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Repeated convergence and fractional differences

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1. Introduction. In [3] the notion of Cesàro summability was investigated by first considering the Cesàro summability classes C_y, i.e. C_y is the set of all series which are summable (C, y), and extending y to include all real values, rather than values greater than -1 as is commonly done. The extended definition, discussed fully in [3] is as follows:

Let $\sum_0^\infty a_n$ be a series of numbers,

$$(1)\qquad A_n^y = \frac{(y+1)(y+2)\dots(y+n)}{n!}, \quad A_0^y = 1,$$

the *n-th Cesàro numbers*, and

$$s_n^y = \sum_{\nu=0}^{n} A_{n-\nu}^y a_\nu,$$

the *n-th Cesàro sums*.

The series $\sum_0^\infty a_n$ is said to be *summable* (C, y) *to sum* $a^{(0)}$, y an arbitrary real number, if

$$(2)\qquad s_n^{y+r} = a^{(0)} A_n^{y+r} + o(n^{y+r}) \quad \text{for } r = 0, 1, 2, \dots$$

This family of sets was then enlarged to a family $\{R_{x,y}\}$ defined for all real values of x and y by a procedure involving the Cesàro sums, which will be reproduced below. This enlarged family, where $R_{0,y} = C_y$, is called the *repeated convergence classes*.

However, given the family $\{R_{k,y}\}$ for all integral k and real y, there is another way of enlarging this family, to classes with fractional k by the use of fractional differences instead of Cesàro sums. The resulting family is called the *fractional difference classes* and will be denoted by $F_{x,y}$.

In this paper, these classes will be discussed and, in particular, their close relation to the classes $R_{x,y}$ will be investigated. For convenience we define here the repeated convergence classes $R_{x,y}$ and give their characterization in terms of Cesàro sums, see [3].

Suppose $\sum a_n$ is a convergent series. If we let

$$a_n^{(1)} = \sum_{\nu=n+1}^{\infty} a_\nu,$$

then $a_n^{(1)}$ is defined for each n and we may consider the series $\sum a_n^{(1)}$. If this series converges, we say, following Zygmund [9], p. 373, Vol. 1, that $\sum a_n$ has convergence of order 1. In general a series $\sum a_n$ has convergence of order k, k a non-negative integer, if $\sum a_n^{(k)}$ converges, where

$$a_n^{(0)} = a_n,$$

$$a_n^{(k)} = \sum_{\nu=n+1}^{\infty} a_\nu^{(k-1)}, \qquad k = 1, 2, 3, \ldots$$

The series $\sum a_n^{(k)}$ is called the *k-th iterate of* $\sum a_n$.

The above definition does not permit us to consider the k-th iterate of a series unless we first assume that the $(k-1)$-st iterate converges. This is an unnecessary restriction that can be overcome by the following procedure; a procedure that will, in addition, suggest a way of extending repeated convergence to fractional orders.

Suppose a series $\sum a_n$ has convergence of order k and let

$$a^{(0)} = \sum_0^{\infty} a_n,\ a^{(1)} = \sum_0^{\infty} a_n^{(1)}, \ldots, a^{(k)} = \sum_0^{\infty} a_n^{(k)}.$$

Then

$$a_n^{(0)} = a_n,$$

$$a_n^{(1)} = \sum_{\nu=n+1}^{\infty} a_\nu = a^{(0)} - s_n^{(0)}, \qquad \text{where } s_n^{(0)} = \sum_{\nu=0}^{n} a_\nu,$$

$$a_n^{(2)} = a^{(1)} - \sum_{\nu=0}^{n} a_\nu^{(1)} = a^{(1)} - a^{(0)} A_n^{(1)} + s_n^{(1)},$$

where $s_n^{(1)}$ is the n-th Cesàro sum of order 1. In general

$$a_n^{(k)} = (-1)^k [s_n^{k-1} - a^{(0)} A_n^{k-1} + a^{(1)} A_n^{k-2} - \ldots + (-1)^k a^{(k-1)}].$$

It is not necessary for $\sum a_n$ to converge in order to define $a_n^{(1)}$. If $\sum a_n$ is summable by any method of summation to $a^{(0)}$, we may define $a_n^{(1)} = a^{(0)} - s_n^{(0)}$ and likewise for the terms $a_n^{(k)}$, $k > 1$. Also, since both s_n^k and A_n^k are defined for fractional orders we can define $a_n^{(k)}$ for k fractional since it consists of terms involving s_n^k and A_n^k only. Finally, if the x-th iterate is not convergent but is summable (C, y), we express this by saying that the original series is in the repeated convergence class $R_{x,y}$. Formally, we have the following definition:

DEFINITION. A series $\sum a_n$ is said to be *in the repeated convergence class* $R_{x,y}$, $x = 0$ or $x = a+k-1$, $0 < a \leqslant 1$, k a positive integer, if there exist numbers $a^{(0)}, a^{(1)}, \ldots, a^{(k)}$ such that $a_n^{(x)}$ is summable (C, y) to $a^{(x)}$, where

$$a_n^{(0)} = a_n \quad \text{if } x = 0,$$

$$a_n^{(x)} = (-1)^k[s_n^{a+k-2} - a^{(0)}A_n^{a+k-2} + a^{(1)}A_n^{a+k-3} - \ldots + (-1)^k a^{(k-1)}A_n^{a-1}] \quad \text{if } x = a+k-1.$$

In addition

$$a^{(x)} = \begin{cases} (-1)^k a^{(k)} & \text{if } x = k, \\ 0 & \text{if } x \text{ is not an integer.} \end{cases}$$

If $x = \gamma - k$, where $0 \leqslant \gamma < 1$, k a positive integer, then the series is said to be *in class* $R_{x,y}$ if $\sum \Delta^k a_n \in R_{\gamma,y}$ and $\sum \Delta^k a_n$ is summable (C, y) to $\Delta^{k-1}a_0$. x is called the *order of convergence* and y, the order of summability of the series.

Note. The definition of repeated convergence classes of negative order arises as a consequence of requiring Lemma 3, see [3], to be valid for all orders of convergence. Also, the requirement that $a^{(x)} = 0$ if x is not an integer is natural. If $\sum a_n \in R_{x,0}$, $x > 0$, then $a^{(x')} = 0$ for $0 < x' < x$ if x' is non-integral. Thus we have defined classes $R_{x,y}$ for all ordered pairs (x, y) of the Euclidean plane with the classes $R_{0,y}$ being the Cesàro summability classes.

