

## An invariance method for constructing fundamental solutions for $P(\square_{mn})$

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**Abstract.** Fundamental solutions are constructed for the operator  $P(\square_{mn})$ , where  $P$  is an arbitrary polynomial and  $\square_{mn}$  is the ultrahyperbolic operator in  $\mathbb{R}^{m+n}$ . The solutions are series of distributions of the form  $s_{\pm}^l \ln^h s_{\pm} [F_{\star} \varphi]$ ,  $\delta_0^k [F_{\star} \varphi]$ ,  $\varphi \in C_0^{\infty}(\mathbb{R}^{m+n})$ , where  $F_{\star} \varphi$  is the push-forward operation (integration over the level sets) for the function  $F = |x|^2 - |y|^2$ ,  $(x, y) \in \mathbb{R}^{m+n}$ . The method consists in establishing asymptotic expansions at zero for the functions of the form  $F_{\star} \varphi$  and performing computations in the dual space of those functions.

**Introduction.** This paper is a completion and simplification of our previous paper [4]. It covers all cases, including those omitted in [4]. The paper is divided into four sections. Section 1 is devoted to proving that for Schwartz test functions  $\varphi$  in  $\mathbb{R}^{n+m}$ ,  $F_{\star} \varphi$ , for  $F = |x|^2 + |y|^2$ ,  $(x, y) \in \mathbb{R}^{n+m}$ , equals asymptotically (see (39), (20))

$$(i) \quad \sum_{h=0}^1 \sum_{j=0}^{\infty} (Y(s) a_{jh}^+ s^{\mu+j/2} \ln^h s + Y(-s) a_{jh}^- |s|^{\mu+j/2} \ln^h |s|), \quad \mu = \frac{1}{2}(m+n) - 1.$$

We present a simple and complete proof of the asymptotic expansion (i) (different from the general method given in [3]; also cf. Tengstrand [7]). In Section 2 we establish the  $F$ -invariance of the operator  $P(\square_{mn})$  and state that under the notation of Section 1

$$(ii) \quad L_r^r(F_{\star} D) \subset F_{\star} D.$$

In Section 3 we find a fundamental solution of the operator  $(\square_{mn})^r$  in the form  $s_{\pm}^{l_0} \ln^h s_{\pm} [F_{\star} \varphi]$  or  $\delta_0^{\mu-r} [F_{\star} \varphi]$  <sup>(1)</sup>. In Section 4 we represent a fundamental solution of  $P(\square_{mn})$  as a series of distributions of the form  $s_{\pm}^l \ln^h s_{\pm} [F_{\star} \varphi]$  and  $\delta^{(k)} [F_{\star} \varphi]$ , the leading term being the fundamental solution of  $\square_{mn}^r$  constructed in Section 3. The method consists in solving the equation

$$(iii) \quad L_r E = a_{jh}^{\pm}$$

<sup>(1)</sup> Concerning the fundamental solution for  $(\square_{mn})^r$  see also [1].

for certain  $j$  and  $h$  in the dual space of smooth functions (outside zero) having asymptotic expansion of the form (i). That (iii) is always solvable is due to inclusion (ii). For a general discussion see [6].

**Notation and definitions.**  $\mathbb{R}^m$  will denote the  $m$ -dimensional Euclidean space.  $N_0$  stands for the set of non-negative integers,  $N$  — for the set of positive integers. We apply the notation commonly used in the theory of distributions and of differential operators. In particular,  $C_0^k(\Omega)$  stands for the set of compactly supported  $C^k$  ( $0 \leq k \leq \infty$ ) functions with support in an open set  $\Omega \subset \mathbb{R}^m$ . The value of a distribution  $u$  on a test function  $\varphi \in C_0^\infty(\Omega)$  will be written as  $u[\varphi]$ . By  $\delta$  we denote the Dirac measure at zero,  $Y$  is the Heaviside function and  $\check{Y}(s) = Y(-s)$ .

In this paper we assume  $m \geq 1$ ,  $n \geq 1$  and put

$$a = \frac{1}{2}(m-2), \quad b = \frac{1}{2}(n-2), \quad \mu = a+b+1 = \frac{1}{2}(m+n-2).$$

By  $S_k$  we denote the set  $S_k = \{(\xi_1, \dots, \xi_k): \xi_1^2 + \dots + \xi_k^2 = 1\}$ ,  $\omega_k$  is the Lebesgue measure on this surface and  $|S_k| = \int_{S_k} d\omega_k$ . If  $k=1$ , we set  $S_1 = \{+1, -1\}$ ,  $\omega_1 = \delta_1 + \delta_{-1}$ , and consequently  $|S_1| = 2$ .

We denote by  $F$  the function

$$F(x, y) = |x|^2 - |y|^2 = x_1^2 + \dots + x_m^2 - (y_1^2 + \dots + y_n^2),$$

playing a fundamental role in the study of the operator  $\square_{mn}$  (and its iterations).

**1. The operation of averaging and its properties.** We shall relate to the function  $F$  a linear operation  $F_*$  (operation  $K$  in [3] and [4]) called the operation of averaging. The name is motivated by condition (2), which appears in Lemma 1, in which the existence of the operation  $F_*$  and some of its simple but important properties are established. The proof of Lemma 1 differs from the proof of an analogous lemma in [3]. Also the continuity of the operation  $F_*: D(\mathbb{R}^{m+n}) \rightarrow D^k(\mathbb{R}^1)$  will be established in another way than in [3] (cf. Lemmas 3 and 4 with Lemmas 2 and 4 from [3]).

LEMMA 1. *There exists a unique linear operation  $F_*$*

$$(1) \quad C_0^\infty(\mathbb{R}^{m+n}) \ni \varphi \mapsto F_* \varphi \in C^\infty(\mathbb{R}^1 \setminus \{0\})$$

*such that for every function  $f \in C^0(\mathbb{R}^1)$  and  $\varphi \in C_0^0(\mathbb{R}^{m+n})$*

$$(2) \quad \int_{\mathbb{R}^{m+n}} (f \circ F)(x, y) \varphi(x, y) d(x, y) = \int_{-\infty}^{+\infty} f(s) (F_* \varphi)(s) ds;$$

*supp  $F_* \varphi$  is bounded; moreover, if  $\lim_{j \rightarrow \infty} \varphi_j = 0$  in  $D(\mathbb{R}^{m+n})$ , then  $F_* \varphi_j$  ( $j = 1, 2, \dots$ ) have commonly bounded supports and, for every  $r > 0$  and  $p \in N_0$ ,  $\lim_{j \rightarrow \infty} D^p F_* \varphi_j = 0$  exists uniformly outside the interval  $[-r, r]$ .*

Proof. First we prove the uniqueness of  $F_*$ . If  $L_*$  were another linear operation satisfying (1) and (2), then we would have

$$\int_{-\infty}^{+\infty} f(s)(F_* \varphi)(s) ds = \int_{-\infty}^{+\infty} f(s)(L_* \varphi)(s) ds \quad \text{for every } f \in C^0(\mathbf{R}^1).$$

Therefore, on account of the regularity of  $F_*$ ,  $L_*$ , we infer from the above that  $(F_* \varphi)(s) = (L_* \varphi)(s)$  for  $s \in \mathbf{R}^1 \setminus \{0\}$ . Now we shall establish the existence of the operation  $F_*$ . To this aim choose arbitrarily functions  $f \in C^0(\mathbf{R}^1)$  and  $\varphi \in C_0^\infty(\mathbf{R}^{m+n})$ . Introducing the homogeneous coordinates in  $\mathbf{R}^m$  and  $\mathbf{R}^n$

$$\mathbf{R}_+^1 \times S_m \ni (u, \xi) \mapsto u \cdot \xi \in \mathbf{R}^m \setminus \{0\}, \quad \mathbf{R}_+^1 \times S_n \ni (v, \eta) \mapsto v \cdot \eta \in \mathbf{R}^n \setminus \{0\},$$

we conclude that

$$(3) \quad \int_{\mathbf{R}^{m+n}} (f \circ F)(x, y) \varphi(x, y) d(x, y) = \int_0^\infty \int_0^\infty u^{m-1} v^{n-1} f(u^2 - v^2) H_\varphi(u, v) du dv,$$

where

$$(4) \quad H_\varphi(u, v) = \int_{S_m} \int_{S_n} \varphi(u\xi, v\eta) d\omega_\xi d\omega_\eta.$$

Put  $a = \frac{1}{2}(m-2)$ ,  $b = \frac{1}{2}(n-2)$ . By the change of variables  $t = u^2$ ,  $s = u^2 - v^2$  in the integral on the right in (3) we conclude that

$$(5) \quad \int_{\mathbf{R}^{m+n}} (f \circ F)(x, y) \varphi(x, y) d(x, y) = \frac{1}{4} \int_{-\infty}^{+\infty} f(s) \left\{ \int_{\max(s, 0)}^\infty t^a (t-s)^b H_\varphi(t^{1/2}, (t-s)^{1/2}) dt \right\} ds.$$

Write

$$(6) \quad (F_* \varphi)(s) = \frac{1}{4} \int_{\max(s, 0)}^\infty t^a (t-s)^b H_\varphi(t^{1/2}, (t-s)^{1/2}) dt.$$

Hence (5) implies (2). Denote by the same symbol  $H_\varphi$  the extension of function (4) to all  $u \in \mathbf{R}^1$ ,  $v \in \mathbf{R}^1$  given by the same formula (4). It is clear that the function  $H_\varphi$  so obtained is an even function with compact support. Moreover,  $H_\varphi \in C_0^\infty(\mathbf{R}^2)$  if  $\varphi \in C_0^\infty(\mathbf{R}^{m+n})$ . Write

$$(7) \quad h_\varphi(u, v) = H_\varphi(\sqrt{u}, \sqrt{v}) = \int_{S_m} \left\{ \int_{S_n} \varphi(\sqrt{u}\xi, \sqrt{v}\eta) d\omega_\xi \right\} d\omega_\eta \quad \text{for } u, v \geq 0$$

and observe that by the properties of  $H_\varphi$ , the function  $h_\varphi$  is smooth<sup>(2)</sup> and has compact support. From (6) and (7) we infer that

$$(8) \quad (F_* \varphi)(s) = \frac{1}{4} \int_{\max(s, 0)}^\infty t^a (t-s)^b h_\varphi(t, t-s) dt.$$

<sup>(2)</sup> We omit the proof, analogous to that of the smoothness of the function  $\Phi_\varphi$  given in Lemma 1 in [5] (cf. also [7], Lemma 4.1).

It is easy to see that  $F_* \varphi$  has compact support and that  $F_* \varphi \in C^\infty(\mathbb{R}^1_-)$ . To show that  $F_* \varphi \in C^\infty(\mathbb{R}^1_+)$  <sup>(3)</sup> choose arbitrarily a number  $p \in \mathbb{N}$ . Since  $b \geq -\frac{1}{2}$ , integrating the last integral by parts  $p$ -times we arrive at the formula

$$(9) \quad (F_* \varphi)(s) = \frac{1}{4(b+1) \dots (b+p)} \int_s^\infty (t-s)^{b+p} \frac{d^p}{dt^p} (t^a h_\varphi(t, t-s)) dt,$$

from which it is easily seen that  $F_* \varphi \in C^p(\mathbb{R}^1_+)$ . Thus we have shown assertion (1).

Take a sequence  $\{\varphi_j\}$  with  $\lim_{j \rightarrow \infty} \varphi_j = 0$  in  $D(\mathbb{R}^{m+n})$ . It follows from (7) and (8) that  $F_* \varphi_j$  ( $j = 1, 2, \dots$ ) have commonly bounded supports. Choose arbitrarily  $p \in \mathbb{N}_0$  and  $r > 0$ . Then it is easily seen from (8) that  $\lim_{j \rightarrow \infty} D^p F_* \varphi_j = 0$  exists uniformly for  $s \leq -r$ . The convergence for  $s > 0$  can be deduced from formulas (9) written for  $\varphi_j$  ( $j = 1, 2, \dots$ ) instead of  $\varphi$ . This ends the proof of Lemma 1.

