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On the existence and uniqueness of analytic solutions of the functional equation $\varphi(z) = h(z, \varphi[f(z)])$

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We consider the problem of the existence and uniqueness of solutions analytic at a fixed point ζ of f for the equation

$$\varphi(z) = h(z, \varphi[f(z)]) , \tag{1}$$

where $\varphi(z)$ is the unknown function and $f(z)$ and $h(z, w)$ are known complex-valued functions of complex variables. This problem was investigated by A. H. Read [7] under more restrictive hypotheses concerning the functions h and f. Equation (1) is a generalization of the Schröder equation

$$\varphi[f(z)] = s\varphi(z)$$

and the Abel equation

$$\varphi[f(z)] = \varphi(z)+1 .$$

An analogous existence and uniqueness theorem for the Schröder equation was already given by G. Koenigs [4]. Some particular cases of equation (1) have been treated by P. J. Myrberg [6] and V. Ganapathy Iyer [3]. The method applied in this paper is similar to that used in the proof of an analogous theorem by B. Choczewski in paper [1] and is due to M. Kuczma [5].

We assume that the functions $f(z)$ and $h(z, w)$ have the following properties:

(I) The function $f(z)$ is analytic at the point ζ and

$$f(z) = \zeta + \sum_{n=1}^{\infty} b_n (z-\zeta)^n$$

for $|z-\zeta| \leqslant r_0$ where $0 \leqslant |b_1| < 1$.

(II) $h(z, w)$ is an analytic function of two complex variables z and w at the point (ζ, β) and for $|z-\zeta| \leqslant r_0$, $|w-\beta| \leqslant R_0$:

$$h(z, w) = \sum_{n,m=0}^{\infty} a_{nm}(z-\zeta)^n (w-\beta)^m , \qquad a_{00} = \beta .$$

The necessary condition of the existence of an analytic solution φ of equation (1) fulfilling $\varphi(\zeta)=\beta$ is the existence of a formal solution of the form

$$\varphi(z)=\beta+\sum_{n=1}^{\infty}c_n(z-\zeta)^n. \tag{2}$$

Suppose that (2) is a formal solution of equation (1). Then

$$\beta+\sum_{n=1}^{\infty}c_n(z-\zeta)^n=\sum_{n,m=0}^{\infty}a_{nm}(z-\zeta)^n\Big\{\sum_{l=1}^{\infty}c_l\Big[\sum_{k=1}^{\infty}b_k(z-\zeta)^k\Big]^l\Big\}^m,$$

whence by simple calculations we get

$$c_1=\frac{a_{10}}{1-b_1a_{01}},$$

$$c_2=\frac{F_2(c_1)}{1-b_1^2a_{01}},$$

$$\cdots\cdots\cdots\cdots$$

$$c_n=\frac{F_n(c_1,\ldots,c_{n-1})}{1-b_1^na_{01}},$$

where $F(c_1,\ldots,c_{n-1})$ $(n=2,3,\ldots)$ are polynomials of the variables $(c_1,\ldots,c_{n-1})$ with coefficients which are functions of a_{ij} and b_k $(i,j,k=1,2,\ldots,n)$.

The following three cases are possible:

(A) For every n we have $1-b_1^na_{01}\neq 0$.

(B) There exists an n such that $1-b_1^na_{01}=0$ and $F_n(c_1,\ldots,c_{n-1})=0$.

(C) There exists an n such that $1-b_1^na_{01}=0$ and $F_n(c_1,\ldots,c_{n-1})\neq 0$.

In case (A) there exists exactly one formal solution. In case (B) there exists a one-parameter family of formal solutions of equation (1). No formal solution exists in case (C). In the sequel we shall assume that (A) or (B) holds, i.e. that

(III) There exists a formal solution of equation (1) in form (2).

Let p be a positive integer such that

$$|h'_w(\zeta,\beta)|\,|f'(\zeta)|^p<1.$$

Such a number exists because $|f'(\zeta)|=|b_1|<1$. Hence there exists a number $\theta<1$ such that

$$|h'_w(\zeta,\beta)|\,|f'(\zeta)|^p<\theta.$$

It follows from the continuity of the functions $h'_w(z, w)$ and $f'(z)$ that there exist numbers r_1 and R such that the inequality

$$(3) \qquad |h'_w(z, w)|\,|f'(z)|^p < \theta$$

is fulfilled for $|z-\zeta| \leqslant r_1$ and $|w-\beta| \leqslant R$.

