $u \in L^2(Q)$ by Lemma 6. The result now follows by the Monotone Convergence Theorem.

In order to prove (V) $\Rightarrow$ (IV) note that by (V) and repeated use of Lemma 6, $|Du| \in L^2(Q)$ and $u \in L^2(Q)$. From (16) and (17) and Young's inequality

$$N(\delta)\leqslant \gamma \int\limits_{\partial Q_\delta} |Du|^2 \Phi dS_x \leqslant C \int\limits_{Q_\delta} \Big\{ \sum_{i,j} |D_{ij}u|^2(\varrho-\delta)+|Du|^2+u^2+f^2\Big\}$$

$$\leqslant C\int\limits_{Q} \Big\{ \sum_{i,j=1}^{n} |D_{ij}u|^2 \varrho+|Du|^2+u^2+f^2\Big\}dx$$

as required.

We now show that the analogue of Theorem 5 holds. As in Section 2 we need the following preliminary results.

Let $H_i(\delta) = \langle D_i u(x_\delta), g\rangle_1$ for $g\in L^2(\partial Q)$.

THEOREM 9. *Let $u\in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1). *Assume one of conditions* (IV), (V), (VI) *holds. There is a sequence $\delta_\nu\to 0$ as $\nu\to\infty$ and functions $\chi_i\in L^2(\partial Q)$ such that*

$$\lim_{\nu\to\infty} H_i(\delta_\nu) = \langle \chi_i, g\rangle_1,$$

for each $g\in L^2(\partial Q)$ and $i=1,\ldots,n$.

THEOREM 10. *Under the assumptions of Theorem* 9, *$H_i(\delta)$, $i=1,\ldots,n$, are continuous on $[0,\delta_0]$ for each $g\in L^2(\partial Q)$.*

Proof. By Lemma 4, $H_i(\delta)$ is absolutely continuous on $[\delta_1,\delta_0]$ for any $\delta_1>0$ so it suffices to prove continuity at $\delta=0$. As in the proof of Theorem 4 it suffices to show

$$\bar{H}_k(\delta) = \int\limits_{\partial Q_\delta} D_k u g \Phi dS_x$$

is continuous for each $g\in C^1(\bar{Q})$.

Setting

$$v(x) = \begin{cases} g(x)(\varrho-\delta) & \text{for } x\in Q_\delta,\\ 0 & \text{for } x\in Q-Q_\delta,\end{cases}$$

in (15) we obtain, applying Green's formula,

$$\bar{H}_k(\delta) = \int\limits_{Q_\delta}\Big\{ \sum_{i,j=1}^{n} \big[D_k u D_i(a_{ij} g D_j\varrho) - a_{ij} D_i(D_k u) D_j g(\varrho-\delta) -$$
$$- D_k a_{ij} D_i u D_j\big(g(\varrho-\delta)\big)\big] + b(x,u,Du) D_k\big(g(\varrho-\delta)\big)\Big\} dx.$$

The continuity of $\bar{H}_k(\delta)$ now follows from the Dominated Convergence Theorem applied to the right-hand side.

THEOREM 11. *Under the assumptions of Theorem* 9 *there is $\chi_k\in L^2_1(\partial Q)$ and $D_k u(x_\delta)$ converges to χ_k in L^2_2.*

Proof. It suffices to show $D_k u(x_\delta)$ converges to χ_k in L^2_1. By Theorem 9, $D_k u(x_\delta)$ converges weakly to χ_k in L^2_2 and it suffices to show that $\lim\limits_{\delta\to 0} \|D_k u(x_\delta)\|_2 = \|\chi_k\|_2$.