The following theorem characterizes repeated convergence classes by means of asymptotic expansions of Cesàro summability. Theorem A* below makes the characterization particularly lucid.

THEOREM A. *A necessary and sufficient condition for a series* $\sum a_n$ *to be in class* $R_{x,y}$, $x = a+k-1$, k *any integer*, $0 < a \leqslant 1$, y *arbitrary, is that there exist constants* $c_0, c_1, \ldots, c_k$ *such that for all non-negative integers* r,

$$(3)\quad s_n^{x+y+r} = c_0 A_n^{x+y+r} + c_1 A_n^{x+y+r-1} + \ldots + c_{k-1}A_n^{a+y+r} + c_k A_n^{y+r} + o(n^{y+r})$$

if $x \geqslant 0$,

$$s_n^{x+y+r} = o(n^{y+r}) \quad \textit{if } x < 0.$$

The $c_j = (-1)^j a^{(j)}$, $j = 0, 1, \ldots, k-1$,

$$c_k = \begin{cases} (-1)^k a^{(k)} & \textit{if } x = k, \\ 0 & \textit{otherwise.} \end{cases}$$

By changing the first few terms of any series, which does not affect the repeated convergence class to which it belongs, we can then state that

THEOREM A*. $\sum a_n \in R_{x,y}$ *if and only if* $s_n^{x+y+r} = o(n^{y+r})$ *for all non-negative integers* r *and arbitrary* x *and* y.

It was proved in [3] that $\sum a_n \in R_{x,y}$ if and only if $\sum A_n^\gamma \Delta^k a_n \in R_{x+k-\gamma,y+\gamma}$ for integral $k \geqslant 0$, real x, y, γ except for certain special values of the parameters, where the sufficiency condition is meant in the sense that there exists a unique series $\sum a_n^*$ satisfying $\Delta^k a_n^* = \Delta^k a_n$ which is in $R_{x,y}$.

We can also replace A_n^γ by n^γ.

References in this paper to Theorem 2* or Theorem 6 of [3] refer to different parts of this result.

Finally, it was shown in [3], Theorem 3 that if $\sum a_n \in R_{x,y}$ and $x' \leqslant x$, $y' \geqslant y$, then $\sum a_n \in R_{x',y'}$.

2. Given a sequence $\{a_n\}$. The fractional difference $\Delta^\gamma a_n$ is defined by

$$\Delta^\gamma a_n = \sum_{n=0}^{\infty} A^{-\gamma-1} a_{n+p},$$

i.e. it is a formal infinite series.

We have the following theorem concerning fractional differences.

THEOREM 1. *Let γ be any real number and let $\{a_n\}$ be any sequence of numbers. If*

$$\Delta^{-\gamma} a_n = \sum_{p=0}^{\infty} A_p^{\gamma-1} a_{n+p} \in R_{x,y}$$

for a particular n, then it is in $R_{x,y}$ for all n, where x and y are arbitrary.

Proof. First observe that if γ is an integer $\leqslant 0$, the theorem is trivial since $\Delta^{-\gamma} a_n$ is a finite series. Thus suppose $\gamma \neq 0, -1, -2, \ldots$ We suppose further that $\Delta^{-\gamma} a_n \in R_{x,y}$ for a particular n. Clearly it suffices to show that

$$\Delta^{-\gamma} a_{n+1} = \sum_{p=0}^{\infty} A_p^{\gamma-1} a_{n+1+p} \in R_{x,y}$$

and

$$\Delta^{-\gamma} a_{n-1} = \sum_{p=0}^{\infty} A_p^{\gamma-1} a_{n-1+p} \in R_{x,y}.$$

The following formula can be easily derived:

$$(1)\quad \sum_{p=0}^{R-1} p A_p^{\gamma-2} a_{n+1+p} = (\gamma-1) \sum_{p=0}^{R-1} A_p^{\gamma-1} a_{n+p} + (\gamma-1) \sum_{p=0}^{R-1} \Delta(A_p^{\gamma-1} a_{n+p}) - (\gamma-1) a_n.$$

Since $\sum_{p=0}^{\infty} A_p^{\gamma-1} a_{n+p} \in R_{x,y}$ it follows from Theorem 2*, [3], that $\sum_{p=0}^{\infty} \Delta(A_p^{\gamma-1} a_{n+p}) \in R_{x+1,y}$ and therefore is in $R_{x,y}$. By (1)

$$\sum_{p=0}^{\infty} p A_p^{\gamma-2} a_{n+1+p} \in R_{x,y}.$$

It follows from Theorem 4, [3], that

$$\sum_{p=0}^{\infty} p^{\gamma-1} a_{n+1+p} \in R_{x,y}$$

and again from the same theorem, that

$$\sum_{p=0}^{\infty} A_p^{\gamma-1} a_{n+1+p} \in R_{x,y}.$$

The proof that $\Delta^{-\gamma} a_{n+1} \in R_{x,y}$ is similar.

Given a series $\sum_{n=0}^{\infty} a_n$. Suppose $\Delta^{-a-k} a_n = \sum_{p=0}^{\infty} A_p^{a+k-1} a_{n+p}$ converges, where $0 \leqslant a < 1$, k a non-negative integer.

Then $\Delta^{-a-j} a_n$ converges for $0 \leqslant j \leqslant k$. Moreover, we have

$$\Delta^{-a-1} a_{n+1} = \Delta^{-a-1} a_0 - \sum_{\nu=0}^{n} \Delta^{-a} a_\nu = a^{\langle a \rangle} - \sum_{\nu=0}^{n} \Delta^{-a} a_\nu,$$

where

$$a^{\langle a \rangle} = \sum_{n=0}^{\infty} \Delta^{-a} a_n,$$

$$\Delta^{-a-2} a_{n+2} = a^{\langle a+1 \rangle} - a^{\langle a \rangle} A_n^{(1)} + t_n^{(1)}(a),$$

where

$$a^{\langle a+1 \rangle} = \Delta^{-a-1} a_1 = \sum_{n=0}^{\infty} \Delta^{-a-1} a_{n+1}$$

and

$t_n^{(1)}(a)$ is the Cesàro sum of order 1 of the series $\sum_{n=0}^{\infty} \Delta^{-a} a_n$.