Let us define the functions $h_k(z, w, w_1, \ldots, w_k)$ by the recurrent relations:

$$(4) \qquad \begin{aligned} h_1(z, w, w_1) &= \frac{\partial h(z, w)}{\partial z} + f'(z)\frac{\partial h(z, w)}{\partial w} w_1 , \\ h_{k+1}(z, w, w_1, \ldots, w_{k+1}) &= \frac{\partial h_k}{\partial z} + f'(z)\left[\frac{\partial h_k}{\partial w} w_1 + \frac{\partial h_k}{\partial w_1} w_2 + \ldots + \frac{\partial h_k}{\partial w_k} w_{k+1}\right], \\ & \qquad k = 1, \ldots, p-1 . \end{aligned}$$

LEMMA 1. *Let hypotheses* (I), (II) *and* (III) *be fulfilled. Then*

$$(5) \qquad k!c_k = h_k(\zeta, \beta, c_1, 2!c_2, \ldots, k!c_k)$$

for $k = 1, \ldots, p$, *where* $c_1, \ldots, c_p$ *are the coefficients in expansion* (2).

Proof. Since there exists a formal solution (2) of equation (1), the coefficients of equal powers in the expansions

$$\beta + \sum_{n=1}^{\infty} c_n(z-\zeta)^n \quad \text{and} \quad \sum_{n,m=0}^{\infty} a_{nm}(z-\zeta)^n \Big\{\sum_{l=1}^{\infty} c_l\Big[\sum_{k=1}^{\infty} b_k(z-\zeta)^k\Big]^l\Big\}^m$$

are equal. Comparing these series and the series obtained by differentiating formally k times, we obtain for $z = \zeta$

$$k!c_k = h_k(\zeta, \beta, c_1, 2!c_2, \ldots, k!c_k) , \qquad k = 1, \ldots, p .$$

LEMMA 2. *Let hypotheses* (I) *and* (II) *be fulfilled. Then the expressions* $h_k(z, w, w_1, \ldots, w_k)$, $k = 1, \ldots, p$, *are analytic functions of the variables* $(z, w, w_1, \ldots, w_k)$ *defined in*

$$\{z\colon |z-\zeta| \leqslant r_0\} \times \{w\colon |w-\beta| \leqslant R_0\} \times \underbrace{C \times \ldots \times C}_{k\,\text{times}}.$$

Moreover,

$$(6) \qquad \begin{aligned} & h_k(z, w, w_1, \ldots, w_k) \\ & = G_k(z, w, w_1, \ldots, w_{k-1}) + \frac{\partial h(z, w)}{\partial w}[f'(z)]^k w_k , \qquad k = 1, 2, \ldots, p , \end{aligned}$$

where $G_k(z, w, w_1, \ldots, w_{k-1})$ *is an analytic function defined in*

$$\{z\colon |z-\zeta| \leqslant r_0\} \times \{w\colon |w-\beta| \leqslant R_0\} \times \underbrace{C \times \ldots \times C}_{(k-1)\,\text{times}}.$$

Proof. From the definition of the functions h_k and from (I) and (II) it follows that the functions h_k are analytic for $|z-\zeta| \leqslant r_0$, $|w-\beta| \leqslant R_0$

and $w_i \in C$, $i = 1, \ldots, k$. For $k = 1$ formula (6) follows from (4). We assume that (6) holds for $k = l$. On account of (4) and of the inductive hypothesis we obtain

$$h_{l+1}(z, w, w_1, \ldots, w_{l+1}) = \frac{\partial h_l}{\partial z} + f'(z)\left(\frac{\partial h_l}{\partial w} w_1 + \ldots + \frac{\partial h_l}{\partial w_l} w_{l+1}\right)$$

$$= \left[\frac{\partial G_l}{\partial z} + \frac{\partial^2 h}{\partial z \partial w}\big(f'(z)\big)^l w_l + l w_l \frac{\partial h}{\partial w}\big(f'(z)\big)^{l-1} f''(z) + \right.$$

$$\left. + \frac{\partial^2 h}{\partial w^2}\big(f'(z)\big)^{l+1} w_1 w_l + f'(z)\left(\frac{\partial G_l}{\partial w} w_1 + \frac{\partial G_l}{\partial w_1} w_2 + \ldots + \frac{\partial G_l}{\partial w_{l-1}} w_l\right)\right] +$$

$$+ \frac{\partial h}{\partial w}\big(f'(z)\big)^{l+1} w_{l+1}.$$

Denoting by $G_{l+1}(z, w, w_1, \ldots, w_l)$ the expression in the square bracket in the above relation we get (6) for $k = l+1$.