In order to investigate the regularity of  $F_* \varphi$  at zero fix a function  $\tilde{\chi} \in C_0^\infty(\mathbb{R}^{m+n})$ ,  $\tilde{\chi}(u\xi, v\eta) = 1$  for  $\xi \in S_m$ ,  $\eta \in S_n$ ,  $0 \leq u \leq 2$ ,  $0 \leq v \leq 2$ . Denote by  $W_k \varphi$  the Taylor polynomial for  $\varphi \in C_0^\infty(\mathbb{R}^{m+n})$  of degree  $k$  at zero and put

$$(10) \quad g = \varphi - \tilde{\chi} W_{k\varphi}, \quad g_{\alpha\beta}(x, y) = D_x^\alpha D_y^\beta g(x, y) \quad \text{for } (x, y) \in \mathbb{R}^{m+n}.$$

Then

$$(11) \quad g \in C_0^\infty(\mathbb{R}^{m+n}), \quad g_{\alpha\beta}(0, 0) = 0 \quad \text{if } |\alpha| + |\beta| \leq k, \\ (F_* \varphi)(s) = (F_* g)(s) + F_*(\tilde{\chi} W_{k\varphi})(s).$$

We begin to investigate the regularity of the function

$$F_*(\tilde{\chi} W_{k\varphi}) = \sum_{0 \leq |\alpha| + |\beta| \leq k} \frac{1}{\alpha! \beta!} (D_x^\alpha D_y^\beta \varphi)(0) F_*(\tilde{\chi} x^\alpha y^\beta).$$

Put

$$\psi(x, y) = \tilde{\chi}(x, y) x^\alpha y^\beta \quad \text{for } (x, y) \in \mathbb{R}^{m+n}$$

and retain the notation  $h_\psi$ ,  $F_* \psi$  for the corresponding functions defined by (7) and (8), respectively.

By Lemma 1 support  $F_* \psi$  is bounded, depends only on the support of  $\tilde{\chi}$  and  $F_* \psi \in C^\infty(\mathbb{R}^1 \setminus \{0\})$ .

Write

$$\lambda_{\alpha\beta}(t, t-s) = \int_{S_m} \int_{S_n} \xi^\alpha \eta^\beta \tilde{\chi}(\sqrt{t} \xi, \sqrt{t-s} \eta) d\omega_n \} d\omega_m$$

$$\text{for } t \geq \max(s, 0), \quad i = |\alpha|, \quad j = |\beta|, \quad a_i = a + \frac{1}{2}i, \quad b_j = b + \frac{1}{2}j.$$

<sup>(3)</sup> This is evident if  $b \in \mathbb{N}_0$ .

Then

$$h_{\psi}(t, t-s) = t^{i/2} (t-s)^{j/2} \lambda_{\alpha\beta}(t, t-s),$$

$$(F_{\star} \psi)(s) = \frac{1}{4} \int_{\max(s, 0)}^{\infty} t^{a_i} (t-s)^{b_j} \lambda_{\alpha\beta}(t, t-s) dt.$$

We examine  $F_{\star} \psi$  in  $U = \{s: |s| < 1\}$ . To this end write

$$(F_{\star} \psi)(s) = \frac{1}{4} \int_{\max(s, 0)}^1 t^{a_i} (t-s)^{b_j} \lambda_{\alpha\beta}(t, t-s) dt + \frac{1}{4} \int_1^{\infty} t^{a_i} (t-s)^{b_j} \lambda_{\alpha\beta}(t, t-s) dt,$$

$$C_{\alpha\beta} = \int_{S_m} \left\{ \int_{S_n} \xi^{\alpha} \eta^{\beta} d\omega_n \right\} d\omega_m, \quad h_{ij}(s) = \int_{\max(s, 0)}^1 t^{a_i} (t-s)^{b_j} dt \quad \text{for } 0 < |s| < 1.$$

It follows from the choice of the function  $\tilde{\chi}$  that

$$\lambda_{\alpha\beta}(t, t-s) = C_{\alpha\beta} \quad \text{where } 0 \leq t \leq 1, s \in U,$$

and therefore

$$(12) \quad (F_{\star} \psi)(s) = \frac{1}{4} C_{\alpha\beta} h_{ij}(s) + \frac{1}{4} \int_1^{\infty} t^{a_i} (t-s)^{b_j} \lambda_{\alpha\beta}(t, t-s) dt \quad \text{for } s \in U.$$

Let  $\chi$  be an arbitrary function in  $C_0^{\infty}(\mathbf{R}^1)$  equal to 1 for  $|s| < \frac{1}{4}$  and zero for  $|s| > \frac{1}{2}$ . From (12) and the definition of the function  $\psi$  we get

$$(13) \quad F_{\star}(\tilde{\chi} x^{\alpha} y^{\beta})(s) = \chi(s) F_{\star}(\tilde{\chi} x^{\alpha} y^{\beta})(s) + (1 - \chi(s)) F_{\star}(\tilde{\chi} x^{\alpha} y^{\beta})(s)$$

$$= \frac{1}{4} C_{\alpha\beta} \chi(s) h_{ij}(s) + \frac{1}{4} \chi(s) \int_1^{\infty} t^{a_i} (t-s)^{b_j} \lambda_{\alpha\beta}(t, t-s) dt +$$

$$+ (1 - \chi(s)) F_{\star}(\tilde{\chi} x^{\alpha} y^{\beta})(s).$$

Denote

$$(14) \quad \delta_{\alpha\beta}(s) = \frac{1}{4} \chi(s) \int_1^{\infty} t^{a_i} (t-s)^{b_j} \lambda_{\alpha\beta}(t, t-s) dt + (1 - \chi(s)) F_{\star}(\tilde{\chi} x^{\alpha} y^{\beta})(s).$$

It is clear that  $\delta_{\alpha\beta} \in C_0^{\infty}(\mathbf{R}^1)$  and its support is contained in a compact  $K$  depending only on the supports of  $\chi$  and  $\tilde{\chi}$ . From (13) and (14) we get<sup>(4)</sup>

$$F_{\star}(\tilde{\chi} x^{\alpha} y^{\beta})(s) = \frac{1}{4} \chi(s) C_{\alpha\beta} h_{ij}(s) + \delta_{\alpha\beta}(s) \quad \text{for } s \in \mathbf{R}^1.$$

Hence

$$(15) \quad F_{\star}(\tilde{\chi} W_{k\varphi})(s) = \sum_{0 \leq i+j \leq k} A_{ij}(\varphi) \chi(s) h_{ij}(s) + \delta_k(\varphi, s),$$

where

$$(16) \quad A_{ij}(\varphi) = \frac{1}{4} \sum_{|\alpha|=i} \sum_{|\beta|=j} \frac{1}{\alpha!} \frac{1}{\beta!} (D_x^{\alpha} D_y^{\beta} \varphi)(0) C_{\alpha\beta},$$

<sup>(4)</sup> We put  $\chi(s) h_{ij}(s) = 0$  for  $|s| \geq 1$ .

and

$$\delta_k(\varphi, s) = \sum_{0 \leq |\alpha| + |\beta| \leq k} \frac{1}{\alpha!} \frac{1}{\beta!} (D_x^\alpha D_y^\beta \varphi)(0) \delta_{\alpha\beta}(s)$$

is a smooth function and  $\text{supp } \delta_k(\varphi, \cdot) \subset K$  independently of the choice of the function  $\varphi$ . Clearly

$$(17) \quad \delta_k(\varphi, \cdot) \in C_0^\lambda(\mathbf{R}^1), \quad A_{00}(\varphi) = \frac{1}{4} |S_m| |S_n| \varphi(0).$$

It is seen from (15) that in order to study the regularity of  $F_*(\tilde{\chi} W_{k\varphi})$  it is enough to investigate the functions  $h_{ij}$  for  $i, j \in N_0$ , and  $|s| < 1$ . For this purpose we shall prove the following

LEMMA 2. Assume  $\lambda = -\frac{1}{2} + \frac{1}{2}i$ ,  $v = -\frac{1}{2} + \frac{1}{2}j$ ,  $i, j \in N_0$  and let  $h(s) = \int_{\max(s, 0)}^1 t^\lambda (t-s)^v dt$  for  $s \neq 0$ ,  $|s| < 1$ . Then there exists a function  $Jh$  such that

$$(18) \quad h - Jh \text{ extends to a smooth function on } (-1, 1).$$

More precisely, if:

(i)  $v \in N_0$ , then  $Jh(s) = A(\lambda, v) Y(s) s^{\lambda+v+1}$ , where

$$A(\lambda, v) = (-1)^{v+1} \Gamma(\lambda+1) \Gamma(v+1) / \Gamma(\lambda+v+2);$$

(ii)  $\lambda \in N_0$ ,  $v \notin N_0$ , then  $Jh(s) = A(\lambda, v) Y(-s) (-s)^{\lambda+v+1}$ , where

$$A(\lambda, v) = (-1)^{\lambda+1} \Gamma(\lambda+1) \Gamma(v+1) / \Gamma(\lambda+v+2);$$

(iii)  $\lambda \notin N_0$ ,  $v \notin N_0$ , then  $Jh(s) = A(\lambda, v) s^{\lambda+v+1} \ln |s| + C(\lambda, v) Y(s) s^{\lambda+v+1}$ , where

$$(19) \quad A(\lambda, v) = (-1)^{\lambda+v} \binom{v}{\lambda+v+1}.$$

The value of  $C(\lambda, v)$  is of no importance for us.

Proof of Lemma 2. ad (i). Suppose  $v \in N_0$  and define the polynomial  $p(s) = \int_0^1 t^\lambda (t-s)^v dt$  for  $s \in \mathbf{R}^1$ . Note that  $h(s) = p(s)$  for  $s < 0$  and that for  $s > 0$  we have

$$h(s) = p(s) - \int_0^s t^\lambda (t-s)^v dt = p(s) + (-1)^{v+1} \frac{\Gamma(\lambda+1) \Gamma(v+1)}{\Gamma(\lambda+v+2)} s^{\lambda+v+1}.$$

Hence follows assertion (i).

ad (ii). Put  $\tilde{p}(s) = \int_s^1 t^\lambda (t-s)^v dt$  for  $s \in (-1, 1)$ . By the change of variables  $t-s = x$  it is easily seen that  $\tilde{p} \in C^\infty((-1, 1))$ . Moreover,  $h(s) = \tilde{p}(s)$  for  $s > 0$

and  $h(s) = \tilde{p}(s) - \int_s^0 t^\lambda (t-s)^\nu dt = \tilde{p}(s) + (-1)^\lambda (-s)^{\lambda+\nu+1} \Gamma(\lambda+1) \Gamma(\nu+1) / \Gamma(\lambda+\nu+2)$  for  $s < 0$ . Hence follows our assertion.

ad (iii). In this case  $\lambda + \nu \in N_0$ . Note that for  $s > 0$ :

$$\begin{aligned} h(s) &= \int_s^1 t^\lambda (t-s)^\nu dt = \int_s^1 t^{\lambda+\nu} (1-s/t)^\nu dt = s^{\lambda+\nu+1} \int_s^1 ((1-x)^\nu / x^{\lambda+\nu+2}) dx \\ &= s^{\lambda+\nu+1} \lim_{\varepsilon \rightarrow 0} \int_s^{1-\varepsilon} (1/x^{\lambda+\nu+2}) \sum_{n=0}^{\infty} \binom{\nu}{n} (-x)^n dx \\ &= A(\lambda, \nu) s^{\lambda+\nu+1} \ln s + \lim_{\varepsilon \rightarrow 0} f(1-\varepsilon) s^{\lambda+\nu+1} - f(s) s^{\lambda+\nu+1}, \end{aligned}$$

where  $A(\lambda, \nu)$  is defined by (19) and

$$f(s) = \sum_{\substack{n=0 \\ n \neq \lambda+\nu+1}}^{\infty} \binom{\nu}{n} \frac{(-1)^n s^{n-\lambda-\nu-1}}{n-\lambda-\nu-1}.$$

In an analogous way we prove that for  $s < 0$

$$\begin{aligned} h(s) &= A(\lambda, \nu) s^{\lambda+\nu+1} \ln |s| - f(s) s^{\lambda+\nu+1} + \\ &\quad + \left( \lim_{\varepsilon \rightarrow 0} f(\varepsilon-1) + (-1)^{\lambda+\nu+1} \int_1^{\infty} ((1+x)^\nu / x^{\lambda+\nu+2}) dx \right) s^{\lambda+\nu+1} \end{aligned}$$

and therefore (iii) follows.

Denote

$$\begin{aligned} (20) \quad \gamma_{ij}(s) &= A(a_i, b_j) Y(s) s^{a_i+b_j+1} && \text{if } b_j \in N_0, \\ &= A(a_i, b_j) Y(-s) |s|^{a_i+b_j+1} && \text{if } a_i \in N_0, b_j \notin N_0, \\ &= (A(a_i, b_j) \ln |s| + C(a_i, b_j) Y(s)) s^{a_i+b_j+1} && \text{if } a_i, b_j \notin N_0 \end{aligned}$$

with constants  $A, C$  defined in Lemma 2. Write

$$\begin{aligned} (21) \quad \mu &= \mu_{00} = a+b+1, \quad \mu_{ij} = a_i+b_j+1 \quad (i, j = 1, 2, \dots), \\ \tilde{\mu} &= [\mu] \quad \text{if } \mu > [\mu], \quad \tilde{\mu} = \mu - 1 \quad \text{if } \mu = [\mu]. \end{aligned}$$

It is easily seen that  $\gamma_{ij} \in C^k(\mathbb{R}^1) \setminus C^{k+1}(\mathbb{R}^1)$  if  $k < \mu_{ij} \leq k+1$  and consequently  $\gamma_{ij} \in C^{\tilde{\mu}}$  ( $C^{-1} \stackrel{\text{df}}{=} \ln s \cdot C^0$  with the topology induced from  $C^0$ ).