. . . .

(2) $$\Delta^{-a-k} a_{n+k} = (-1)^k [t_n^{k-1}(a) - a^{\langle a \rangle} A_n^{k-1} + a^{\langle a+1 \rangle} A_n^{k-2} - \ldots + (-1)^k a^{\langle a+k-1 \rangle}],$$

where

$$a^{\langle a+j \rangle} = \sum_{n=0}^{\infty} \Delta^{-a-j} a_{n+j}, \qquad 0 \leqslant j \leqslant k,$$

and $t_n^{(k)}(a)$ is the k-th Cesàro sum of $\sum_{n=0}^{\infty} \Delta^{-a} a_n$.

It is not necessary for $\Delta^{-a} a_n$ to converge in order to define a series corresponding to $\Delta^{-a-1} a_n$. If $\Delta^{-a} a_n$ is summable by any linear method of summation to sum $a^{\langle a \rangle}$ we can define $\Delta^{(-a-1)} a_{n+1} = a^{\langle a \rangle} - \sum_{\nu=0}^{n} \Delta^{-a} a_\nu$ and likewise for higher orders. The parenthesis will indicate this definition.

Note also that $\Delta^{(-a-1)}a_n = \Delta^{-1}\Delta^{-a}a_n$. Thus we have the following definition:

DEFINITION. Given a series $\sum_{n=0}^{\infty} a_n$ and let $0 \leqslant a < 1$, k a non-negative integer. If there exist numbers

$$a^{\langle a\rangle}, a^{\langle a+1\rangle} \ldots, a^{\langle a+k\rangle}$$

such that $\sum_{p=0}^{\infty} \Delta^{(-a-k)}a_{n+k}$ is summable (C, y) to $a^{\langle a+k\rangle}$ and $\Delta^{-a}a_n = \sum_{p=0}^{\infty} A_p^{a-1}a_{n+p}$ is summable (C, y), then we shall say that the series $\sum_{n=0}^{\infty} a_n$ is *in the fractional difference class* $F_{a+k,y}$, where $\Delta^{(-a-k)}a_n$ is defined by

$$\Delta^{(-a-k)}a_{n+k} = (-1)^k[t_n^{k-1}(a) - a^{\langle a\rangle}A_n^{k-1} + \ldots + (-1)^k a^{\langle a+k-1\rangle}]$$

and $t_n^k(a)$ is the k-th Cesàro sum of $\sum_{n=0}^{\infty} \Delta^{-a}a_n$.

If $x = a-k$, where $0 \leqslant a < 1$, k a positive integer, then a series is said to be *in class* $F_{x,y}$ if

$$\sum_{n=0}^{\infty} \Delta^k a_n \in F_{a,y}$$

and

$$\sum_{n=0}^{\infty} \Delta^k a_n \overset{(C,y)}{=} \Delta^{k-1}a_0.$$

Notice that here we have a ranging from $0 \leqslant a < 1$ and not $0 < a \leqslant 1$ as in the case of repeated convergence. This means that the fractional order $k+a$ is analogous to the integral order k and not $k+1$ as in the case of the repeated convergence classes.

Let $x = k+a$, where $0 < a < 1$, k a non-negative integer. If $\Delta^{-x}a_n$ is summable (C), i.e., summable (C, y) for some real y, then it is summable to $\Delta^{(-x)}a_n$. Thus, if $\sum a_n \in F_{x,y}$, then $\sum \Delta^{-x}a_n$ is summable (C, y).

Conversely, if $\sum \Delta^{-x}a_n$ is summable (C, y) and $\Delta^{-a}a_n$ is summable (C, y), then $\sum a_n \in F_{x,y}$.

The classes $F_{x,y}$ have the following properties:

(i) $R_{k,y} = F_{k,y}$ *for all* y *and all integers* k;

(ii) $\sum_{n=0}^{\infty} a_n \in F_{a+k,y}$ *if and only if* $\sum_{n=0}^{\infty} \Delta^{-a}a_n \in R_{k,y}$, *where* $\Delta^{-a}a_n = \sum_{p=0}^{\infty} A_p^{a-1}a_{n+p}$ *is summable* (C, y), $k \geqslant 0$;

(iii) *If* $\sum_{n=0}^{\infty} a_n \in F_{k,y}$, *then* $\Delta^{-k}a_n = \sum_{p=0}^{\infty} A_p^{k-1}a_{n+p}$ *is summable* $(C, y+k-1)$ *or all integers* k *and all* y.

Note that (i) is a special case of (ii) (the case where $a = 0$), for $k \geqslant 0$. For $k < 0$ (i) follows immediately from the definition.

In order to prove (ii) it is only necessary to observe that if $b_n = \Delta^{-a} a_n$, then

$$\Delta^{-a-k} a_{n+k} = b_n^{(k)},$$

where $b_n^{(k)}$ is the n-th term of the k-th iterate of $\sum_{n=0}^{\infty} b_n$ (see (3), [3]).

If k is a non-positive integer, then (iii) is trivial. Suppose k is positive. From (i) it follows that $\sum_{n=0}^{\infty} a_n \epsilon R_{k,y}$ and by Theorem 2*, [3]

$$\sum_{n=0}^{\infty} A_n^{k-1} a_n \epsilon R_{1,y+k-1}.$$

But

$$\Delta^{-k} a_0 = \sum_{n=0}^{\infty} A_n^{k-1} a_n$$

and by Theorem 1

$$\Delta^{-k} a_{n+k} \epsilon R_{1,y+k-1}.$$

Thus $\Delta^{-k} a_{n+k}$ is summable $(C, y+k-1)$.

3. The following theorem is due to Isaacs [5]:

THEOREM. *If* $r < 0$, $r+s \neq 0, 1, \ldots$, $\lambda \geqslant \max(-s-1, -1)$, $k > s$, *and if* $\Delta^{r+s} a_n$ *is summable* (C, λ), *then* $\Delta_{(C,\lambda)}^{r+s} a_n = \Delta_{(C,\lambda+k)}^{r} (\Delta_{(C,\mu)}^{s} a_n)$, *where* $\mu = \max(\lambda + r, -1)$.