LEMMA 3. *Suppose that hypotheses* (I) *and* (II) *are fulfilled. Then there exist constants* $L_0, \ldots, L_{p-1}, L_p = \theta$ *independent of* z *and such that for* $(z, w', w_1', \ldots, w_p')$, $(z, w'', w_1'', \ldots, w_p'') \in Z$ *we have*

$$(7) \qquad |h_p(z, w', w_1', \ldots, w_p') - h_p(z, w'', w_1'', \ldots, w_p'')| \leqslant L_0|w' - w''| + \sum_{k=1}^{p} L_k |w_k' - w_k''|,$$

where

$$(8) \qquad Z = \{z\colon |z - \zeta| \leqslant r_0\} \times \{w\colon |w - \beta| \leqslant R\} \times \bar{K}(z_1, \varrho_1) \times \ldots \times \bar{K}(z_p, \varrho_p).$$

Here z_i *are arbitrary complex numbers and* ϱ_i *are arbitrary positive numbers,* $i = 1, \ldots, p$.

Proof. It follows from Lemma 2 that the function $h_p(z, w, w_1, \ldots, w_p)$ fulfils a Lipschitz condition. The equality $L_p = \theta$ follows from (6) and (3).

THEOREM. *If hypotheses* (I), (II) *and* (III) *are fulfilled, then there exists a solution of equation* (1) *that is analytic at the point* ζ *and fulfils the condition* $\varphi(\zeta) = \beta$. *In case* (A) *the solution is unique and in case* (B) *the solutions form a one-parameter family.*

Proof. Let us fix a positive number K. It follows from Lemma 2 that the function $h_p(z, \beta, c_1, \ldots, p!c_p)$ is continuous at the point ζ. There exists a number $r_2 > 0$ such that the inequality

$$(9) \qquad |h_p(z, \beta, c_1, \ldots, p!c_p) - h_p(\zeta, \beta, c_1, \ldots, p!c_p)| < \frac{(1-\theta)K}{2}$$

holds for $|z - \zeta| \leqslant r_2$.

Let us put

$$(10)\qquad M_k = \sum_{i=k+1}^{p} i!\,|c_i| + K\,, \quad k = 1, \dots, p-1\,,\ M_p = K\,,$$

$$(11)\qquad z_i = i!\,c_i\,, \quad \varrho_i = M_i\,, \quad i = 1, \dots, p\,.$$

On account of Lemma 3 the function $h_p(z, w, w_1, \dots, w_p)$ fulfils a Lipschitz condition with respect to the variables $(w, w_1, \dots, w_p)$ in the set Z defined by (8) and (11).

Since $|f'(\zeta)| < 1$, there exists an r_3 such that for $|z-\zeta| < r_3$ we have $|f'(z)| < 1$. Hence

$$|f(z)-f(\zeta)| \leqslant \sup_{|\xi-\zeta|<r_3} |f'(\xi)|\,|z-\zeta| \leqslant |z-\zeta| < r_3\,.$$

Thus for $z \in K(\zeta, r_3)$ we have $f(z) \in K(\zeta, r_3)$.

We choose a positive number r such that

$$(12)\qquad r \leqslant \min(1, r_0, r_1, r_2, r_3)\,,$$

$$(13)\qquad \sum_{k=1}^{p} |c_k|\,r^k + K\frac{r^p}{p!} < R\,,$$

$$(14)\qquad \sum_{k=0}^{p-1}\sum_{i=1}^{p-k} \frac{(k+i)!}{i!} L_k\,|c_{k+i}|\,r^i + K\sum_{k=1}^{p} L_{p-k}\frac{r^k}{k!} < \frac{(1-\theta)K}{2}$$

and

$$(15)\qquad \sum_{k=0}^{p-1} L_k \frac{r^{p-k}}{(p-k)!} + \theta < \vartheta\,,$$

where $\theta < \vartheta < 1$.