Take any sequence  $\{\varphi_p\}$  such that

$$(22) \quad \lim_{p \rightarrow \infty} \varphi_p = 0 \quad \text{in } D(\mathbb{R}^{m+n}).$$

Therefore

$$(23) \quad \lim_{p \rightarrow \infty} A_{ij}(\varphi_p) = 0 \quad (0 \leq i+j \leq k)$$

and

$$(24) \quad \lim_{p \rightarrow \infty} \delta_k(\varphi_p, \cdot) = 0 \quad \text{in } D(\mathbb{R}^1)$$

because the supports of  $\delta_k(\varphi_p)$  ( $p = 1, 2, \dots$ ) are contained in a compact set containing the supports of the functions  $\delta_{\alpha\beta}$  ( $0 \leq |\alpha| + |\beta| \leq k$ ). Note that by (23) and (18)

$$(25) \quad \lim_{p \rightarrow \infty} \chi(s) \sum_{0 \leq i+j \leq k} A_{ij}(\varphi_p)(h_{ij} - \gamma_{ij})(s) = 0 \quad \text{in } D(\mathbb{R}^1)$$

for every  $\chi \in C_0^\infty(\mathbb{R}^1)$ .

Now we pass to the study of  $F_* g$  and of the sequence  $\{F_* g_p\}$ , where  $g, g_p$  ( $p = 1, 2, \dots$ ) are defined by (10) with  $\varphi \in D(\mathbb{R}^{m+n})$  and  $\varphi_p \in D(\mathbb{R}^{m+n})$  ( $p = 1, 2, \dots$ ), respectively. Our aim is to prove that

$$(26) \quad F_* g \in C_0^k(\mathbb{R}^1), \quad \lim_{p \rightarrow \infty} F_* g_p = 0 \quad \text{in } D^k(\mathbb{R}^1) \quad \text{if} \quad \lim_{p \rightarrow \infty} \varphi_p = 0 \quad \text{in } D(\mathbb{R}^{m+n}).$$

By Lemma 1 it is enough to show that

$$(27) \quad F_* g \in C^k(U), \quad \lim_{p \rightarrow \infty} (F_* g_p)^{(q)}(s) = 0 \quad \text{uniformly on } U,$$

where  $U = \{s: |s| < 1\}$  ( $q = 0, 1, \dots, k$ ).

For the proof observe first that the function  $h_g$  given by (7) is a smooth function with compact support and that by (11) we have

$$(28) \quad (h_g)_{\alpha\beta}(0, 0) = D_u^\alpha D_v^\beta(h_g)(0, 0) = 0 \quad \text{for } |\alpha| + |\beta| \leq k.$$

Write

$$(29) \quad (F_* g)(s) = \frac{1}{4} I(s) + \frac{1}{4} w(s),$$

where

$$I(s) = \int_{\max(s, 0)}^1 h_g(t, t-s) t^a (t-s)^b dt, \quad w(s) = \int_1^\infty h_g(t, t-s) t^a (t-s)^b dt.$$

Note that  $w \in C^\infty(U)$ . By (8) we have

$$(F_* g_p)(s) = \frac{1}{4} \int_{\max(s, 0)}^\infty t^a (t-s)^b h_{g_p}(t, t-s) dt \quad (p = 1, 2, \dots),$$

where

$$h_{g_p}(t, t-s) = \int_{S_n} \left\{ \int_{S_m} g_p(\sqrt{t} \xi, \sqrt{t-s} \eta) d\omega_\xi \right\} d\omega_\eta.$$

It is easy to see that for every  $q, r \in N_0$  (22) implies

$$(30) \quad \lim_{p \rightarrow \infty} D_u^q D_v^r h_{g_p}(u, v) = 0 \quad \text{uniformly for } u, v \geq 0.$$



Moreover, there exists a number  $\varrho$ ,  $0 < \varrho < \infty$  such that  $\text{supp } F_* g_p \subset [-\varrho, \varrho]$  ( $p = 1, 2, \dots$ ). From (30) and formula (29), this time involving  $g_p$ ,  $I_p$ ,  $w_p$  instead of  $g$ ,  $I$ ,  $w$ , we infer that  $\lim_{p \rightarrow \infty} D^a w_p = 0$  uniformly on  $U$  ( $q = 0, 1, \dots, k$ ). From (28) and the Taylor formula in the integral form we obtain

$$I(s) = \sum_{|\beta| + |\gamma| = k+1} \int_{\max(s, 0)}^1 t^{a+|\beta|} (t-s)^{b+|\gamma|} A_{\beta\gamma}(t, t-s) dt,$$

where  $A_{\beta\gamma}(t, t-s) = (k+1) \int_0^1 D_x^\beta D_y^\gamma h_\theta(t\tau, (t-s)\tau) (1-\tau)^k d\tau$  are smooth functions. To end the proof of (27) it remains to show that  $I \in C^k(U)$  when  $a, b > -1$  and that (22) implies the relations

$$\lim_{p \rightarrow \infty} D^a I_p = 0 \quad \text{uniformly on } U \quad (q = 0, 1, \dots, k).$$

This results from Lemma 3 below. First we fix some notation.

Let  $\lambda \geq -\frac{1}{2}$ ,  $\nu \geq -\frac{1}{2}$  and denote by  $C^0(Q)$  the space of continuous functions  $A$  defined on  $Q = [0, \infty) \times [0, \infty)$ . We shall denote by  $h_{\lambda\nu}(A)$  the function given by

$$h_{\lambda\nu}(A)(s) = \int_{\max(s, 0)}^1 t^\lambda (t-s)^\nu A(t, t-s) dt \quad \text{for } s \in U.$$

**Remark 1.** 1° If  $\lambda + \nu > -1$ , then the function  $h_{\lambda\nu}(A)$  is continuous on  $U$  independently of the choice of the function  $A \in C^0(Q)$ .

2° If  $\{A^j\}$  is a sequence of functions  $A^j \in C^0(Q)$  ( $j = 1, 2, \dots$ ) uniformly convergent to zero on  $Q$ , then  $\lim_{j \rightarrow \infty} h_{\lambda\nu}(A^j) = 0$  uniformly on  $U$ .

3° If  $h_{\lambda\nu}(A) \in C^m(U)$  for  $A \in C^0(Q)$ , then<sup>(5)</sup>  $h_{\lambda, \nu+1}(A) \in C^m(U)$  for  $A \in C^0(Q)$ .

**LEMMA 3.** Let

$$(31) \quad \lambda = -\frac{1}{2} + \frac{1}{2}i \quad (i = 0, 1, \dots),$$

$$\nu = -\frac{1}{2} + \frac{1}{2}j \quad (j = 0, 1, \dots), \quad \lambda + \nu \geq k.$$

Then  $h_{\lambda\nu}(A) \in C^k(U)$ . If  $\{A^j\}$  is a sequence of functions  $A^j \in C^0(Q)$  ( $j = 1, 2, \dots$ ) such that for every  $q, r \in N_0$

$$(32) \quad \lim_{j \rightarrow \infty} D_u^q D_v^r A^j(u, v) = 0 \quad \text{uniformly on } Q,$$

then

$$(33) \quad \lim_{j \rightarrow \infty} D^a h_{\lambda\nu}(A^j) = 0 \quad \text{uniformly on } U \quad (q = 0, 1, \dots, k).$$

<sup>(5)</sup>  $h_{\lambda, \nu+1}(A) = h_{\lambda\nu}(A_1)$ , where  $A_1(u, v) = vA(u, v)$ .

**Proof.** We first consider two cases,  $v = 0$  and  $v = -\frac{1}{2}$ . By Remark 1,  $h_{\lambda 0}(\Lambda) \in C^0(U)$  and  $\lim_{j \rightarrow \infty} h_{\lambda 0}(\Lambda^j) = 0$  uniformly on  $U$ . If  $\lambda > m-1$ ,  $m \in \mathbb{N}$ , it is easy to show<sup>(6)</sup> that

(i) there exists a smooth function  $f_m(\Lambda)$  such that

$$(34) \quad \frac{d^m h_{\lambda 0}(\Lambda)}{ds^m}(s) - s^{\lambda-m+1} (f_m(\Lambda))(s) Y(s) = (-1)^m \left( h_{\lambda 0} \left( \frac{\partial^m \Lambda(t, v)}{\partial v^m} \right) \Big|_{v=t-s} \right)(s).$$

(ii) If  $\{\Lambda^j\}$  is a sequence satisfying (32), then for the corresponding sequence  $f_m(\Lambda^j)$  we have

$$(35) \quad \lim_{j \rightarrow \infty} f_m(\Lambda^j) = 0 \quad \text{uniformly on } [0, 1].$$

Observe that the function  $U \ni s \mapsto s^{\lambda-m+1} Y(s)$  is continuous, whence by (i) and Remark 1

$$(36) \quad h_{\lambda 0}(\Lambda) \in C^m(U) \quad \text{if } \lambda > m-1.$$

Moreover, by (34)

$$\lim_{j \rightarrow \infty} h_{\lambda 0} \left( \frac{\partial^m \Lambda^j}{\partial v^m} \Big|_{v=t-s} \right) = 0 \quad \text{uniformly on } [0, 1].$$

From (34) with  $\Lambda^j$  instead of  $\Lambda$  and from (35), (36) follows at once the last assertion of Lemma 3 in the case  $v = 0$ .

We now pass to the second case,  $v = -\frac{1}{2}$ . By Remark 1 the function  $h_{\lambda, -1/2}(\Lambda)$  is continuous if  $\lambda > 0$  and  $\lim_{j \rightarrow \infty} h_{\lambda, -1/2}(\Lambda^j) = 0$  uniformly on  $U$ . Suppose that  $\lambda > 1$ . By the change of variables  $t = s + \tau$  we can write

$$\begin{aligned} (h_{\lambda, -1/2}(\Lambda))(s) &= \int_0^{1-s} (s+\tau)^\lambda \tau^{-1/2} \Lambda(s+\tau, \tau) d\tau \quad \text{for } s > 0, \\ &= \int_{-s}^{1-s} (s+\tau)^\lambda \tau^{-1/2} \Lambda(s+\tau, \tau) d\tau \quad \text{for } s < 0. \end{aligned}$$

Differentiating the last formula with respect to  $s$  and then going back to the initial integral variable  $t$ , we obtain

$$(37) \quad \frac{d}{ds} (h_{\lambda, -1/2}(\Lambda))(s) = -(1-s)^{-1/2} \Lambda(1, 1-s) + \lambda (h_{\lambda-1, -1/2}(\Lambda))(s) + h_{\lambda, -1/2} \left( \frac{\partial \Lambda(u, t-s)}{\partial u} \Big|_{u=t} \right)(s).$$

<sup>(6)</sup> Leaving the general case to the reader, we note that  $(f_1(\Lambda))(s) = -\Lambda(s, 0)$  for  $s \geq 0$ ,  $(f_2(\Lambda))(s) = -\lambda \Lambda(s, 0) - s \frac{\partial \Lambda(s, 0)}{\partial s} + s \frac{\partial \Lambda(s, v)}{\partial v} \Big|_{v=0}$  for  $s \geq 0$ . Substituting in these formulas  $\Lambda^j$  ( $j = 1, 2, \dots$ ) for  $\Lambda$ , by (32) we get (34) for  $m = 1, 2$ .

Since  $\Lambda$  is a smooth function and  $\lambda - 1 > 0$ , the right-hand side of (37) is a continuous function. Therefore  $h_{\lambda, -1/2}(\Lambda) \in C^1(U)$  when  $\lambda > 1$ . Writing formula (37) for  $\Lambda^j$  ( $j = 1, 2, \dots$ ) instead of  $\Lambda$  because of (32), we easily see that (33) is satisfied for  $\nu = -\frac{1}{2}$ ,  $q = 1$ . Thus we have proved Lemma 3 in the case  $\nu = -\frac{1}{2}$ ,  $k = 0, 1$ . To prove it for  $\nu = -\frac{1}{2}$ ,  $k = 2$  it is enough, supposing that  $\lambda > 2$ , to differentiate (37) in the same way as we effected the first differentiation. It is clear that by induction we get Lemma 3 in the case  $\nu = -\frac{1}{2}$ .