If s is an integer we may take $k = s$ but if s is non-integral the expression on the right need not exist for $k = s$. When s is a non-negative integer, the condition $r < 0$ may be omitted.

This theorem enables us to establish connection between the classes $F_{x,0}$ and the Dirichlet series $\sum_{n=0}^{\infty} A_n^x a_n$.

THEOREM 2. *Let* $x = a + k$, $0 < a < 1$, k *a non-negative integer. If* $\sum_{n=0}^{\infty} a_n \epsilon F_{x,0}$, *then* $\sum_{n=0}^{\infty} A_n^x a_n$ *is summable* $(C, x+\varepsilon)$ *for any* $\varepsilon > 0$. *Conversely, if* $\sum_{0}^{\infty} A_n^x a_n$ *is summable* $(C, x-\varepsilon)$, *then* $\sum_{0}^{\infty} a_n \epsilon F_{x,0}$.

Proof. For x integral the theorem is a consequence of Theorem 2*, [3]. Assume x is non-integral and that $\sum_{0}^{\infty} a_n \epsilon F_{x,0}$.

Claim.

$$\Delta_{(C,0)}^{x} (\Delta^{-x} a_n) = a_n, \quad \text{where } x = a + k. \tag{3}$$

This was proved by Andersen in [1] whenever the inner difference exists as a convergent series. If $\Delta^{-x}a_n$ is summable (C), then it is summable to sum $\Delta^{(-x)}a_n$. Therefore,

$$\Delta^{x}(\Delta^{-x}a_n) = \Delta^{a}\Delta^{1}\ldots\Delta^{1}(\Delta^{-1}\Delta^{-1}\ldots\Delta^{-1}\Delta^{-a}a_n) = \Delta^{a}\Delta^{-a}a_n = a_n$$

since $\Delta^{-a}a_n$ is a convergent series.

From Isaacs' theorem we can easily show that if $\sum a_n \in F_{x,0}$, then $\Delta^{-x}a_n$ is summable (C).

To see this, first suppose $x = a$, where $0 < a < 1$. Then from the definition of $F_{a,0}$ we have that $\Delta^{-a}a_n$ is summable (C).

Suppose $x = a+1$. Then from Isaacs' theorem

$$\Delta^{-1}_{(C,0)}(\Delta^{-a}a_n) = \Delta^{-1-a}_{(C)}\Delta^{a}_{(C)}(\Delta^{-a}a_n) = \Delta^{-1-a}_{(C)}a_n$$

since $\Delta^{a}(\Delta^{-a}a_n) = a_n$, where $r = -1-a$, $s = a$.

Thus $\Delta^{-1-a}a_n$ is summable (C), and by induction, the result follows in general.

Now $\sum\limits_{0}^{\infty} a_n \in F_{x,0}$ implies $\Delta^{-1}_{(C,0)}(\Delta^{-x}a_n)$. But

$$\Delta^{-1}_{(C,0)}(\Delta^{-x}a_n) = \Delta^{-1-x}_{(C,x+\varepsilon)}\Delta^{x}_{(C,0)}(\Delta^{-x}a_n)$$

by the theorem of Isaacs with

$$r = -1-x, \quad s = x.$$

However, by the claim

$$\Delta^{x}_{(C,0)}(\Delta^{-x}a_n) = a_n.$$

Hence

$$\Delta^{-1}_{(C,0)}(\Delta^{-x}a_n) = \Delta^{-1-x}_{(C,x+\varepsilon)}(a_n)$$

which means $\sum\limits_{p=0}^{\infty} A^{x}_{p-n}a_p$ is summable $(C, x+\varepsilon)$ for each n and in particular, $\sum\limits_{p=0}^{\infty} A^{x}_{p}a_p$ is summable $(C, x+\varepsilon)$.

The converse follows immediately from Isaacs' theorem since $\sum A^{x}_{n}a_n = \Delta^{-x-1}a_0$.

Corollary 1. *If* $\sum\limits_{p=0}^{\infty} a_n \in F_{x,0}$, *then* $\Delta^{-x}a_n = \sum\limits_{p=0}^{\infty} A^{x-1}_{p}a_{n+p}$ *is summable* $(C, x+\varepsilon-1)$ *for any* $\varepsilon > 0$.

The next theorem shows that the fractional difference classes have the inclusion property, at least for zero order of summability.

Theorem 3. *If* $\sum\limits_{n=0}^{\infty} a_n \in F_{x,0}$, *then* $\sum\limits_{n=0}^{\infty} a_n \in F_{x',0}$ *for* $x' < x$.

Proof. We first assume that $x \geqslant 0$. We may also assume $k < x' < x < k+1$, all other cases follow from this case and property (ii). Now

$$\sum_{n=0}^{m} \Delta^{-x'} a_n = \sum_{n=0}^{m} \Delta^{-x'}[\Delta^x (\Delta^{-x} a_n)]$$

by the claim in the proof of Theorem 2. Hence

$$\sum_{n=0}^{m} \Delta^{-x'} a_n = -\sum_{n=0}^{m} \Delta^{-x'}\Big[\Delta^x\Big(\Delta \sum_{\nu=0}^{n-1} \Delta^{-x} a_\nu\Big)\Big] = -\sum_{n=0}^{m} \Delta^{-x'}\Big[\Delta^{x+1}\Big(\sum_{\nu=0}^{n-1} \Delta^{-x} a_\nu\Big)\Big]$$

since x and 1 are both non-negative and $\sum_{n=0}^{\infty} \Delta^{-x} a_\nu$ converges. But

$$\Delta_{(C,0)}^{-x'}\Big(\Delta_{(C,0)}^{x+1} \sum_{\nu=0}^{n-1} \Delta^{-x} a_\nu\Big) = \Delta_{(C,0)}^{-x'+x+1}\Big(\sum_{\nu=0}^{n-1} \Delta^{-x} a_\nu\Big)$$

by Isaacs' theorem with $r = -x'$, $s = x+1$. Since $k < x' < x < k+1$, $r+s \neq 0, 1, 2, \ldots$ Thus

$$(4) \quad \sum_{n=0}^{m} \Delta^{-x'} a_n = -\sum_{n=0}^{m} \sum_{p=0}^{\infty} A_{p-n+1}^{x'-x-2}\Big(\sum_{\nu=0}^{p} \Delta^{-x} a_\nu\Big)$$

$$= -\sum_{p=0}^{\infty}\Big(\sum_{\nu=0}^{p} \Delta^{-x} a_\nu\Big) \sum_{n=0}^{m} A_{p-n+1}^{x'-x-2} - \sum_{p=0}^{\infty}\Big(\sum_{\nu=0}^{p} \Delta^{-x} a_\nu\Big)(A_{p+1}^{x'-x-1} - A_{p-m}^{x'-x-1}).$$