We denote by A_r the set of all functions which fulfil the following conditions:

(i) $\varphi(z)$ is analytic for $|z-\zeta| < r$ and $\varphi^{(p)}(z)$ is continuous for $|z-\zeta| \leqslant r$.

$$\text{(ii)}\qquad \varphi(z) = \beta + \sum_{n=1}^{p} c_n (z-\zeta)^n + \sum_{n=p+1}^{\infty} d_n (z-\zeta)^n$$

where c_n $(n = 1, \dots, p)$ are the coefficients occurring in expansion (2).

$$\text{(iii)}\qquad |\varphi^{(p)}(z) - p!\,c_p| \leqslant K \quad \text{for} \quad |z-\zeta| \leqslant r\,.$$

We define a metric ϱ in the set A_r putting

$$(16)\qquad \varrho(\varphi_1, \varphi_2) = \sup_{|z-\zeta|\leqslant r} |\varphi_1^{(p)}(z) - \varphi_2^{(p)}(z)| \quad \text{for} \quad \varphi_1, \varphi_2 \in A_r\,.$$

One can easily verify that the metric postulates are fulfilled. Applying the Taylor formula (cf. [2]) for the function $\varphi_1^{(k)}(z)-\varphi_2^{(k)}(z)$ we get

$$\varphi_1^{(k)}(z)-\varphi_2^{(k)}(z)$$
$$=\frac{1}{(p-k-1)!}\int_0^1(1-t)^{p-k-1}\left[\varphi_1^{(p)}\big(\zeta+t(z-\zeta)\big)-\varphi_2^{(p)}\big(\zeta+t(z-\zeta)\big)\right]dt\,(z-\zeta)^{p-k},$$

whence

$$(17)\qquad |\varphi_1^{(k)}(z)-\varphi_2^{(k)}(z)|\leqslant\frac{r^{p-k}}{(p-k)!}\sup_{|z-\zeta|\leqslant r}|\varphi_1^{(p)}(z)-\varphi_2^{(p)}(z)|\,,\quad k=0,1,\ldots,p-1.$$

On account of the definition of the metric and from (17) we have

$$(18)\qquad \sup_{|z-\zeta|\leqslant r}|\varphi_1^{(k)}(z)-\varphi_2^{(k)}(z)|\leqslant\frac{r^{p-k}}{(p-k)!}\varrho(\varphi_1,\varphi_2)\,.$$

It follows from relation (18) that the convergence in the sense of metric (16) is equivalent to the uniform convergence of functions and their derivatives up to the order p in the disc $|z-\zeta|\leqslant r$. The space A_r with metric (16) is complete.

We consider the transformation $\psi=T[\varphi]$ defined by the formula

$$\psi(z)=h\big(z,\varphi[f(z)]\big)\,.$$

We shall prove that the transformation T maps A_r into itself and that

$$\varrho\big(T[\varphi_1],T[\varphi_2]\big)\leqslant\vartheta\varrho(\varphi_1,\varphi_2)\,.$$

Hence, on account of Banach's fixed-point theorem it follows that there exists exactly one function $\varphi(z)$ from the set A_r fulfilling equation (1) for $|z-\zeta|\leqslant r$.

We shall prove that $\varphi\in A_r$ implies $T[\varphi]\in A_r$. For this purpose we prove by induction the formula

$$(19)\qquad \psi^{(k)}(z)=h_k\big(z,\varphi[f(z)],\ldots,\varphi^{(k)}[f(z)]\big)\,,\quad k=1,2,\ldots$$

Let $k=1$. Differentiating the relation $\psi(z)=h\big(z,\varphi[f(z)]\big)$ we get

$$\psi'(z)=\frac{\partial h\big(z,\varphi[f(z)]\big)}{\partial z}+\frac{\partial h\big(z,\varphi[f(z)]\big)}{\partial w}\varphi'[f(z)]f'(z)$$

and in view of (4)

$$\psi'(z)=h_1\big(z,\varphi[f(z)],\varphi'[f(z)]\big)\,.$$

We assume that (19) holds for $k=l$. Accordingly

$$\psi^{l+1}(z)=\frac{\partial h_l}{\partial z}+\left(\frac{\partial h_l}{\partial w}\varphi'[f(z)]+\frac{\partial h_l}{\partial w_1}\varphi''[f(z)]+\ldots+\frac{\partial h_l}{\partial w_l}\varphi^{(l+1)}[f(z)]\right)f'(z)\,.$$