From (15), taking $v = g\varrho$ with $g \in C^1(\bar{Q})$, we obtain

$$\langle \chi_k, g\rangle_2 = \int_Q \Big\{ \sum_{i,j=1}^n [-a_{ij}D_{ik}uD_j g\varrho + D_k u D_i(a_{ij} g D_j \varrho) - \\ - D_k a_{ij} D_i u D_j (g\varrho)] + b(x, u, Du) D_k(g\varrho)\Big\} dx = \int_Q R\big(g(x)\big)\, dx .$$

The last equality remains true for all $g \in W^{1,2}(Q)$. As $(D_k u)\big(x^\delta(x)\big) \in W^{1,2}(Q)$ we have

$$\langle \chi_k, (D_k u)(x^\delta)\rangle_2 = \int_{Q_\delta} R\big((D_k u)(x^\delta)\big)\, dx + \int_{Q-Q_\delta} R\big((D_k u)(x^\delta)\big)\, dx .$$

Note that

$$(18)\qquad \int_{Q_\delta} R\big((D_k u)(x^\delta)\big)\, dx = \int_{Q_\delta} R\big((D_k u)(x)\big)\, dx \\ = -\int_{Q_\delta} \Big\{ \sum_{i,j=1}^n [-a_{ij} D_{ik} u D_{jk} u \varrho + D_k u D_i (a_{ij} D_k u D_j \varrho) - \\ - D_k a_{ij} D_i u D_j (D_k u \varrho)] + b(x, u, Du) D_k (D_k u \varrho)\Big\} dx .$$

On the other hand we have by (16)

$$(19)\qquad \int_{Q_\delta} D_k u \sum_{i,j=1}^n D_i (a_{ij} D_k u D_j \varrho)\, dx \\ = \int_{Q_\delta} \sum_{i,j=1}^n [D_i (a_{ij} D_j \varrho)\, |D_k u|^2 + a_{ij} D_j \varrho D_i (D_k u) D_k u]\, dx \\ = \int_{Q_\delta} \Big\{ \sum_{i,j=1}^n \big[D_i (a_{ij} D_j \varrho)\, |D_k u|^2 - a_{ij} D_{ik} u D_{jk} u (\varrho - \delta) - \\ - D_k a_{ij} D_j u D_j\big(D_k u(\varrho - \delta)\big)\big] + b(x, u, Du) D_k\big(D_k u(\varrho-\delta)\big)\Big\} dx .$$

It follows from (16) and (17)

$$(20)\qquad \int_{\partial Q_\delta} |D_k u|^2 \Phi\, dS_x = \int_{Q_\delta} \Big\{ \sum_{i,j=1}^n \big[D_i (a_{ij} D_j \varrho)\, |D_k u|^2 - \\ - 2 D_k a_{ij} D_i u D_j\big(D_k u(\varrho-\delta)\big) - 2 a_{ij} D_{ik} u D_{jk} u (\varrho - \delta)\big] + \\ + 2b(x, u, Du) D_k\big(D_k u(\varrho - \delta)\big)\Big\} dx .$$

Comparing (18), (19) and (20) we obtain

$$\lim_{\delta\to 0} \int_{Q_\delta} R\big(D_k u(x)\big)\, dx = \lim_{\delta\to 0} \|(D_k u)(x^\delta)\|_2^2 .$$

Since

$$\langle \chi_k, (D_k u)(x^\delta)\rangle_2 = \int_{\partial Q} \chi_k (D_k u)(x_{\delta/2}) \Phi dS_k$$

hence

$$\lim_{\delta\to 0} \langle \chi_k, (D_k u)(x^\delta)\rangle_2 = \|\chi_k\|_2^2.$$

To complete the proof it suffices to prove that

$$\lim_{\delta\to 0} \int_{Q-Q_\delta} R\big((D_k u)(x^\delta)\big)\,dx = 0.$$

The proof now is similar to that in Theorem 5.

4. The results of the previous sections remain valid for the diagonal system of the elliptic type

$$\sum_{i,j=1}^{n} D_j\big(a_{ij}^k(x) D_i u^k\big) - b^k(x, u, Du^k) = 0, \tag{21}$$

$k = 1, \ldots, N$. We only show how to extend some results of Section 2 to the solution of (21).

We assume

(A′) There is a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}^k(x)\xi_i\xi_j \leqslant \gamma|\xi|^2, \qquad k = 1, \ldots, N,$$

for all $x \in Q$ and $\xi \in \boldsymbol{R}_n$.