We may suppose without loss of generality that $\sum_{n=0}^{\infty} \Delta^{-x} a_n$ converges to 0. From (4) we see that

$$\sum_{n=0}^{m} \Delta^{-x'} a_n = \sum_{p=0}^{\infty} A_{p+1}^{x'-x-1} o(1) + \sum_{p=m}^{\infty}\Big(\sum_{\nu=0}^{p} \Delta^{-x} a_\nu\Big) A_{p-m}^{x'-x-1}.$$

Since $x' < x$ the first series has terms $o(p^{x'-x-1})$ and hence converges as $m \to \infty$. As for the second series,

$$\Big|\sum_{p=m}^{\infty}\Big(\sum_{\nu=0}^{p} \Delta^{-x} a_\nu\Big) A_{p-m}^{x'-x-1}\Big| \leqslant \max_{p \geqslant m}\Big|\sum_{\nu=0}^{p} \Delta^{-x} a_\nu\Big| \sum_{p=0}^{\infty} |A_p^{x'-x-1}| = o(1) \quad \text{as } m \to \infty$$

since we assumed that $\sum_{n=0}^{\infty} \Delta^{-x} a_n = 0$.

We now assume $x < 0$,

$$x = -k+a, \quad x' = -k+a'$$

say, where k is a positive integer and $0 < a' < a < 1$. From the definition of fractional difference classes of negative order of convergence we see that

$$\sum_{n=0}^{\infty} a_n \in F_{x,0} \Leftrightarrow \sum_{n=0}^{\infty} \Delta^k a_n \in F_{a,0}$$

and

$$\sum_{n=0}^{\infty} a_n \in F_{x',0} \Leftrightarrow \sum_{n=0}^{\infty} \Delta^k a_n \in F_{\alpha',0}.$$

By what was proved above, the theorem easily follows in this case.

Since the classes $R_{x,y}$ and $F_{x,y}$ are the same for x integral one may suspect that, though they are different when x is not integral, there may still be close connection between them. We have in this regard the following theorem:

THEOREM 4. *If* $\sum_{n=0}^{\infty} a_n \in F_{x,0}$, *then* $\sum_{n=0}^{\infty} a_n \in R_{x-\varepsilon,0}$ *and* $\sum_{n=0}^{\infty} a_n \in R_{x,\varepsilon}$ *for any* $\varepsilon > 0$.

Conversely, if $\sum_{n=0}^{\infty} a_n \in R_{x,0}$, *then* $\sum_{n=0}^{\infty} a_n \in F_{x-\varepsilon,\varepsilon}$.

Proof. We shall first prove that if $\sum_{n=0}^{\infty} a_n \in F_{x,0}$, then $\sum_{n=0}^{\infty} a_n \in R_{x',0}$ for any $x' < x$. For x integral $F_{x,0} = R_{x,0}$ by (i) and the proof is an immediate consequence of Theorem 3, [3]. We may suppose that $k \leq x' < x < k+1$, k an integer.

Case (i) $0 < x' < x < 1$. Since $\sum_{n=0}^{\infty} a_n$ converges, we have the identity

$$s_n^{x'} = A_n^{x'} a^{(0)} + \sum_{\nu=0}^{\infty} (A_{n-\nu}^{x'} - A_n^{x'}) a_\nu$$

and since $\Delta^{-x} a_n$ converges and $\sum_{n=0}^{\infty} \Delta^{-x} a_n$ converges it follows that

$$\Delta^x (\Delta^{-x} a_n) = a_n = -\Delta^x \Big[\Delta \sum_{\nu=0}^{n-1} \Delta^{-x} a_\nu\Big] = -\Delta^{x+1} \Big(\sum_{\nu=0}^{n-1} \Delta^{-x} a_\nu\Big).$$

Hence

$$s_n^{x'} - a^{(0)} A_n^{x'} = \sum_{\nu=0}^{\infty} (A_n^{x'} - A_{n-\nu}^{x'}) \sum_{p=0}^{\infty} A_{p-\nu+1}^{-x-2} \Big(\sum_{r=0}^{p} \Delta^{-x} a_r\Big)$$

$$= \sum_{p=0}^{\infty} \Big(\sum_{r=0}^{p} \Delta^{-x} a_r\Big) \sum_{\nu=0}^{\infty} (A_n^{x'} - A_{n-\nu}^{x'}) A_{p-\nu+1}^{-x-2}.$$

The interchange of the order of summation is justified as follows: letting

$$S_n^{(x)} = \sum_{\nu=0}^{n} \Delta^{-x} a_\nu$$

we have

$$\sum_{p=0}^{\infty}\sum_{\nu=0}^{\infty} A_{p-\nu+1}^{-x-2} S_p^{(x)} = \sum_{p=0}^{\infty} A_{p+1}^{-x-1} S_p^{(x)} = \sum_{p=0}^{\infty} A_p^{-x-1} S_{p-1}$$

$$= \sum_{p=0}^{\infty} A_p^{-x-2}(S_{p-1}^{(x)}) = -\Delta_{(C,0)}^{x-1}(\Delta^{-x} a_0) = -\Delta_{(C,0)}^{-1} a_0$$

$$= -\sum_{n=0}^{\infty} a_n = -a^{(0)}.$$

On the other hand,

$$\sum_{\nu=0}^{\infty}\sum_{p=0}^{\infty} A_{p-\nu+1}^{-x-2} S_p^{(x)} = \sum_{\nu=0}^{\infty} \Delta^{x+1} S_{\nu-1}^{(x)} = \sum_{\nu=0}^{\infty} \Delta^{x}[\Delta S_{\nu-1}^{(x)}]$$

$$= -\sum_{\nu=0}^{\infty} \Delta^{x}(\Delta^{-x} a_\nu) = -\sum_{\nu=0}^{\infty} a_\nu = -a^{(0)}.$$