Passing to the general case, choose an arbitrary pair of indices  $\lambda, \nu$  fulfilling conditions (31) with  $\nu \neq 0$ ,  $\nu \neq -\frac{1}{2}$ . Hence we have  $\nu > 0$  (in fact  $\nu \geq \frac{1}{2}$ ) and

$$\frac{dh_{\lambda\nu}(\Lambda)}{ds}(s) = -\nu(h_{\lambda, \nu-1}(\Lambda))(s) - \left( h_{\lambda\nu} \left( \frac{\partial \Lambda(t, v)}{\partial v} \Big|_{v=t-s} \right) \right)(s).$$

If  $\nu > 1$ , we can differentiate once more, getting

$$\begin{aligned} \frac{d^2 h_{\lambda\nu}(\Lambda)}{ds^2}(s) &= \nu(\nu-1)(h_{\lambda, \nu-2}(\Lambda))(s) + 2\nu \left( h_{\lambda, \nu-1} \left( \frac{\partial \Lambda(t, v)}{\partial v} \Big|_{v=t-s} \right) \right)(s) + \\ &\quad + \left( h_{\lambda, \nu} \left( \frac{\partial^2 \Lambda(t, v)}{\partial v^2} \Big|_{v=t-s} \right) \right)(s). \end{aligned}$$

Let  $p$  be a non-negative integer such that  $p-1 < \nu \leq p$ . An attentive look at the above formulas permits us to discover that all the derivatives up to the order  $p$  (inclusively) of the function  $h_{\lambda\nu}(\Lambda)$  are linear combinations with constant coefficients of the functions

$$(38) \quad h_{\lambda, \nu}, h_{\lambda, \nu-1}, \dots, h_{\lambda, \nu-p}$$

with the argument  $\Lambda$  in each of them replaced by an adequate derivative of  $\Lambda$  with respect to the second variable. If  $\nu = p$ , we get by (31),  $\lambda \geq k-p > k-p-1$ , and therefore (36) implies  $h_{\lambda 0}(\lambda) \in C^{k-p}$ . Now from part 3° of Remark 1 we conclude that all the functions (38) are of class  $C^{k-p}$ . Hence it follows that  $h_{\lambda p}(\Lambda) \in C^k(U)$ . If  $p > \nu > p-1$  then  $\nu-p = -\frac{1}{2}$  and by (31)  $\lambda \geq k-p+\frac{1}{2} > k-p$ . Recall that we have just proved Lemma 3 in the case  $\nu = -\frac{1}{2}$ . Hence  $h_{\lambda, -1/2}(\Lambda) \in C^{k-p}$  for  $\Lambda = C^0(Q)$ . As before, by part 3° of Remark 1, all the functions (38) are of class  $C^{k-p}$  and therefore  $h_{\lambda\nu}(\Lambda) \in C^k(U)$ .

To end the proof of Lemma 3 suppose (32). Recall that we have just proved (33) in the cases where  $\nu = 0$  and  $\nu = -\frac{1}{2}$ . Applying these results and Remark 1, we prove in the standard way assertion (33) for  $\nu > 0$ .

LEMMA 4. Let  $\Lambda_{ij}$ ,  $\gamma_{ij}$  be the functions defined by (16) and (20), respectively. In particular,

$$\Lambda_{00}(\varphi) = B_0 \varphi(0), \quad \text{where } B_0 = \frac{1}{4} |S_m| |S_n|.$$

Let  $\chi \in C_0^\infty(\mathbf{R}^1)$ ,  $\chi = 1$  for  $|s| < \frac{1}{4}$ ,  $\chi = 0$  for  $|s| > \frac{1}{2}$ . Then for every function  $\varphi \in C_0^\infty(\mathbf{R}^{m+n})$  and every  $k \in \mathbf{N}$  we have<sup>(7)</sup>

$$(39) \quad (F_* \varphi)(s) = \chi(s) \sum_{0 \leq i+j \leq 2k} A_{ij}(\varphi) \gamma_{ij}(s) + h_k(\varphi, s),$$

where  $h_k(\varphi, \cdot) \in C_0^k(\mathbf{R}^1)$  and  $\gamma_{ij} \in C^\mu$  ( $C^{-1} \stackrel{\text{df}}{=} \ln s \cdot C^0$ ) and  $\tilde{\mu}$  given in (21). Moreover, convergence (22) implies

$$(40) \quad h_k(\varphi_p, \cdot) \xrightarrow{p \rightarrow \infty} 0 \quad \text{in } D^k(\mathbf{R}^1),$$

$$(41) \quad \chi(\cdot) \sum_{0 \leq i+j \leq 2k} A_{ij}(\varphi_p) \gamma_{ij}(\cdot) \xrightarrow{p \rightarrow \infty} 0 \quad \text{in } D^{\tilde{\mu}}(\mathbf{R}^1) \quad (D^{-1}(\mathbf{R}^1) \stackrel{\text{df}}{=} \ln s \cdot D^0(\mathbf{R}^1)).$$

Consequently  $F_*$  is a continuous operation  $F_*: D(\mathbf{R}^{m+n}) \rightarrow D^{\tilde{\mu}}(\mathbf{R}^1)$ .

Proof. Note that, by (11) and (15), for every  $\varphi \in D(\mathbf{R}^{m+n})$  and every  $k \in \mathbf{N}$  we have the decomposition

$$(F_* \varphi)(s) = (F_* g)(s) + \chi(s) \sum_{0 \leq i+j \leq 2k} A_{ij}(\varphi) h_{ij}(s) + \delta_k(\varphi, s).$$

Hence follows (39) with

$$h_k(\varphi, s) = (F_* g)(s) + \delta_k(\varphi, s) + \chi(s) \sum_{0 \leq i+j \leq 2k} A_{ij}(\varphi) (h_{ij} - \gamma_{ij})(s).$$

By (26), (17) and Lemma 2,  $h_k(\varphi, \cdot) \in C_0^k(\mathbf{R}^1)$ . Suppose now (22). We see that (23) implies (41) and together with (24) and (26) proves (40). That is the end of the proof.

Applying the Taylor formula to  $h_k(\varphi, s)$ , we get

Remark 2. Let  $k \geq 2r + [\mu]$ ,  $r \in \mathbf{N}$ . Formula (39) can be written in the following form:

$$(42) \quad (F_* \varphi)(s) = \tilde{\chi}(s) \sum_{j=0}^{[\mu]+2r} b_j s^j + \chi(s) \sum_{0 \leq i+j \leq 2k} A_{ij}(\varphi) \gamma_{ij}(s) + R(s),$$

where  $\tilde{\chi} \in C_0^\infty(\mathbf{R}^1)$ ,  $\tilde{\chi} = 1$ , on some neighbourhood of zero,  $b_j$  are adequate constants and  $R \in C_0^{2r+[\mu]}(\mathbf{R}^1)$  is flat at zero up to order  $2r + [\mu]$ .

**2.  $F$ -invariance of the operator  $P(\square_{mn})$ .** Consider the operator  $P(\square_{mn}) = \sum_{j=0}^r a_j (\square_{mn})^j$ , where  $\square_{mn}^0 = \text{id}$  in  $D'(\mathbf{R}^{m+n})$ ,  $r \geq 1$  and  $a_j$  are constants with  $a_r = 1$ . For brevity we shall write  $P_r = P(\square_{mn})$ .

THEOREM 1. 1° The operator  $P_r$  of order  $2r$  is  $F$ -invariant. More precisely,

$$P_r(f \circ F) = L_r f \circ F \quad \text{for } f \in C^{2r}(\mathbf{R}^1),$$

<sup>(7)</sup> We then say that  $F_* \varphi$  equals asymptotically  $\sum_{0 \leq i+j \leq \infty} A_{ij}(\varphi) \gamma_{ij}(s)$ .

where

$$L_r = \sum_{k=0}^r a_k L^k, \quad L^0 = \text{Id} \quad \text{in } D'(\mathbf{R}^1),$$

$$L = 2(n+m) \frac{d}{ds} + 4s \frac{d^2}{ds^2}.$$

2° Let  $\tilde{\mu}$  be given by (21) and suppose that a distribution  $E \in (D^0)'(\mathbf{R}^1)$  satisfies the equation

$$(43) \quad L_r E[F_* \varphi] = C \varphi(0) \quad \text{for } \varphi \in C_0^\infty(\mathbf{R}^{m+n}),$$

where  $C \neq 0$  is a constant. Then the distribution  $u$  defined by

$$(44) \quad u[\varphi] = \frac{1}{C} E[F_* \varphi] \quad \text{for } \varphi \in D(\mathbf{R}^{m+n})$$

is a fundamental solution of the operator  $P_r$ .

**Proof.** The first part of Theorem 1 is obvious. To prove the second part denote by  $(P_r)^{\text{tr}}$   $((L_r)^{\text{tr}}, L^{\text{tr}})$  the formal transpose of the operator  $P_r$   $(L_r, L)$ . Then

$$(L_r)^{\text{tr}} = \sum_{k=0}^r a_k (L^k)^{\text{tr}} = \sum_{k=0}^r a_k (L^{\text{tr}})^k, \quad L^{\text{tr}} = 2(4-n-m) \frac{d}{ds} + 4s \frac{d^2}{ds^2}.$$

It is easy to show<sup>(8)</sup> that for every  $\varphi \in C_0^\infty(\mathbf{R}^{m+n})$

$$F_*((P_r)^{\text{tr}} \varphi)(s) = (L_r)^{\text{tr}}(F_* \varphi)(s) \quad \text{for } s \neq 0$$

and that by Lemma 4 the functional  $u$  defined by (44) belongs to  $D'(\mathbf{R}^{m+n})$ . Therefore (43) implies the last assertion of Theorem 1.

In Section 3 we construct a fundamental solution of the homogeneous operator  $P_r = \square_{mn}^r$ , and then in Section 4 we consider the general case.

**3. Fundamental solution of the operator  $\square_{mn}^r$ ,  $r \geq 1$ .** Let  $P = (\square_{mn}^r)^r$ ,  $r \geq 1$ . Observe that in this case the corresponding one-dimensional operators  $L_r = L$  and  $(L_r)^{\text{tr}} = (L)^{\text{tr}}$  are homogeneous of order  $r$ . This means that for any real number  $\lambda$

$$E(s^\lambda) = p(\lambda) s^{\lambda-r}, \quad (L^{\text{tr}})^r(s^\lambda) = w(\lambda) s^{\lambda-r},$$

<sup>(8)</sup> The proof is given in [3], p. 241. We accept the following convention: Let  $g \in C^{2r}(\mathbf{R}^1 \setminus \{0\})$ ; by  $L^{\text{tr}} g$  we denote the function given by

$$(L^{\text{tr}} g)(s) = 2(4-n-m) \frac{dg(s)}{ds} + 4s \frac{d^2 g(s)}{ds^2} \quad \text{for } s \in \mathbf{R}^1 \setminus \{0\}$$

and its continuous extension to the whole of  $\mathbf{R}^1$  (if it exists).

where

$$p(\lambda) = 4^r \lambda(\lambda-1) \dots (\lambda-r+1)(\lambda+\mu)(\lambda+\mu-1) \dots (\lambda+\mu-r+1),$$

$$w(\lambda) = 4^r \lambda(\lambda-1) \dots (\lambda-r+1)(\lambda-\mu)(\lambda-\mu-1) \dots (\lambda-\mu-r+1).$$

The polynomials  $p$  and  $w$  are called the *characteristic polynomials* of the operators  $L$  and  $(L')^r$ , respectively. Note that  $\lambda = \mu$  is a root of the polynomial  $w$  and that  $r - \mu - 1$  is a root of  $p$ .