(B′) The coefficients a_{ij}^k are of the class $C^1(\bar{Q})$.

(C′) The functions $b^k(x, u, s)$, $k = 1, \ldots, N$, defined for $(x, u, s) \in Q\times \boldsymbol{R}_{n+1}$, $s = (s_1, \ldots, s_n)$ satisfy inequalities

$$|b^k(x, u, s)| \leqslant f^k(x) + L(|u| + |s|),$$

where L is a positive constant and f^k are measurable non-negative functions such that

$$\int_Q f^k(x)^p r(x)^\theta dx < \infty.$$

Here we assume that b^k satisfy the Carathéodory conditions (see conditions (i) and (ii) of Assumption (C) in Section 1). $1 < p < \infty$, $p \leqslant \theta < 2p-1$. Additionally we suppose that for every index k the inequalities $u^j \leqslant \boldsymbol{u}^j$ $(j \neq k)$ imply

$$b^k(x, u, s) \leqslant b^k(x, \boldsymbol{u}^1, \ldots, \boldsymbol{u}^{k-1}, u^k, \boldsymbol{u}^{k+1}, \ldots, \boldsymbol{u}^N, s) \tag{22}$$

for all $x \in Q$ and $s \in \boldsymbol{R}_n$.

As in Section 1 we introduce the concept of a weak solution of system (21). The system of the functions $\{u^k\}$, $k=1,\ldots,N$, is called a *weak solution* of (21) if the u^k belong to $W^{1,p}_{\mathrm{loc}}(Q)$ and for any system of functions $\{v^k\}$, $k=1,\ldots,N$, with the v^k in $W^{1,p'}(Q)$ having compact supports in Q, $1/p+1/p'=1$, we have

$$(23)\qquad \int_Q \Big\{\sum_{i,j=1}^n a^k_{ij}(x)D_iu^kD_jv^k+b^k(x,u,Du^k)v^k\Big\}dx=0,\qquad k=1,\ldots,N.$$

Define the surface integrals

$$M_k(u)=\int_{\partial Q_\delta}|u^k_+(x)|^p\,dS_x,\qquad M^1_k(u)=\int_{\partial Q}\big|u^k_+\big(x_\delta(x)\big)\big|^p\,dS_x,$$

$k=1,\ldots,N$, where $u^k_+(x)=\max\big(0,u^k(x)\big)$.

THEOREM 11. *Let* $\{u^k\}$, $k=1,\ldots,N$, *be a weak solution* (21) *belonging to* $W^{1,p}_{\mathrm{loc}}(Q)$ *for fixed* $2\leqslant p<\infty$. *Then the following conditions are equivalent*:

(I′) $M_k(\delta)$ $(k=1,\ldots,N)$ *are bounded on* $(0,\delta_0]$;

(II′) $\int_Q |Du^k_+(x)|^2\big(u^k_+(x)\big)^{p-2}r(x)\,dx<\infty$;

(III′) $M^1_k(\delta)$, $k=1,\ldots,N$, *are continuous functions on* $[0,\delta_0]$.

Proof. Set

$$v^k(x)=\begin{cases}u^k_+(x)\big(u^k_+(x)\big)^{p-2}\big(\varrho(x)-\delta\big) & \text{for } x\in Q_\delta,\\ 0 & \text{for } x\in Q-Q_\delta.\end{cases}$$

By standard properties of distributional derivatives we verify that v^k are admissible test functions in (23). As in the proof of Theorem 1 we obtain

$$(24)\qquad \int_{Q_\delta}|Du^k_+|^2(u^k_+)^{p-2}(\varrho-\delta)\,dx$$
$$\leqslant C\Big(M_k(\delta)+\int_{Q_\delta}\{(u^k_+)^p(\varrho-\delta)+(u^k_+)^p+b(x,u,Du^k)(u^k_+)^{p-1}(\varrho-\delta)\}\,dx\Big),$$

$k=1,\ldots,N$, where C is independent of δ. We estimate the last integral as follows.