Hence both double series converge to the same sum. As for the other term,

$$\sum_{\nu=0}^{\infty} A_{n-\nu}^{x'} \sum_{p=0}^{\infty} A_{p-\nu+1}^{-x-2} S_p^{(x)} = \sum_{\nu=0}^{n} A_{n-\nu}^{x'} \sum_{p=0}^{\infty} A_{p-\nu+1}^{-x-2} S_p^{(x)}$$

$$= \sum_{p=0}^{\infty} S_p^{(x)} \sum_{\nu=0}^{n} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2}$$

$$= \sum_{p=0}^{\infty} S_p^{(x)} \sum_{\nu=0}^{\infty} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2}.$$

Thus we finally get

$$s_n^{x'} - a^{(0)} A_n^{x'} = \sum_{p=0}^{\infty} S_p^{(x)} \sum_{\nu=0}^{\infty} (A_n^{x'} - A_{n-\nu}^{x'}) A_{p-\nu+1}^{-x-2} \tag{5}$$

$$= \sum_{p=0}^{\infty} S_p^{(x)} \Big[A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{n} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2}\Big].$$

We will show that Toeplitz's conditions are satisfied, except that in this case

$$\sum_{p=0}^{\infty}\Big[A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{n} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2}\Big] \to 0 \quad \text{as } n \to \infty$$

instead of 1, implying that the transformed series $s_n^{x'} - a^{(0)} A_n^{x'}$ converges to 0 as $n \to \infty$:

(a) $$A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{n} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2}$$

$$= A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{n} A_n^{x'} A_{p-\nu+1}^{-x-2} + \sum_{\nu=0}^{n} (A_n^{x'} - A_{n-\nu}^{x'}) A_{p+1-\nu}^{-x-2}$$

$$= \sum_{\nu=0}^{n} (A_n^{x'} - A_{n-\nu}^{x'}) A_{p+1-\nu}^{-x-2} \quad \text{for } n > p \text{ and this sum is}$$

$$\leqslant |A_n^{x'} - A_{n-p-1}^{x'}| O(1) = O(1) \sum_{\nu=0}^{p-1} A_{n-\nu}^{x'-1} = O(n^{x'-1}) = o(1)$$

as $n \to \infty$ since $0 < x' < 1$.

(b) $$\sum_{p=0}^{\infty} A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{n} \sum_{p=0}^{\infty} A_{n-\nu}^{x} A_{p-\nu+1}^{-x-2} = -A_n^{x'} - \sum_{\nu=0}^{n} A_{n-\nu}^{x'} \sum_{p=0}^{\infty} A_{p-\nu+1}^{-x-2}.$$

The second term is zero for all $\nu > 0$ and for $\nu = 0$, it is $A_n^{x'}$. Hence

$$\sum_{p=0}^{\infty} A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{n} \sum_{p=0}^{\infty} A_{n-\nu}^{x} A_{p-\nu+1}^{-x-2} = -A_n^{x'} + A_n^{x'} = 0$$

for all n.

(c) $$\sum_{p=0}^{\infty} \left| A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{p-1} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2} \right| = \sum_{p=0} + \sum_{p=n+1} = Q_1 + Q_2.$$

Now

$$Q_2 = \sum_{p=n+1}^{\infty} \left| A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{n} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2} \right|.$$

Since each term inside the absolute value signs is negative we have

$$Q_2 = - \sum_{p=n+1}^{\infty} \left(A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{n} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2} \right).$$

Now

$$A_n^{x'} \sum_{p=n+1}^{\infty} A_{p+1}^{-x-1} = O(n^{x'})\, o(n^{-x}) = o(1) \quad \text{as } n \to \infty.$$

Also

$$\sum_{p=n+1}^{\infty} \sum_{\nu=0}^{n} A_{n-\nu}^{x'} A_{p-\nu+1}^{-x-2} = \sum_{\nu=0}^{n} A_{n-\nu}^{x'} \sum_{p=n+1}^{\infty} A_{p+1-\nu}^{-x-2} = \sum_{\nu=0}^{n} A_{n-\nu}^{x'} O\big((n-\nu)\big)^{-x-1}$$

$$= O(1) \quad \text{for all } n.$$

$$Q_1 = \sum_{p=0}^{n} \left| A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{p-1} A_{n-\nu}^{x'} A_{p+1-\nu}^{-x-2} \right|.$$

But

$$\sum_{\nu=0}^{p+1} A_{n-\nu}^{x'} A_{p+1-\nu}^{-x-2} = \sum_{\nu=0}^{p} A_{n-\nu}^{x'-1} \sum_{r=0}^{\nu} A_{p+1-r}^{-x-2} - A_{n-p-1}^{x'} A_{p+1}^{-x-1}$$

$$= A_{p+1}^{-x-1}(A_n^{x'} - A_{n-p-1}^{x'}) - \sum_{\nu=0}^{p} A_{n-\nu}^{x'-1} + A_{n-p-1}^{x'} A_{p+1}^{-x-1}$$

$$= A_n^{x'} A_{p+1}^{-x-1} - \sum_{\nu=0}^{p} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1}.$$

Hence

$$Q_1 = \sum_{p=0}^{n} \Big| \sum_{\nu=0}^{p} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1} \Big|.$$

Claim. $\sum_{\nu=0}^{p} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1}$ is positive for $0 \leqslant p \leqslant n$, all n.

$$\sum_{\nu=0}^{p} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1} = \sum_{\nu=0}^{p-1} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1} + A_{n-p}^{x'-1}.$$

Since the first term is negative it follows that

$$\sum_{\nu=0}^{p-1} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1} + A_{n-p}^{x'-1} > A_{n-p+1}^{x'-1} \sum_{\nu=0}^{p-1} A_{p-\nu}^{-x-1} + A_{n-p}^{x'-1}$$

$$= A_{n-p+1}^{x'-1}(A_p^{-x} - 1) + A_{n-p}^{x'-1} = A_p^{-x} A_{n-p+1}^{x'-1} + A_{n-p}^{x'-1} - A_{n-p+1}^{x'-1}$$

$$= A_p^{-x} A_{n-p+1}^{x'-1} - A_{n-p+1}^{x'-2}.$$

But

$$A_{n-p+1}^{x'-2} < 0 \qquad \text{for all } p < n+1$$

and so

$$A_p^{-x} A_{n-p+1}^{x'-1} - A_{n-p+1}^{x'-2} > 0 \qquad \text{for } 0 \leqslant p \leqslant n$$

and the claim is proved.