We state without proof (which is of a technical character) the following simple properties:

**PROPOSITION 1.** Let  $\chi \in C_0^\infty(\mathbf{R}^1)$ ,  $\chi = 1$ , in a neighbourhood of zero. Then

1° there exist constants  $c_j(\lambda)$ ,  $\tilde{c}_j(\lambda)$  ( $j = 1, \dots, 2r$ ) such that

$$(45) \quad (L)^r (s^\lambda \chi(s)) = w(\lambda) s^{\lambda-r} \chi(s) + \sum_{j=1}^{2r} c_j(\lambda) s^{\lambda-r+j} \chi^{(j)}(s), \quad s > 0,$$

$$(L')^r (|s|^\lambda \chi(s)) = (-1)^r w(\lambda) |s|^{\lambda-r} \chi(s) + \sum_{j=1}^{2r} \tilde{c}_j(\lambda) |s|^{\lambda-r+j} \chi^{(j)}(s)$$

if  $s < 0$ ;

2° there exist constants  $c_{ij}$ ,  $\tilde{c}_{ij}$  ( $i = 0, 1$ ;  $j = 1, \dots, 2r$ ) such that

$$(46) \quad (L)^r (\chi(s) s^\lambda \ln s) = (w(\lambda) \ln s + w'(\lambda)) s^{\lambda-r} \chi(s) + \sum_{j=1}^{2r} (c_{0j} \ln s + c_{1j}) \chi^{(j)}(s) s^{\lambda-r+j}, \quad s > 0,$$

$$(L')^r (\chi(s) |s|^\lambda \ln |s|) = (-1)^r (w(\lambda) \ln |s| + w'(\lambda)) |s|^{\lambda-r} \chi(s) + (-1)^r \sum_{j=1}^{2r} (\tilde{c}_{0j} \ln |s| + \tilde{c}_{1j}) |s|^{\lambda-r+j} \chi^{(j)}(s), \quad s < 0.$$

**DEFINITION 1.** For  $\lambda \geq 1$  we define<sup>(9)</sup> for  $\varphi \in C_0^\infty(\mathbf{R}^1)$ :

$$\begin{aligned} s_+^{-\lambda} \ln^k s_+ [\varphi] &= Pf \int_0^\infty \frac{(\ln s)^k}{s^\lambda} \varphi(s) ds \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_\varepsilon^\infty \frac{(\ln s)^k}{s^\lambda} \varphi(s) ds + \sum_{j=0}^{[\lambda]-1} \frac{\varphi^{(j)}(0)}{j!} v_j^{\lambda,k}(\varepsilon) \right\}, \\ s_-^{-\lambda} \ln^k s_- [\varphi] &= Pf \int_{-\infty}^0 \frac{(\ln |s|)^k}{|s|^\lambda} \varphi(s) ds \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{-\infty}^{-\varepsilon} \frac{(\ln |s|)^k}{|s|^\lambda} \varphi(s) ds + \sum_{j=0}^{[\lambda]-1} (-1)^j \frac{\varphi^{(j)}(0)}{j!} v_j^{\lambda,k}(\varepsilon) \right\}, \end{aligned}$$

<sup>(9)</sup> This notation agrees with that of [1].

where

$$\begin{aligned} v_j^{\lambda,k}(\varepsilon) &= \sum_{q=0}^k (-1)^q \frac{k!}{(k-q)!} \frac{\varepsilon^{j-\lambda+1}}{(j-\lambda+1)^{q+1}} (\ln \varepsilon)^{k-q} \quad \text{for } j < \lambda-1, \\ &= \frac{1}{k+1} (\ln s)^{k+1} \quad \text{for } j = \lambda-1, \lambda \text{ an integer.} \end{aligned}$$

**PROPOSITION 2.** Let  $\chi \in C_0^\infty(\mathbf{R}^1)$ ,  $\chi = 1$ , in a neighbourhood of zero. If  $k \in N_0$  and  $\lambda$  is a non-negative real number, then the following identities hold:

$$\begin{aligned} Pf \int_0^\infty \frac{(\ln s)^k}{s^\lambda} \chi'(s) ds &= -Pf \int_0^\infty \frac{d}{ds} \left( \frac{(\ln s)^k}{s^\lambda} \right) \chi(s) ds, \\ Pf \int_{-\infty}^0 \frac{(\ln |s|)^k}{|s|^\lambda} \chi'(s) ds &= -Pf \int_{-\infty}^0 \frac{d}{ds} \left( \frac{(\ln |s|)^k}{|s|^\lambda} \right) \chi(s) ds \end{aligned}$$

provided  $k$  and  $\lambda$  are not simultaneously equal to zero.

The proof of this proposition can be found in Section 3 of [6]. Note that for  $\chi = \varphi \in C_0^\infty(\mathbf{R}^1)$ ,  $\lambda \in N$ ,  $\varphi^{(\lambda)}(0) \neq 0$ , the above formula of integration by parts fails to be true if  $k = 0$ .

Recall that  $\mu = a + b + 1 \geq 0$  is a root of the characteristic polynomial  $w$  of the operator  $(L')^*$  and denote its multiplicity by  $\tau$ . It is clear that  $\tau = 1$  or  $\tau = 2$ . The second case occurs iff  $\mu \in N_0$ ,  $\mu < r$ . Moreover,

$$t_0 = r - \mu - 1$$

is a root of the polynomial  $p$  of the same multiplicity  $\tau$ .

Now we shall consider equation (43). To find its solution we distinguish five possibilities, depending on the value of  $\tau$  and on the function  $\gamma_{00}$  in formula (42).

(I)  $b \in N_0$ ,  $\tau = 1$ . Then

$$(47) \quad T[\alpha] = Pf \int_0^\infty s^{t_0} \alpha(s) ds \quad \text{for } \alpha \in D^{(\mu)}(\mathbf{R}^1).$$

(II)  $b \in N_0$ ,  $\tau = 2$ . Then

$$T[\alpha] = Pf \int_0^\infty s^{t_0} \ln s \alpha(s) ds \quad \text{for } \alpha \in D^{(\mu)}(\mathbf{R}^1).$$

(III)  $b \notin N_0$ ,  $a \notin N_0$ ,  $\tau = 2$ . Then  $T$  is defined by the same formula (47).

(IV)  $a = N_0$ ,  $b \notin N_0$ . Then

$$T[\alpha] = Pf \int_{-\infty}^0 |s|^{t_0} \alpha(s) ds \quad \text{for } \alpha \in D^{(\mu)}(\mathbf{R}^1).$$

(V)  $a \notin N_0$ ,  $b \notin N_0$ ,  $\tau = 1$  (hence  $\mu \geq r$ ,  $\mu \in N$ ). Then

$$T = \frac{\delta^{(\mu-r)}}{(\mu-r)!}.$$

It turns out that cases (I)–(III) can be considered jointly. To this end denote by  $h$  the number equal to the power of the logarithm appearing in the function  $\gamma_{00}$ . So we have

$$h = 0 \quad \text{if } b \in N_0 \text{ or } a \in N_0,$$

$$h = 1 \quad \text{if } a \notin N_0, b \notin N_0.$$

Since  $h = 0, 1$ ,  $\tau = 1, 2$ , there are only two possibilities:  $h < \tau$  or  $h = 1 = \tau$ . Observe that  $h = 1 = \tau$  occurs iff  $a \notin N_0$ ,  $b \notin N_0$ ,  $\mu \geq r$ , that is, in case (V). If  $h < \tau$ , define  $q \in N$  such that  $h + q = \tau$  and put

$$(48) \quad T[\alpha] = Pf \int_0^\infty s'^0 (\ln s)^{q-1} \alpha(s) ds \quad \text{for } \alpha \in D^{(\mu)}(\mathbb{R}^1).$$

It is clear that the above formula includes all the functionals defined in (I)–(III) and that  $\mu$  and  $t_0$  are roots of the polynomials  $w$  and  $p$ , respectively, of multiplicities  $h + q$  precisely. Note that<sup>(10)</sup>

$$E(s'^0 (\ln s)^{q-1}) = 0$$

and therefore

$$(49) \quad ET[R] = 0 \quad \text{for } R \in C_0^{\{\mu\}+2r} \text{ flat at zero up to order } [\mu] + 2r.$$

As  $\mu$  is a root of  $w$  of multiplicity  $h + q$  precisely, there exists a polynomial  $v$  such that

$$w(\lambda) = (\lambda - \mu)^{h+q} v(\lambda), \quad v(\mu) \neq 0.$$

We shall prove that

$$(50) \quad ET[F_* \varphi] = \Lambda_{00}(\varphi)(-1)^q A(a, b) v(\mu);$$

thus by (17) distribution (48) satisfies equation (43) with

$$C = \frac{1}{4} |S_m| |S_n| (-1)^q A(a, b) v(\mu).$$

Observe that (50) follows at once from (42), (49) and the four groups of relations below:

$$(51) \quad ET[\chi(s) Y(s) s^\lambda] = 0 \quad \text{if } \lambda < \mu, \lambda \in N_0,$$

$$(52) \quad ET[\chi(s) Y(s) s^\lambda (\ln s)^p] = 0 \quad \text{if } \lambda > \mu \text{ or } \lambda = \mu \text{ and } p < h,$$

$$(53) \quad ET[\chi(s) Y(-s) |s|^\lambda (\ln |s|)^p] = 0 \quad \text{for arbitrary } \lambda \in \mathbb{R}^1, p \in N_0,$$

$$(54) \quad ET[\chi(s) Y(s) s^\mu (\ln s)^h] = (-1)^q v(\mu).$$

<sup>(10)</sup>  $q = 2$  implies  $\tau = 2$ , and therefore  $E(s'^0 \ln s) = (p(t_0) \ln s + p'(t_0)) s'^0 = 0$ .



To prove (51) apply (45) and then, by integration by parts, observe that there exist constants  $b_i(\lambda)$  ( $i = 1, \dots, q$ ) such that:

$$\begin{aligned} E T[\chi(s) Y(s) s^\lambda] &= w(\lambda) Pf \int_0^\infty s^{\lambda-\mu-1} (\ln s)^{q-1} \chi(s) ds + \sum_{j=1}^{2r} c_j(\lambda) \int_0^\infty s^{\lambda-\mu-1+j} (\ln s)^{q-1} \chi^{(j)}(s) ds \\ &= w(\lambda) Pf \int_0^\infty s^{\lambda-\mu-1} (\ln s)^{q-1} \chi(s) ds + \sum_{j=0}^{q-1} b_j(\lambda) \int_0^\infty \chi'(s) s^{\lambda-\mu} (\ln s)^j ds. \end{aligned}$$

Hence by Proposition 2 we obtain

$$\begin{aligned} E T[\chi(s) Y(s) s^\lambda] &= w(\lambda) Pf \int_0^\infty s^{\lambda-\mu-1} (\ln s)^{q-1} \chi(s) ds - \\ &\quad - \sum_{j=0}^{q-1} b_j(\lambda) Pf \int_0^\infty \frac{d}{ds} \left( \frac{(\ln s)^j}{s^{\mu-\lambda}} \right) \chi(s) ds. \end{aligned}$$

There are two possibilities:  $q = 1$  and  $q = 2$ . If  $q = 1$ , then

$$\begin{aligned} E T[\chi(s) Y(s) s^\lambda] &= w(\lambda) Pf \int_0^\infty s^{\lambda-\mu-1} \chi(s) ds + (\mu - \lambda) b_0(\lambda) Pf \int_0^\infty s^{\lambda-\mu-1} \chi(s) ds \\ &= (w(\lambda) + (\mu - \lambda) b_0(\lambda)) Pf \int_0^\infty s^{\lambda-\mu-1} \chi(s) ds. \end{aligned}$$

The left-hand side being independent of  $\chi$ , we conclude that  $w(\lambda) + (\mu - \lambda) b_0(\lambda) = 0$  and therefore (51) holds. If  $q = 2$ , then

$$\begin{aligned} E T[\chi(s) Y(s) s^\lambda] &= w(\lambda) Pf \int_0^\infty s^{\lambda-\mu-1} \ln s \chi(s) ds + \\ &\quad + (\mu - \lambda) b_0(\lambda) Pf \int_0^\infty s^{\lambda-\mu-1} \chi(s) ds - b_1(\lambda) Pf \int_0^\infty s^{\lambda-\mu-1} \chi(s) ds - \\ &\quad - b_1(\lambda)(\lambda - \mu) Pf \int_0^\infty \ln s s^{\lambda-\mu-1} \chi(s) ds \\ &= (w(\lambda) - b_1(\lambda)(\lambda - \mu)) Pf \int_0^\infty s^{\lambda-\mu-1} \ln s \chi(s) ds + \\ &\quad + ((\mu - \lambda) b_0(\lambda) - b_1(\lambda)) Pf \int_0^\infty s^{\lambda-\mu-1} \chi(s) ds. \end{aligned}$$

The left-hand side being again independent of the function  $\chi$ , we conclude that  $w(\lambda) - b_1(\lambda)(\lambda - \mu) = 0$ ,  $(\mu - \lambda) b_0(\lambda) - b_1(\lambda) = 0$  and therefore (51) holds.

We omit an analogous proof of relations (52)–(54), which, moreover, can be found in Section 4 of [6]. Observe that (53) is evident.

The proof of (43) in case (IV) is based on the following set of equations analogous to (51)–(54):

$$\begin{aligned}ET[\chi(s)Y(-s)|s|^\lambda] &= 0 \quad \text{if } \lambda < \mu, \lambda \in N_0, \\ET[\chi(s)Y(-s)|s|^\lambda(\ln|s|)^p] &= 0 \quad \text{if } \lambda > \mu, p \in N_0, \\ET[\chi(s)Y(s)s^\lambda(\ln s)^p] &= 0 \quad \text{if } \lambda \in \mathbf{R}, p \in N_0, \\ET[\chi(s)Y(-s)s^\mu] &= v[\mu],\end{aligned}$$

where  $w(\lambda) = (\lambda - \mu)v(\lambda)$ ,  $v(\mu) \neq 0$ . This time the constant  $C$  in (43) is equal to  $\frac{1}{4}|S_m||S_n|v(\mu)A(a, b)$ .