Setting $\boldsymbol{u}^i=\max(-u^i,0)$, $i=1,\ldots,N$, it follows from Assumption (C′) that

$$(25)\qquad \int_{Q_\delta}b^k(x,u,Du^k)(u^k_+)^{p-1}(\varrho-\delta)\,dx$$
$$\leqslant\int_{Q_\delta}b^k(x,u^1+\boldsymbol{u}^1,\ldots,u^{k-1}+\boldsymbol{u}^{k-1},u^k,u^{k+1}+\boldsymbol{u}^{k+1},\ldots,u^N+$$
$$+\boldsymbol{u}^N,Du^k)(u^k_+)^{p-1}(\varrho-\delta)\,dx\leqslant\int_{Q_\delta}\Big\{f^k(u^k_+)^{p-1}(\varrho-\delta)+$$
$$+L\sum_{i=1}^N u^i_+(u^k_+)^{p-1}(\varrho-\delta)+L|Du^k_+|(u^k_+)^{p-1}(\varrho-\delta)\Big\}\,dx.$$

Note that

$$\sum_{j\neq k} u_+^j (u_+^k)^{p-1} \leqslant \sum_{j\neq k} \left[\frac{1}{p}(u_+^j)^p + \frac{1}{p'}(u_+^k)^p\right] \tag{26}$$

and

$$|Du_+^k|(u_+^k)^{p-1} \leqslant \varepsilon |Du_+^k|^2 (u_+^k)^{p-2} + \frac{1}{\varepsilon}(u_+^k)^p. \tag{27}$$

Summing inequalities (24) over k from 1 to N and using (25), (26) and (27) we obtain

$$\begin{aligned}\int_{Q_\delta} \sum_{k=1}^{n} |Du_+^k|^2 (u_+^k)^{p-2}(\varrho-\delta)\,dx \\ \leqslant C\Big(\sum_{k=1}^{N} M_k(\delta) + \int_{Q_\delta}\Big\{\sum_{k=1}^{N}(u_+^k)^p(\varrho-\delta) + \sum_{k=1}^{N}(u_+^k)^p + \\ + \sum_{k=1}^{N}(f^k)^p(\varrho-\delta)^\theta + \sum_{k=1}^{N}(u_+^k)^p(\varrho-\delta)^{-(\theta-1)/(p-1)+1}\Big\}dx\Big),\end{aligned}$$

where C is independent of δ. The proof of the equivalence of conditions (I′), (II′) and (III′) is similar to that of Theorem 1, therefore we omit the details.

Finally we state without proof two theorems on traces of solutions.

THEOREM 12. *Let $\{u^k\}$, $k = 1, \ldots, N$, be a weak solution of* (21) *for fixed $2 \leqslant p < \infty$. Assume that one of conditions* (I′), (II′) *or* (III′) *holds. Then there is a sequence $\delta_\nu \to 0$ as $\nu \to \infty$ and functions $\varphi^i \in L^p(\partial Q)$ such that*

$$\lim_{\nu\to\infty} \int_{\partial Q} u_+^i\big(x_{\delta_\nu}(x)\big) g(x)\,dS_x = \int_{\partial Q} \varphi^i(x) g(x)\,dS_x,$$

$i = 1, \ldots, N$, for each function $g \in L^{p'}(\partial Q)$.

Remark 8. If (A′), (B′) and (C′) hold with $p = 2$, then $u^k(x_\delta) \to \varphi^k$, $k = 1, \ldots, N$, in $L^p(\partial Q)$. The proof is similar to that of Theorem 5.

THEOREM 13. *Let $\{u^k\}$, $k = 1, \ldots, N$, be a solution of* (21) *satisfying one of conditions* (I′), (II′) *or* (III′) *for fixed p, $2 < p < \infty$; then $u^k(x_\delta)$ converges to φ_k in $L^p(\partial Q)$.*

Remark 9. If in condition (C′) we omit (22), then the analogues of results of the Sections 2 and 3 hold under the modified Assumptions (A′), (B′) and (C′); for example, Theorems 11, 12 and 13 hold, where we replace u_+^k by u^k in the appropriate conditions.

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