It is now easy to complete the proof of (b)

$$\sum_{p=0}^{n} \Big| \sum_{\nu=0}^{p} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1} \Big| = \sum_{p=0}^{n} \sum_{\nu=0}^{p} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1}$$

$$= \sum_{p=0}^{n} \sum_{\nu=0}^{n} A_{n-\nu}^{x'-1} A_{p-\nu}^{-x-1} = \sum_{\nu=0}^{n} A_{n-\nu}^{x'-1} \sum_{p=0}^{n} A_{p-\nu}^{-x-1}$$

$$= \sum_{\nu=0}^{n} A_{n-\nu}^{x'-1} A_{n-\nu}^{-x} = O(1).$$

Hence Toeplitz's conditions are satisfied and

$$s_n^{x'} - a^{(0)} A_n^{x'} = o(1).$$

The proof of the general case of this part of the theorem is an immediate consequence of what was proved above and of the following lemma:

LEMMA 1. $\sum_{n=0}^{\infty} a_n \epsilon F_{a+k,0}$ *if and only if* $\sum_{n=0}^{\infty} \Delta^{-k} a_n \epsilon F_{a,0}$, *where* $0 < a < 1$ *and* k *is any integer.*

Proof. First, suppose $k \geqslant 0$. If $\sum_{n=0}^{\infty} \Delta^{-k} a_n \epsilon F_{a,0}$ we must show that

(1) $\sum_{n=0}^{\infty} \Delta^{-a-k} a_n$ converges,

(2) $\Delta^{-a} a_n = \sum_{p=0}^{\infty} A_p^{a-1} a_{n+p}$ converges.

Now

$$\Delta_{(C,0)}^{-a}(\Delta^{-k} a_n) = \Delta_{(C,k)}^{-a-k} a_n$$

by Andersen [2] and $\sum_{n=0}^{\infty} \Delta_{(C,k)}^{-a-k} a_n$ converges by hypothesis. Also

$$\Delta_{(C,k)}^{-a-k} a_n = \sum_{p=0}^{\infty} A_p^{a+k-1} a_{n+p} \epsilon R_{0,k}$$

if and only if

$$\sum_{p=0}^{\infty} A_p^{a-1} a_{n+p} \epsilon R_{k,0}$$

and hence $\Delta^{-a} a_n$ converges.

Conversely suppose $\sum_{n=0}^{\infty} a_n \epsilon F_{a+k,0}$. This implies $\sum_{n=0}^{\infty} \Delta^{-a-k} a_n$ converges and by the corollary to Theorem 2, $\Delta^{-a-k} a_n$ is summable (C, k). But

$$\Delta_{(C,k)}^{-a-k} a_n = \Delta_{(C,0)}^{-a}(\Delta^{-k} a_n)$$

again by Andersen [2]. Thus $\sum_{n=0}^{\infty} \Delta_{(C,0)}^{-a}(\Delta^{-k} a_n)$ converges and this means that

$$\sum_{n=0}^{\infty} \Delta^{-k} a_n \epsilon F_{a,0}.$$

For $k < 0$ the lemma follows from the definition of fractional difference classes of negative order.

If $\sum_{n=0}^{\infty} a_n \epsilon F_{x,0}$, then $\sum_{n=0}^{\infty} A_n^x a_n \epsilon R_{0,x+\varepsilon}$ by Theorem 2. From Theorem 6*, [3] it follows that $\sum_{n=0}^{\infty} a_n \epsilon R_{x,\varepsilon}$.

Conversely, if $\sum_{n=0}^{\infty} a_n \in R_{x,0}$, then

$$\sum_{n=0}^{\infty} A_n^{x'} a_n \in R_{x-x',x'} \quad \text{for } 0 < x - x' < 1,$$

by Theorem 6*, [3] and $\sum_{n=0}^{\infty} A_n^{x'} a_n \in R_{0,x'}$. But

$$\Delta_{(C,x')}^{-x'-1} a_n = \Delta_{(C,\varepsilon)}^{-1}(\Delta_{(C,x'-1)}^{-x'} a_n)$$

by Isaacs' theorem with $r = -1$, $s = -x'$, Thus $\sum_{n=0}^{\infty} a_n \in F_{x',\varepsilon}$ or equivalently $\sum_{n=0}^{\infty} a_n \in F_{x-\varepsilon,\varepsilon'}$. This completes the proof of the theorem.

In [3] we saw that $\sum_{n=0}^{\infty} a_n \in R_{k,0}$, k a positive integer, if and only if $\sum_{n=0}^{\infty} A_n^k a_n$ is summable (C, k). We also saw that, if $\sum_{n=0}^{\infty} a_n \in R_{x,0}$, x positive but not an integer, then it does not follow that $\sum_{n=0}^{\infty} A_n^x a_n$ is summable (C, x). However, as we have noted, when x is not integral we can extend the classes $R_{k,0}$ in two separate directions to classes $R_{x,0}$ and $F_{x,0}$ each of which have distinct properties. Although neither $\sum_{n=0}^{\infty} a_n \in R_{x,0}$ or $\sum_{n=0}^{\infty} a_n \in F_{x,0}$ implies that $\sum_{n=0}^{\infty} A_n^x a_n$ is summable (C, x), if we allow the series to be in both classes, then the implication is true. In fact, we even have the following theorem:

THEOREM 5. *A necessary and sufficient condition for* $\sum_{n=0}^{\infty} A_n^x a_n$ *to be summable* (C, x) *is that* $\sum_{n=0}^{\infty} a_n \in R_{x,0}$ *and* $\sum_{n=0}^{\infty} a_n \in F_{x,\varepsilon}$ *for* $0 < \varepsilon < 1$, $x = k + a$, k *a non-negative integer*, $0 < a < 1$.