Now we shall prove that in case V

$$(55) \quad ET[F_*\varphi] = \frac{1}{4}|S_m||S_n|w'(\mu)\varphi(0)A(a, b).$$

To this end observe that from (45) we obtain<sup>(11)</sup>

$$(56) \quad D^{\mu-r}((L)^r(\chi(s)|s|^\lambda))(0) = 0 \quad \text{if } \lambda \in N_0 \text{ or } \lambda > \mu.$$

Because of (46) we get

$$(L)^r(\chi(s)s^\mu \ln|s|) = w'(\mu)s^{\mu-r}\chi(s) + \sum_{j=0}^{2r}(\tilde{c}_{0j}\ln|s| + \tilde{c}_{1j})(-1)^j s^{\mu-r+j}\chi^{(j)}(s),$$

whence

$$(57) \quad D^{\mu-r}((L)^r(\chi(s)s^\mu \ln|s|))(0) = w'(\mu)(\mu-r)!.$$

From (46) follows analogously that

$$(58) \quad D^{\mu-r}((L)^r(\chi(s)|s|^\lambda \ln|s|))(0) = 0 \quad \text{if } \lambda > \mu.$$

Assertion (55) follows at once from (42) and formulas (56)–(58).

**Remark 3.** In case (II) ( $b \in N_0$ ,  $\tau = 2$ ),  $T = s_+^{t_0} \ln s_+$  and (50) yields

$$Es_+^{t_0} \ln s_+ [F_*\varphi] = \frac{1}{2}A_{00}(\varphi)A(a, b)w''(\mu),$$

$Es_+^{t_0} \ln s_+$  vanishes on all the summands of the right-hand side of (42) except  $A_{00}(\varphi)\chi(s)\gamma_{00}(s)$ .

A similar argument leads in this case to the formula

$$Es_+^{t_0} [F_*\varphi] = 0,$$

$Es_+^{t_0}$ , vanishing separately on all the summands on the right-hand side of (42).

We put together the results of this section in the form of the following

<sup>(11)</sup> See also foot-note <sup>(8)</sup>.

**THEOREM 2.** Let  $m, n \in \mathbb{N}$ ,  $a = \frac{1}{2}(m-2)$ ,  $b = \frac{1}{2}(n-2)$ ,  $\mu = a+b+1$ ,  $t_0 = r - \mu - 1$ ,  $C = \frac{1}{4}|S_m||S_n| \cdot A(a, b)$ , where  $A(a, b)$  are defined in Lemma 2.  $F_*$  stands for the operation of averaging from Lemma 1 and  $E$  denotes the one-dimensional operator related to  $(\square_{mn})'$  as in Theorem 1. Let  $w$  be the characteristic polynomial of the operator  $(E)^\mu$  transposed to  $E$  and suppose  $\varphi \in D(\mathbb{R}^{m+n})$ . Then

$$\begin{aligned} E s_+^{t_0} [F_* \varphi] &= -C w'(\mu) \varphi(0) & \text{if } b \in N_0, w'(\mu) \neq 0, \\ &= -\frac{1}{2} C w''(\mu) \varphi(0) & \text{if } b \notin N_0, a \notin N_0, w'(\mu) = 0, \end{aligned}$$

$$E s_+^{t_0} \ln s_+ [F_* \varphi] = \frac{1}{2} C w''(\mu) \varphi(0) \quad \text{if } b \in N_0, w'(\mu) = 0,$$

$$E s_-^{t_0} [F_* \varphi] = C w'(\mu) \varphi(0) \quad \text{if } a \in N_0, b \notin N_0,$$

$$E \frac{\delta^{(\mu-r)}}{(\mu-r)!} [F_* \varphi] = C w'(\mu) \varphi(0) \quad \text{if } a \notin N_0, b \notin N_0, w'(\mu) \neq 0.$$

A fundamental solution of the operator  $(\square_{mn})'$  is given by

$$u[\varphi] = -\frac{1}{C} \frac{1}{w'(\mu)} s_+^{t_0} [F_* \varphi] \quad \text{for } \varphi \in D(\mathbb{R}^{m+n}) \text{ if } b \in N_0, w'(\mu) \neq 0,$$

$$u[\varphi] = -\frac{2}{C} \frac{1}{w''(\mu)} s_+^{t_0} [F_* \varphi] \quad \text{for } \varphi \in D(\mathbb{R}^{m+n}) \text{ if } b \notin N_0, a \notin N_0, w'(\mu) = 0,$$

$$u[\varphi] = \frac{1}{C} \frac{2}{w''(\mu)} s_+^{t_0} \ln s_+ [F_* \varphi] \quad \text{for } \varphi \in D(\mathbb{R}^{m+n}) \text{ if } b \in N_0, w'(\mu) = 0,$$

$$u[\varphi] = \frac{1}{C} \frac{1}{w'(\mu)} s_-^{t_0} [F_* \varphi] \quad \text{for } \varphi \in D(\mathbb{R}^{m+n}) \text{ if } a \in N_0, b \notin N_0,$$

$$u[\varphi] = \frac{1}{C} \frac{1}{w'(\mu)} \frac{1}{(\mu-r)!} \delta^{(\mu-r)} [F_* \varphi] \quad \text{for } \varphi \in D(\mathbb{R}^{m+n}) \text{ if } a \notin N_0, b \notin N_0, w'(\mu) \neq 0.$$

**4. Fundamental solution of the operator  $P(\square_{mn})$ .** By Theorem 1 in order to find a fundamental solution of the operator  $P(\square_{mn})$  it is enough to find a distribution  $E$  satisfying equation (43) with some constant  $C \neq 0$ . To this end we shall select an adequate solution of the equation

$$(59) \quad L_r y = 0 \quad \text{in } \mathbb{R}_+^1$$

and define its convenient regularization  $E_0$ .

Retain the notation of Section 3 and denote by  $p_h$  the characteristic polynomial of the operator  $L^h$ :

$$p_h(\lambda) = 4^h \lambda(\lambda-1) \dots (\lambda-h+1)(\lambda+\mu)(\lambda+\mu-1) \dots (\lambda+\mu-h+1),$$

$$h = 1, \dots, r.$$

Analogously, we denote by  $w_h$  the characteristic polynomial of the operator  $(E^r)^h$ ,  $h = 1, \dots, r$ . Note that  $p_r$  coincides with the polynomial  $p$  from Section 3 and that  $t_0 = r - \mu - 1$  is a zero of the polynomial  $p_r$ .

Following a method of Frobenius<sup>(12)</sup>, we look for a solution of equation (59) of the form

$$(60) \quad y(s, t) = \sum_{i=0}^{\infty} c_i(t) s^{t+i} \quad \text{for } s > 0,$$

where  $t$  denotes some parameter. Let us formally substitute series (60) in  $L_r y$  and arrange it with respect to the powers of  $s$ :

$$(61) \quad \begin{aligned} \sum_{j=0}^r a_j E^j y(s, t) &= \sum_{k=0}^{\infty} \sum_{j=0}^{\min(k, r)} a_{r-j} c_{k-j}(t) E^{-j} s^{t+k-j} \\ &= \sum_{k=0}^{\infty} s^{t+k-r} \left( \sum_{j=0}^{\min(k, r)} a_{r-j} c_{k-j}(t) p_{r-j}(t+k-j) \right). \end{aligned}$$

Equating to zero the expressions in the brackets, we obtain the following system of equations:

$$(62) \quad c_0(t) p_r(t) = 0,$$

$$(63) \quad \sum_{j=0}^{\min(k, r)} a_{r-j} c_{k-j}(t) p_{r-j}(t+k-j) = 0, \quad k = 1, 2, \dots$$

Neglecting equation (62), which for  $t = t_0$  is satisfied by an arbitrary  $c_0$ , we write system (63) in the following equivalent form:

$$(64) \quad \sum_{i=\max(0, k-r)}^k a_{r-k+i} c_i(t) p_{r-k+i}(t+i) = 0, \quad k = 1, 2, \dots$$

Looking for a non-zero solution of equation (59), we distinguish four cases.

Case (I<sub>1</sub>) <sup>(13)</sup>  $a \notin N_0$ ,  $b \in N_0$ . Then  $w'(\mu) \neq 0$ ,  $t_0$  is not an integer and  $p_r(t_0 + k) \neq 0$  for  $k = 1, 2, \dots$ . We compute  $c_k(t_0)$  from the  $k$ th equation of system (64) with  $t_0$  instead of  $t$  and  $c_0(t) = 1$ . By the Frobenius theorem the radius of convergence of the series  $\sum_{i=0}^{\infty} c_i(t_0) s^i$  is  $+\infty$  and the function

$$(65) \quad y(s; t_0) = \sum_{i=0}^{\infty} c_i(t_0) s^{t_0+i}$$

satisfies equation (59). We define a distribution  $E_0 \in D'(\mathbf{R}^1)$  by putting

$$(66) \quad E_0[\alpha] = \sum_{i=0}^{\infty} c_i(t_0) s_+^{t_0+i}[\alpha] \quad \text{for } \alpha \in C_0^\infty(\mathbf{R}^1).$$

<sup>(12)</sup> See [2].

<sup>(13)</sup> Case (I) from Section 3 is now divided into two subcases, (I<sub>1</sub>) and (I<sub>2</sub>).

Case (II)  $b \in N_0$ ,  $w'(\mu) = 0$ . Then  $t_0$  is a double root of the polynomial  $p_r$ ,  $0 \leq t_0 < r-1$ . Moreover,

$$p_r(t_0+1) = \dots = p_r(t_0+\mu) = 0, \quad p_r(t_0+\mu+k) \neq 0 \quad \text{for } k = 1, 2, \dots$$

Put

$$c_0(t) = p_r(t+1) \dots p_r(t+\mu), \quad B = p'_r(t_0+1) \dots p'_r(t_0+\mu)$$

and note that

$$(67) \quad c_0^{(k)}(t_0) = 0 \quad \text{for } k = 0, 1, \dots, r-2-t_0, \quad c_0^{(r-1-t_0)}(t_0) = B \neq 0.$$

As in the preceding cases we find  $c_k(t)$  ( $k = 1, 2, \dots$ ) from system (64). By the Frobenius theorem the series

$$\begin{aligned} y_{r-t_0}(s; t_0) &= \sum_{i=0}^{\infty} \sum_{j=0}^{r-t_0} \binom{r-t_0}{j} (\ln s)^{r-t_0-j} s^{t_0+i} c_i^{(j)}(t_0) \\ &= (r-t_0) B s^{t_0} \ln s + c_0^{(r-t_0)}(t_0) s^{t_0} + \sum_{i=1}^{\infty} \sum_{j=0}^{r-t_0} \binom{r-t_0}{j} c_i^{(j)}(t_0) (\ln s)^{r-t_0-j} s^{t_0+i} \end{aligned}$$

is a solution of (59). We define a distribution  $E_0$  by putting

$$(68) \quad E_0 = (r-t_0) B s_+^{t_0} \ln s_+ + c_0^{(r-t_0)}(t_0) s_+^{t_0} + \sum_{i=1}^{\infty} \sum_{j=0}^{r-t_0} \binom{r-t_0}{j} c_i^{(j)}(t_0) s_+^{t_0+i} \ln^{r-t_0-j} s_+.$$

Case (III)  $a \notin N_0$ ,  $b \notin N_0$ ,  $w'(\mu) = 0$ . Define  $c_0(t)$  and  $B$  as in case (II). In view of (67) and the Frobenius theorem series<sup>(14)</sup>

$$\begin{aligned} y_{r-t_0-1}(s, t_0) &= \sum_{i=0}^{\infty} \sum_{j=0}^{r-t_0-1} \binom{r-t_0-1}{j} c_i^{(j)}(t_0) (\ln s)^{r-t_0-1-j} s^{t_0+i} \\ &= c_0^{(r-t_0-1)}(t_0) s^{t_0} + \sum_{i=1}^{\infty} \sum_{j=0}^{r-t_0-1} \binom{r-t_0-1}{j} c_i^{(j)}(t_0) (\ln s)^{r-t_0-1-j} s^{t_0+i} \end{aligned}$$

is a solution of (59). We define a distribution  $E_0$  by putting

$$(69) \quad E_0 = c_0^{(r-t_0-1)}(t_0) s_+^{t_0} + \sum_{i=1}^{\infty} \sum_{j=0}^{r-t_0-1} \binom{r-t_0-1}{j} c_i^{(j)}(t_0) s_+^{t_0+i} \ln^{r-t_0-1-j} s_+.$$