By (4) we have

$$\psi^{(l+1)}(z) = h_{l+1}\big(z, \varphi[f(z)], \ldots, \varphi^{(l+1)}[f(z)]\big) .$$

Condition (iii) implies the relation

$$(20) \qquad |\varphi^{(p)}(z)| \leqslant p!\,|c_p| + K \quad \text{for} \quad |z-\zeta| \leqslant r .$$

On account of Taylor's theorem and after simple transformations we get for $|z-\zeta| \leqslant r$

$$|\varphi^{(k)}(z) - k!\,c_k| \leqslant |\varphi^{(k+1)}(\zeta)|\,r + \ldots + \sup_{|z-\zeta|\leqslant r} |\varphi^{(p)}(z)| \frac{r^{p-k}}{(p-k)!} ,$$

$$k = 0, 1, \ldots, p-1 , \qquad c_0 = \beta .$$

Hence and by (20) we have

$$(21) \qquad |\varphi^{(k)}(z) - k!\,c_k| \leqslant \sum_{i=1}^{p-k} \frac{(k+i)!}{i!} |c_{k+i}|\, r^i + K \frac{r^{p-k}}{(p-k)!}$$

for $|z-\zeta| \leqslant r$. We obtain from (21) for $k = 0$

$$|\varphi(z) - \beta| \leqslant \sum_{i=1}^{p} |c_i|\, r^i + K \frac{r^p}{p!}$$

and hence by (13)

$$(22) \qquad |\varphi(z) - \beta| \leqslant R \quad \text{for} \quad |z-\zeta| \leqslant r .$$

Since the functions $f(z)$ and $h(z, w)$ are analytic for $|z-\zeta| \leqslant r$, $|w-\beta| \leqslant R$, thus according to (i), (12), (19), and (22) the function $\psi(z)$ is analytic for $|z-\zeta| < r$ and the function $\psi^{(p)}(z)$ is continuous for $|z-\zeta| \leqslant r$.

It follows from (19) that

$$\psi^{(k)}(\zeta) = h_k\big(\zeta, \varphi(\zeta), \ldots, \varphi^{(k)}(\zeta)\big) , \qquad k = 1, 2, \ldots$$

We infer from (ii) that $\varphi(\zeta) = \beta$, $\varphi^{(k)}(\zeta) = k!\,c_k$, $k = 1, \ldots, p$, whence

$$\psi^{(k)}(\zeta) = h_k(\zeta, \beta, c_1, \ldots, k!\,c_k) , \qquad k = 1, \ldots, p .$$

On account of Lemma 1 we obtain

$$\psi^{(k)}(\zeta) = k!\,c_k , \qquad k = 1, \ldots, p .$$

This together with $\psi(\zeta) = \beta$ yields condition (ii).

From (21) and since $r < 1$ (cf. (12)), we get

$$|\varphi^{(k)}(z) - k!\,c_k| \leqslant \sum_{i=1}^{p-k} (k+i)!\,|c_{k+i}| + K , \qquad k = 1, \ldots, p-1 .$$

By (10) and (iii) we have

(23) $$|\varphi^{(k)}(z)-k!\,c_k| \leqslant M_k$$

for $|z-\zeta| \leqslant r$, $k = 1, \ldots, p$.