Proof. We first prove the following:

Claim. If $\sum_{n=0}^{\infty} a_n \in R_{x,0}$ and $\sum_{n=0}^{\infty} a_n \in F_{x,\varepsilon}$, then

$$\Delta_{(C,0)}^{x}(\Delta_{(C,\varepsilon)}^{-x} a_n) = a_n.$$

Suppose first that $x = a$. Since $\sum_{n=0}^{\infty} a_n \in R_{a,0}$ the series converges and therefore $\Delta^{-a} a_n = \sum_{p=0}^{\infty} A_p^{a-1} a_{n+p}$ converges. Hence $\Delta^a(\Delta^{-a} a_n) = a_n$. In general, since $\sum_{n=0}^{\infty} \Delta^{-x} a_n$ is summable (C, ε) it follows that $\Delta^{-x} a_n = o(n^{\varepsilon})$ and $\Delta^x(\Delta^{-x} a_n) = \sum_{p=0}^{\infty} A_p^{-x-1} \Delta^{-x} a_{n+p}$ converges.

Also, summing by parts k times, we find that

$$\Delta^x(\Delta^{-x}a_n) = \Delta^{x-1}(\Delta^{-x+1}a_n) = \ldots = \Delta^a(\Delta^{-a}a_n)$$

and by what was just proved above,

$$\Delta^a(\Delta^{-a}a_n) = a_n.$$

We now apply Isaacs' theorem.

$$\Delta^{-1}_{(C,\varepsilon)}(\Delta^{-x}a_n) = \Delta^{-1-x}_{(C,\varepsilon)}[\Delta^{x}_{(C,x+1)}(\Delta^{-x}_{(C,0)}a_n)],$$

where $r = -1-x$, $s = x$.

By the claim, the expression in brackets is just a_n. Hence $\sum_{p=0}^{\infty} A_p^x a_p$ is summable $(C, x+1)$.

Thus if $\sum_{n=0}^{\infty} a_n \in R_{x,0}$ and $\sum_{n=0}^{\infty} a_n \in F_{x,\varepsilon}$, then $\sum_{n=0}^{\infty} A_n^x a_n$ is summable $(C, x+1)$ and by Theorem 7, [3], $\sum_{n=0}^{\infty} A_n^x a_n$ is summable (C, x).

Conversely, suppose $\sum_{n=0}^{\infty} A_n^x a_n$ is summable (C, x). By Theorem 6*, [3], $\sum_{n=0}^{\infty} a_n \in R_{x,0}$ and by Isaacs' theorem

$$\Delta^{-x-1}_{(C,x)}a_n = \Delta^{-1}_{(C,\varepsilon)}[\Delta^{-x}_{(C,x-1)}a_n]$$

taking $r = -1$, $s = -x$.

This means $\sum_{n=0}^{\infty} \Delta^{-x}a_n$ is summable (C, ε) and since $\Delta^{-a}a_n$ converges, $\sum_{n=0}^{\infty} a_n \in F_{x,\varepsilon}$ proving the theorem.

Observe that for $x = k$, k a positive integer, the theorem reduces to a special case of Theorem 2*, [3] since $F_{k,\varepsilon} = R_{k,\varepsilon}$.

As an application of Theorem 5 and also of the principle:

If a theorem holds for $\sum a_n \in R_{x,y}$, x integral, and is false when x is non-integral, then if we take $\sum a_n$ to be in $F_{x,y}$ as well, the theorem becomes valid again, we state the following result in Fourier series:

Let $f(x)$ be a periodic, integrable function of period 2π. Suppose for $0 < a < 1$

(1) $$f(x+t) = f(x) + o(|t|^a) \quad \text{as } t \to 0,$$

(2) $$\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\infty} \frac{f(x+t)+f(x-t)-2f(x)}{t^{1+a}}\,dt$$

exists for x in a set E of positive measure.

Then $S[f] = \sum n^a(a_n \cos nx + b_n \sin nx)$ is summable (C, a) a.e. in E.

The proof of this theorem, given in [4] follows from Theorem 5, by connecting condition (1) to $S[f]$ being in $R_{a,0}$ a.e. in E and connecting conditions (1) and (2) to $S[f]$ being in $F_{a,\varepsilon}$ for $\varepsilon > 0$.

If $a=1$, then $f(x+t)=f(x)+f'(x)t+o(t)$ already implies that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{f(x+t)+f(x-t)-2f(x)}{t^2}\,dt$$

exists, and the theorem reduces to a well-known classical result. On the other hand, Salem and Zygmund in [8] showed that $f(x+t)=f(x)+$ $+o(|t|^a)$, $0<a<1$, no longer implies that $\sum_{0}^{\infty}n^a(a_n\cos nx+b_n\sin nx)$ is summable (C, a). However, if we assume in addition that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{f(x+t)+f(x-t)-2f(x)}{t^{1+a}}\,dt$$

exists, which is closely related to the Fourier series of $f(x)$ being in $F_{a,\varepsilon}$, $\varepsilon>0$, then the theorem once again becomes valid in the fractional case.

References

[1] A. F. Andersen, *Summation of ikke hel Orden*, Mat. Tidsskrift B (1946), p. 33–52.
[2] — *Om Differenstransformationer*, ibidem (1950), p. 110–122.
[3] M. Barsky, *On extending the theory of Cesàro summability*, Ann. Polon. Math. 22 (1970), p. 179–200.
[4] — *A summability theorem in Fourier series*, Duke Math. J. 35 (1968), p. 117–122.
[5] G. L. Isaacs, *An iteration formula for fractional differences*, Proc. London Math. Soc. 13 (1963), p. 430–460.
[6] S. Izumi, *Einiges über die Differenz Operation in der Theorie der unendlichen Reihen*, Tohoku Math. J. 27 (1926), p. 324–331.
[7] B. Kuttner, *On differences of fractional order*, Proc. London Math. Soc. 7 (1957), p. 453–466.
[8] R. Salem and A. Zygmund, *Capacity of sets and Fourier series*, Trans. Amer. Math. Soc. 59 (1946), p. 23–41.
[9] A. Zygmund, *Trigonometric series*, Vols. I and II, second edition, Cambridge 1959.

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