Case (I<sub>2</sub>)  $a \in N_0$ ,  $b \in N_0$ ,  $w'(\mu) \neq 0$ . Then  $t_0$  is a negative integer, all the roots of  $p_r$  are simple and  $p_r(t_0+j) = 0$  if  $j = |t_0|, |t_0|+1, \dots, |t_0|+r-1$ . Put

$$A = p'_r(0) p'_r(1) \dots p'_r(r-1), \quad c_0(t) = p_r(t-t_0) p_r(t-t_0+1) \dots p_r(t-t_0+r-1)$$

<sup>(14)</sup> We shall see later that the choice of  $y_{r-t_0-1} = y_\mu$  instead of  $y_{r-t_0}$  as in case (II) is connected with the presence in the case in question of the logarithmic multiplier of the power  $s^\mu$  in the asymptotic expansion of  $F_* \varphi$ .

and observe that

$$(70) \quad c_0^{(k)}(t_0) = 0 \quad \text{for } k = 0, 1, \dots, r-1, \quad c_0^{(r)}(t_0) = A \neq 0.$$

We compute  $c_k(t)$  from the  $k$ th equation of system (64) successively for  $k = 1, 2, \dots$ . Denote by  $y_q(s; t)$  the series obtained from  $\sum_{i=0}^{\infty} c_i(t) s^{t+i}$  by formal differentiation  $\partial^q / \partial t^q$  term by term. Following Frobenius, the radius of convergence of the series  $\sum_{i=0}^{\infty} c_i^{(k)}(t_0) s^i$  ( $k = 0, 1, \dots, r$ ) is  $+\infty$  and for every  $q = 0, 1, \dots, r$  the series

$$y_q(s; t_0) = \sum_{i=0}^{\infty} \sum_{k=0}^q \binom{q}{k} c_i^{(k)}(t_0) (\ln s)^{q-k} s^{t_0+i}$$

is a solution of equation (59). We define a distribution  $E_0$  putting

$$(71) \quad E_0 = A s_+^{t_0} + \sum_{i=1}^{\infty} \sum_{k=0}^r \binom{r}{k} c_i^{(k)}(t_0) s_+^{t_0+i} \ln^{r-k} s_+.$$

Only the distribution  $E_0$  from case (I<sub>1</sub>) satisfies equation (43). In the remaining cases some additional terms may appear. To see this we shall use the following

LEMMA 5. Let  $\eta \in C_0^{2r}(\mathbf{R}^1)$ ,  $\alpha(s) = s^\lambda (\ln s)^h \eta(s)$  for  $s > 0$ ,  $\lambda \geq \mu$ ,  $h = 0, 1$ , and take  $c_k$  ( $k = 0, 1, \dots$ ) satisfying (62), (63). Then for all  $k = 1, 2, \dots$  and  $t > r - \mu - 2$  we have

$$(72) \quad \sum_{j=0}^{\min(k, r)} a_{r-j} c_{k-j}(t) L^{-j} s_+^{t+k-j} [\alpha] = 0.$$

If  $\mu$  is not an integer, (72) holds true for  $\alpha(s) = s^p \eta(s)$  with all  $p \in \mathbf{N}_0$ .

Proof. Let  $\alpha(s) = s^\lambda \ln s \eta(s)$  for  $s > 0$  and suppose  $t > r - \mu - 2$ ,  $\eta \in C_0^{2r}(\mathbf{R}^1)$ . Denote for every  $k = 1, 2, \dots$

$$(73) \quad \begin{aligned} K_k(\eta) &= \sum_{j=0}^{\min(k, r)} a_{r-j} c_{k-j}(t) L^{-j} s_+^{t+k-j} [s^\lambda \ln s \eta(s)] \\ &= \sum_{j=0}^{\min(k, r)} a_{r-j} c_{k-j}(t) Pf \int_0^\infty s^{t+k-j} (L^r)^{r-j} (s^\lambda \ln s \eta(s)) ds. \end{aligned}$$

It is easy to see that there exist constants  $\beta_p, \gamma_p$  ( $p = 0, 1, \dots, 2(r-j)$ ) such that

$$(L^r)^{r-j} (s^\lambda \ln s \eta(s)) = \sum_{p=0}^{2(r-j)} \eta^{(p)} s^{\lambda-r+j+p} (\beta_p \ln s + \gamma_p).$$

Substituting in (73) the right-hand side of the above formula and putting  $\kappa = t + k + \lambda - r$ , we find that since  $\kappa > -1$

$$K_k(\eta) = \sum_{j=0}^{\min(k,r)} \sum_{p=0}^{2(r-j)} a_{r-j} c_{k-j}(t) \int_0^\infty s^{\kappa+p} \eta^{(p)}(s) (\beta_p \ln s + \gamma_p) ds.$$

Integrating by parts, we arrive at the formula

$$(74) \quad K_k(\eta) = Q_1(t) \int_0^\infty \eta(s) s^\kappa \ln s ds + Q_2(t) \int_0^\infty \eta(s) s^\kappa ds$$

with adequate functions  $Q_1$  and  $Q_2$ . Take  $\eta = \chi$ . We shall show that  $K_k(\chi)$  is independent of the choice of  $\chi$  equal to one in a neighbourhood of zero. In fact, if  $\chi_1, \chi_2 \in C_0^\infty(\mathbf{R}^1)$  are two such functions and if  $\tilde{\alpha}(s) = (\chi_1(s) - \chi_2(s)) s^\lambda \ln s$  for  $s > 0$ , then  $\tilde{\alpha} \in C_0^\infty(\mathbf{R}_+^1)$  and in view of (73)

$$\begin{aligned} K_k(\chi_1) - K_k(\chi_2) &= K_k(\chi_1 - \chi_2) \\ &= \sum_{j=0}^{\min(k,r)} a_{r-j} c_{k-j}(t) \int_0^\infty s^{t+k-j} (L^{r-j})^r ((\chi_1 - \chi_2) s^\lambda \ln s) ds \\ &= \sum_{j=0}^{\min(k,r)} a_{r-j} c_{k-j}(t) \int_0^\infty (\chi_1 - \chi_2) s^\lambda \ln s L^{-j} s^{t+k-j} ds \\ &= \int_0^\infty (\chi_1 - \chi_2)(s) \ln s s^{t+k-r+\lambda} \sum_{j=0}^{\min(k,r)} a_{r-j} c_{k-j}(t) p_{r-j}(t+k-j) ds = 0. \end{aligned}$$

In order that the right-hand side of (74) with  $\eta = \chi$  be independent of  $\chi$  the functions  $Q_1$  and  $Q_2$  must be equal to zero, whence  $K_k(\eta) = 0$  for every  $\eta \in C_0^{2r}(\mathbf{R}^1)$ ,  $k \in N$ .

A similar proof of the remaining assertions of Lemma 5 is left to the reader. The proof in the case  $\alpha(s) = s^\lambda \eta(s)$ ,  $\lambda \geq \mu$  is simpler.

If  $\mu \notin N_0$ ,  $\alpha(s) = s^\mu \eta(s)$ , it is convenient to apply Proposition 2.

**4.1. Fundamental solution of the operator  $P(\square_{mn})$  in case  $(I_1)$  ( $a \notin N_0$ ,  $b \in N_0$ ).**

Take the distribution  $E_0$  defined by (66) and a function  $\eta \in C_0^{2r}(\mathbf{R}^1)$ . Since  $t_0 > r - \mu - 2$ , Lemma 5 yields the following formulas:

$$(75) \quad L_r E_0 [\eta(s) s^\lambda \ln |s|] = L s_+^{t_0} [\eta(s) s^\lambda \ln |s|] \quad \text{for } \lambda \geq \mu,$$

$$(76) \quad L_r E_0 [\eta(s) Y(s) s^\lambda] = L s_+^{t_0} [\eta(s) Y(s) s^\lambda] \quad \text{for } \lambda \geq \mu \text{ and for } \lambda \in N_0.$$

Moreover, it is clear that

$$(77) \quad L_r E_0 [\chi(s) Y(-s) |s|^\lambda] = L s_+^{t_0} [\chi(s) Y(-s) |s|^\lambda] = 0$$

for  $\lambda$  arbitrary.

We shall show that

$$(78) \quad L_r E_0 [F_* \varphi] = E s_+^{t_0} [F_* \varphi] \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^{m+n}).$$

Let  $R \in C_0^{2r+[\mu]}(\mathbb{R}^1)$  be the function from Remark 2 flat at zero up to order  $2r+[\mu]$ . It is easy to see that

$$(79) \quad \begin{aligned} L_r E_0 [R] &= \sum_{j=0}^r a_j \sum_{i=0}^\infty c_i(t_0) P f \int_0^\infty s^{t_0+i} (L^r)^j R(s) ds \\ &= \sum_{j=0}^r a_j \sum_{i=0}^\infty c_i(t_0) \int_0^\infty R(s) E s^{t_0+i} ds \\ &= \int_0^\infty L_r \left( \sum_{i=0}^\infty c_i(t_0) s^{t_0+i} \right) R(s) ds = 0 = E s_+^{t_0} [R]. \end{aligned}$$

because (65) satisfies equation (59). From (42), (75)–(77) and (79) follows (78). On the other hand, (50) yields

$$E s_+^{t_0} [F_* \varphi] = -A_{00}(\varphi) A(a, b) w'(\mu),$$

and thus by (78) and (17) the distribution  $E_0$  satisfies equation (43) with  $C = -\frac{1}{4} |S_m| |S_n| A(a, b) w'(\mu)$ . Put  $E = E_0$ . By Theorem 1 the distribution  $u$  defined by (44) is a fundamental solution of the operator  $P_r$  in the case in question.

**4.2. Fundamental solution of the operator  $P(\square_{mn})$  in case (II) ( $a \in N_0$ ,  $b \in N_0$ ,  $w'(\mu) = 0$ ).**

Take the distribution  $E_0$  defined by (68), a function  $\eta \in C_0^{2r}(\mathbb{R}^1)$  and the numbers  $\lambda \geq \mu$ , and  $h = 0, 1$ . By Lemma 5 with  $\alpha(s) = s^\lambda (\ln s)^h \eta(s)$  for  $s > 0$ , we get successively:

$$(80) \quad \begin{aligned} L_r E_0 [\alpha] &= \sum_{j=0}^r a_j E \frac{\partial^{r-t_0}}{\partial t^{r-t_0}} \left( \sum_{i=0}^\infty c_i(t) s_+^{t+i} \right) \Big|_{t=t_0} [\alpha] \\ &= \frac{\partial^{r-t_0}}{\partial t^{r-t_0}} \sum_{k=0}^\infty \sum_{j=0}^{\min(k,r)} a_{r-j} c_{k-j}(t) E^{-j} s_+^{t+k-j} \Big|_{t=t_0} [\alpha] \\ &= \frac{\partial^{r-t_0}}{\partial t^{r-t_0}} (c_0(t) E s_+^t) \Big|_{t=t_0} [\alpha] \\ &= \sum_{v=0}^{r-t_0} \binom{r-t_0}{v} c_0^{(v)} \frac{\partial^{r-t_0-v}}{\partial t^{r-t_0-v}} E s_+^t \Big|_{t=t_0} [\alpha] \\ &= (r-t_0) B \frac{\partial}{\partial t} E s_+^t \Big|_{t=t_0} [\alpha] + c_0^{(r-t_0)} E s_+^t \Big|_{t=t_0} [\alpha] \\ &= (r-t_0) B E s_+^{t_0} \ln s_+ [\alpha] + c_0^{(r-t_0)} E s_+^{t_0} [\alpha]. \end{aligned}$$

Remark 3 together with (80) implies that

$$(81) \quad L_r E_0 [\chi(s) Y(s) s^\mu] = (r-t_0) B \frac{1}{2} w''(\mu),$$



and

$$(82) \quad \begin{aligned} L_r E_0 [\chi(s) |s|^\lambda \ln |s|] &= 0 \quad \text{for } \lambda > \mu, \\ L_r E_0 [\chi(s) Y(s) s^\lambda] &= L_r E_0 [\chi(s) Y(-s) |s|^\lambda] = 0 \quad \text{for } \lambda > \mu, \\ L_r E_0 [R] &= 0 \quad \text{for } R \in C_0^{2r+\mu} \text{ flat at zero up to order } 2r + \mu. \end{aligned}$$