According to (19) and Lemma 1 we get

$$\begin{aligned}
|\psi^{(p)}(z)-p!\,c_p| &\\
&= |h_p(z, \varphi[f(z)], \varphi'[f(z)], \ldots, \varphi^{(p)}[f(z)])-p!\,c_p| \\
&\leqslant |h_p(z, \varphi[f(z)], \varphi'[f(z)], \ldots, \varphi^{(p)}[f(z)])-h_p(z, \beta, c_1, \ldots, p!\,c_p)|+ \\
&\qquad + |h_p(z, \beta, c_1, \ldots, p!\,c_p)-h_p(\zeta, \beta, c_1, \ldots, p!\,c_p)| .
\end{aligned}$$

It follows from (22), (23), (12) and (11) that the points $(z, \varphi[f(z)], \varphi'[f(z)], \ldots, \varphi^{(p)}[f(z)])$, $(z, \beta, c_1, \ldots, p!\,c_p)$ belong to the set Z (cf. (8)). Since $r \leqslant r_2$ (cf. (12)), we have by (9) and Lemma 3

$$\begin{aligned}
&|\psi^{(p)}(z)-p!\,c_p| \\
&\leqslant L_0|\varphi[f(z)]-\beta|+\sum_{k=1}^{p-1} L_k|\varphi^{(k)}[f(z)]-k!\,c_k|+\theta\,|\varphi^{(p)}[f(z)]-p!\,c_p|+(1-\theta)K/2 .
\end{aligned}$$

In virtue of (21) and (iii) we have

$$\begin{aligned}
|\psi^{(p)}(z)-p!\,c_p| &\leqslant L_0\sum_{k=1}^{p} c_k r^k+L_0K\frac{r^p}{p!}+ \\
&\quad +\sum_{k=1}^{p-1} L_k\sum_{i=1}^{p-k}|c_{k+i}|(k+i)!\,\frac{r^i}{r!}+\sum_{k=1}^{p-1} KL_k\frac{r^{p-k}}{(p-k)!}+\theta K+\frac{(1-\theta)K}{2} \\
&= \frac{(1+\theta)K}{2}+\sum_{k=0}^{p-1}\sum_{i=1}^{p-k} L_k|c_{k+i}|(k+i)!\,\frac{r^i}{i!}+K\sum_{k=1}^{p} L_{p-k}\frac{r^k}{k!},
\end{aligned}$$

whence we get by (14)

$$|\psi^{(p)}(z)-p\,!\,c_p| \leqslant \frac{(1+\theta)K}{2}+\frac{(1-\theta)K}{2}=K .$$

Thus the function ψ fulfils condition (iii). We have proved that the transformation T maps set A_r into itself.

Let $\psi_1 = T[\varphi_1]$, $\psi_2 = T[\varphi_2]$. It follows by (19) and (7) that

$$\begin{aligned}
&|\psi_1^{(p)}(z)-\psi_2^{(p)}(z)| \\
&\quad = \left|h_p(z, \varphi_1[f(z)], \ldots, \varphi_1^{(p)}[f(z)])-h_p(z, \varphi_2[f(z)], \ldots, \varphi_2^{(p)}[f(z)])\right| \\
&\quad \leqslant L_0\,|\varphi_1[f(z)]-\varphi_2[f(z)]|+\sum_{k=1}^{p-1} L_k|\varphi_1^{(k)}[f(z)]-\varphi_2^{(k}[f(z)]|+ \\
&\qquad\qquad + \theta\,|\varphi_1^{(p)}[f(z)]-\varphi_2^{(p)}[f(z)]| .
\end{aligned}$$

since by (22), (23), (12) and (11) the points $\big(z, \varphi_i[f(z)], \ldots, \varphi_i^{(p)}[f(z)]\big)$ $(i = 1, 2)$ belong to Z. From the above inequality and from the definition of the metric we infer

$$\varrho(\psi_1, \psi_2) \leqslant \theta\varrho(\varphi_1, \varphi_2) + \sum_{k=0}^{p-1} L_k \sup_{z-\zeta\leqslant r} |\varphi_1^{(k)}(z) - \varphi_2^{(k)}(z)| \,.$$

It follows from (18) that

$$\varrho(\psi_1, \psi_2) \leqslant \theta\varrho(\varphi_1, \varphi_2) + \sum_{k=0}^{p-1} L_k \frac{r^{p-k}}{(p-k)!} \varrho(\varphi_1, \varphi_2)$$

and finally we have by (15)

$$\varrho(\psi_1, \psi_2) \leqslant \vartheta\varrho(\varphi_1, \varphi_2) \,.$$

Consequently T is a contraction mapping. Thus there exists exactly one solution of equation (1) for $|z-\zeta| \leqslant r$ belonging to the set A_r. The proof of the theorem is completed.

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