In the case under consideration  $\mu > 0$  and, in general, the distribution  $L_r E_0$  does not vanish on the functions  $\chi(s) s^\lambda$  for  $0 \leq \lambda < \mu$ ,  $\lambda \in N_0$ . Observe that  $r - \mu, \dots, r - 1$  are simple roots of the polynomial  $p_r$ . Put  $c_0(t) = 1$  if  $\mu = 1$  and  $c_0(t) = p_r(t+1) \dots p_r(t+\mu-1)$  if  $\mu > 1$  and compute  $c_k$  ( $k = 1, 2, \dots$ ) from system (64). Define for  $j = 1, \dots, \mu$

$$(83) \quad U_j = \sum_{i=0}^{\infty} c_i^{(j-1)} (r-j) s^{r-j+i} Y.$$

By the Frobenius theory it solves (59) and

$$(84) \quad \begin{aligned} L_r U_j [\chi(s) s^{j-1}] &= -C_0^{(j-1)} (r-j) w'_r(j-1) \neq 0, \\ L_r U_j [\chi(s) s^{j+k}] &= 0 \quad \text{for } k = 0, 1, \dots, \\ L_r U_j [\chi(s) Y(s) s^\lambda] &= L_r U_j [\chi(s) Y(-s) |s|^\lambda] = 0 \quad \text{for } \lambda \geq \mu, \\ L_r U_j [\chi(s) |s|^\lambda \ln |s|] &= 0 \quad \text{for } \lambda \geq \mu, \\ L_r U_j [R] &= 0 \quad \text{for } R \in C_0^{2r+\mu}(\mathbf{R}^1) \text{ flat at zero up to order } 2r + \mu. \end{aligned}$$

Put<sup>(15)</sup>

$$(85) \quad \begin{aligned} V_0 &= E_0 \quad (E_0 \text{ given by (68)}), \\ V_{j+1} &= V_j + \frac{L_r V_j [s^{\mu-j-1} \chi(s)]}{w'_r(\mu-j-1)} U_{\mu-j} \quad \text{for } j = 0, 1, \dots, \mu-1. \end{aligned}$$

From (84), (81), (82) and (42) it easily follows that  $E = V_\mu$  satisfies

$$L_r E [F_* \varphi] = \frac{1}{2} (r - t_0) B A_{00}(\varphi) A(a, b) w''(\mu) \quad \text{for } \varphi \in C_0^\infty(\mathbf{R}^{m+n}),$$

and thus by (17) the distribution  $E$  satisfies equation (43) with  $C = \frac{1}{8} |S_m| |S_n| A(a, b) (r - t_0) B w''(\mu)$ , where  $B$  is defined by (67). By Theorem 1 the distribution  $u$  defined by (44) with  $E = V_\mu$  is a fundamental solution of the operator  $P(\square_{mn})$  in the case under consideration.

**4.3. Fundamental solution of the operator  $P(\square_{mn})$  in case (III) ( $a \notin N_0$ ,  $b \notin N_0$ ,  $w'(\mu) = 0$ ).**

Take the distribution  $E_0$  defined by (69), a function  $\eta \in C_0^{2r}(\mathbf{R}^1)$  and numbers  $\lambda \geq \mu$  and  $\varepsilon = 0, 1$ . Proceeding as in the proof of (80), we show that

$$(86) \quad L_r E_0 [s^\lambda (\ln s)^\varepsilon \eta(s)] = B L s_+^{\lambda_0} [s^\lambda (\ln s)^\varepsilon \eta(s)].$$

<sup>(15)</sup> This definition is correct because  $w'_r(\mu-1-j) \neq 0$  for  $j = 0, 1, \dots, \mu-1$ .

We know that<sup>(16)</sup>

$$Ls_+^{(0)}[s^\mu \ln s \chi(s)] = -\frac{1}{2} w''(\mu).$$

Hence from (86) follows

$$L_r E_0[s^\mu \ln s \chi(s)] = -\frac{1}{2} B w''(\mu).$$

We put  $E = E_0$  if  $\mu = 0$ . If  $\mu > 0$ , we proceed as in case (II) and put  $E = V_\mu$  with  $V_\mu$  given by (85) and  $V_0 = E_0$  defined by (69).

Thus we have constructed fundamental solutions of the operator  $P(\square_{mn})$  in three cases. Besides case (I<sub>2</sub>) just mentioned, there remain precisely two possibilities:

Case (IV)  $a \in N_0$ ,  $b \notin N_0$ .

Case (V)  $a \notin N_0$ ,  $b \in N_0$ ,  $w'(\mu) \neq 0$ .

Observe that in cases (I<sub>2</sub>) and (V),  $\mu$  is an entire, simple root of the polynomial  $w$  and therefore  $\mu \geq r$ . We can define distributions

$$(87) \quad E_j = \frac{\delta^{(\mu-r-j)}}{(\mu-r-j)!}, \quad j = 0, 1, \dots, \mu-r,$$

and observe that by (45) <sup>(17)</sup>

$$(88) \quad L_r E_0[\chi(s) s^\mu Y(s)] = \frac{1}{(\mu-r)!} \delta^{(\mu-r)} \left[ \sum_{k=0}^r a_k w_k(\mu) s^{\mu-k} Y(s) \right] = 0$$

because  $w_k(\mu) = 0$ ,  $k = 1, \dots, r$  and  $\chi \in C_0^\infty(\mathbb{R}^1)$  equals one in a neighbourhood of zero<sup>(18)</sup>. Similarly, by (46) we get<sup>(19)</sup>

$$(89) \quad \begin{aligned} L_r E_0[\chi(s) s^\mu \ln |s|] &= \frac{\delta^{(\mu-r)}}{(\mu-r)!} \left[ \sum_{k=0}^r a_k (w_k(\mu) \ln |s| + w'_k(\mu)) s^{\mu-k} \right] \\ &= (-1)^{\mu-r} w'_r(\mu). \end{aligned}$$

Analogously, we derive the following formulas for  $j = 0, 1, \dots, \mu-r$

$$(90) \quad \begin{aligned} L_r E_j[\chi(s) |s|^\lambda \ln |s|] &= 0 \quad \text{if } \lambda > \mu-j, \\ L_r E_j[\chi(s) s^\lambda Y(s)] &= 0 \quad \text{if } \lambda > \mu-j, \\ L_r E_j[\chi(s) |s|^\lambda Y(-s)] &= 0 \quad \text{if } \lambda > \mu-j, \end{aligned}$$

$L_r E_j[R] = 0$  if  $R \in C_0^p(\mathbb{R}^1)$  is flat at zero up to order  $p > 2\mu$ ,

$$L_r E_j[\chi(s) s^{\mu-j-k}] = a_{r-k} w_{r-k}(\mu-j-k) (-1)^{\mu-r-j} \quad \text{for } k = 0, 1, \dots, r.$$

<sup>(16)</sup> See Section 3, case (III).

<sup>(17)</sup> See foot-note <sup>(8)</sup>.

<sup>(18)</sup> Since  $\text{supp } E_j = \{0\}$  ( $j = 0, 1, \dots, \mu-r$ ), this property of  $\chi$  allows us to neglect the symbol  $\chi$  in formula (88) and in formulas (89) and (90) below.

<sup>(19)</sup> It is worth while to note that  $E_0$  equals  $T$  defined in case (V) of Section 4.

**4.4. Fundamental solution of the operator  $P(\square_{mn})$  in case (V)** ( $a \notin N_0$ ,  $b \notin N_0$ ,  $w'(\mu) \neq 0$ ).

Denote by  $E_j$  the distributions defined by (87). Put  $V_0 = E_0$  if  $\mu = r$ . If  $\mu > r$  we first define the distributions  $\tilde{E}_j$  ( $j = 0, 1, \dots, \mu - r$ ), putting  $\tilde{E}_0 = E_0$  and

$$(91) \quad \tilde{E}_\varrho = \tilde{E}_{\varrho-1} - \frac{L_r \tilde{E}_{\varrho-1} [s^{\mu-\varrho}]}{L_r E_\varrho [s^{\mu-\varrho}]} E_\varrho \quad (\varrho = 1, \dots, \mu - r).$$

This definition is correct because by (90)

$$L_r E_\varrho [s^{\mu-\varrho}] = w_r(\mu - \varrho) (-1)^{\mu-r-\varrho} \neq 0.$$

Define

$$(92) \quad V_0 = \tilde{E}_{\mu-r} \quad \text{if} \quad \mu > r.$$

It is easy to see that

$$(93) \quad L_r V_0 [\chi(s) s^\mu \ln |s|] = (-1)^{\mu-r} w'_r(\mu), \quad L_r V_0 [\chi(s) Y(s) s^\mu] = 0$$

and that for  $\lambda > \mu$

$$(94) \quad L_r V_0 [\chi(s) |s|^\lambda \ln |s|] = 0,$$

$$L_r V_0 [\chi(s) Y(s) s^\lambda] = 0 = L_r V_0 [\chi(s) Y(-s) |s|^\lambda].$$

Moreover,

$$(95) \quad L_r V_0 [\chi(s) s^{r+k}] = 0 \quad \text{for} \quad k = 0, 1, \dots,$$

$$(96) \quad L_r V_0 [R] = 0 \quad \text{for} \quad R \in C_0^p(\mathbf{R}^1) \text{ flat at zero up to order } p \geq \mu + 2r.$$

Now we proceed as in case (II) of this section. We put  $C_0(t) = 1$  if  $r = 1$  and  $c_0(t) = p_r(t+1) \dots p_r(t+r-1)$  if  $r > 1$ , and compute  $c_k$  ( $k = 1, 2, \dots$ ) from system (64). As before, the distributions  $U_j$  ( $j = 1, \dots, r$ ) defined by (83) satisfy relations (84). Put

$$(97) \quad V_{j+1} = V_j + \frac{L_r V_j [s^{r-j-1} \chi(s)]}{-w'_r(r-j-1)} U_{r-j}, \quad j = 0, 1, \dots, r-1.$$

We easily verify that the distribution  $E = V_r$  satisfies equation (43) with  $C = \frac{1}{4} (-1)^{\mu-r} w'_r(\mu) A(a, b) |S_m| |S_n|$ .

**4.5. Fundamental solution of the operator  $P(\square_{mn})$  in case (I<sub>2</sub>)** ( $a \in N_0$ ,  $b \in N_0$ ,  $w'(\mu) \neq 0$ ).

We begin as in case (II). We take the distribution  $E_0$  defined by (71) and a function  $\alpha(s) = \eta(s) s^\lambda (\ln s)^\varepsilon$  for  $s > 0$ , where  $\eta \in C_0^{2r}(\mathbf{R}^1)$ ,  $\varepsilon = 0, 1$ ,  $\lambda \geq \mu$ . This time the calculation from (80) leads to the relation

$$L_r E_0 [\alpha] = A E s_+^{l_0} [\alpha], \quad A = c_0^{(r)}(t_0).$$

Therefore, from case (II) of Section 4 we get formulas (82) and

$$L_r E_0 [\chi(s) s^\mu Y(s)] = -A w'(\mu).$$

Put  $V_0 = E_0$  if  $\mu = r$ . If  $\mu > r$ , the numbers  $\mu - 1, \dots, r$  are not roots of the polynomial  $w_r$  and the numbers  $r - 1, \dots, 0$  are simple roots of  $p_r$ . Define  $E_j$  ( $j = 1, \dots, \mu - r$ ) by (87). Put  $\tilde{E}_0 = E_0$  and define  $\tilde{E}_q$  ( $q = 1, \dots, \mu - r$ ) by (91) and  $V_0$  by (92).

It is easy to see that

$$L_r V_0 [\chi(s) s^\mu Y(s)] = -Aw'(\mu),$$

and that relations (94)–(96) hold.

We now proceed without changes, as in case (V), defining  $U_j$  ( $j = 1, \dots, r$ ) by (83) and  $V_j$  ( $j = 1, \dots, r$ ) by (97). The distribution  $E = V_r$  satisfies equation (43) with  $C = -\frac{1}{4}Aw'(\mu)A(a, b)|S_m||S_n|$ .

**4.6. Fundamental solution of the operator  $P(\square_{mn})$  in case (IV) ( $a \in N_0$ ,  $b \notin N_0$ ).**

This case can be reduced to case (I<sub>1</sub>) ( $a \notin N_0$ ,  $b \in N_0$ ) by interchanging the variables  $x$  and  $y$ . This amounts to changing  $F$  to  $F^1 = -F$  and consequently  $F_* \varphi$  to  $(F_*^1 \varphi)(s) = (F_* \varphi)(-s)$ . Also the operator  $\square_{mn}$  is changed to  $\square_{mn}^1 = -\square_{mn}$  and the polynomial  $P(s)$  to  $P^1(s) = P(-s)$ . Then we easily observe that, if  $E_0^1[\alpha] = \sum_{i=0}^{\infty} c_i(t_0) s_+^{i_0+i}[\alpha]$  given by (66) is such that

$$P^1(\square_{mn}^1) E_0^1[F_*^1 \varphi] = C\delta[\varphi],$$

where  $C = -\frac{1}{4}|S_m||S_n|A(a, b)w'(\mu)$ , then the distribution

$$E_0[\alpha] = E_0^1[\alpha(-s)] = \sum_{i=0}^{\infty} c_i(t_0) s_-^{i_0+i}[\alpha]$$

satisfies the equation

$$P(\square_{mn}) E_0 = C\delta.$$

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