ANNALES
POLONICI MATHEMATICI
LII (1990)

Strong maximum and minimum principles for parabolic functional-differential problems with non-local inequalities $[u^j(t_0, x)-K^j]+\sum_i h_i(x)[u^j(T_i, x)-K^j] \underset{(\geqslant)}{\leqslant} 0$

by LUDWIK BYSZEWSKI (Kraków)

Abstract. The aim of the paper is to give strong maximum and minimum principles for parabolic functional-differential problems with non-local inequalities in relatively arbitrary $(n+1)$-dimensional time-space sets more general than the cylindrical domain. The results obtained can be applied in the theory of diffusion and in the theory of heat conduction.

1. Introduction. Recently, a number of papers on parabolic problems with non-local conditions were published (e.g. [5], [8], [3], [4]).

In this paper, we also consider parabolic non-local problems. Namely, we consider diagonal systems of non-linear parabolic functional-differential inequalities of the form

$$(1.1)\qquad u_t^i(t, x) \underset{(\geqslant)}{\leqslant} f^i(t, x, u(t, x), u_x^i(t, x), u_{xx}^i(t, x), u) \quad (i = 1, \ldots, m)$$

for $(t, x) = (t, x_1, \ldots, x_n) \in D$, where $D \subset (t_0, t_0+T] \times \boldsymbol{R}^n$ is one of six relatively arbitrary sets more general than the cylindrical domain $(t_0, t_0+T] \times D_0 \subset \boldsymbol{R}^{n+1}$ The symbol u denotes the mapping

$$u\colon \tilde{D} \ni (t, x) \to u(t, x) = (u^1(t, x), \ldots, u^m(t, x)) \in \boldsymbol{R}^m,$$

where $\tilde{D}$ is an arbitrary set contained in $(-\infty, t_0+T] \times \boldsymbol{R}^n$ such that $\bar{D} \subset \tilde{D}$. The right-hand sides f^i $(i = 1, \ldots, m)$ of systems (1.1) are functionals of u; $u_x^i(t, x) = \operatorname{grad}_x u^i(t, x)$ $(i = 1, \ldots, m)$ and $u_{xx}^i(t, x)$ $(i = 1, \ldots, m)$ denote the matrices of second order derivatives with respect to x of $u^i(t, x)$ $(i = 1, \ldots, m)$. We give two theorems on strong maximum and minimum principles for problems with inequalities (1.1) and with the non-local inequalities

$$[u^j(t_0, x)-K^j]+\sum_{i \in I_*} h_i(x)[u^j(T_i, x)-K^j] \underset{(\geqslant)}{\leqslant} 0 \quad \text{for } x \in S_{t_0} \ (j = 1, \ldots, m),$$

respectively, where $K = (K^1, \ldots, K^m)$ is a constant function, I_* is a subset of a countable set I of natural indices, $t_0 < T_i \leqslant t_0+T$ $(i \in I)$, $h_i\colon S_{t_0} \to (-\infty, 0]$ $(i \in I_*)$ are some functions and

$$S_{t_0} := \operatorname{int}\{x \in \boldsymbol{R}^n\colon (t_0, x) \in \bar{D}\}.$$

The results obtained are a continuation of those given by the author in [1] and [2], and generalize some results of Chabrowski [3]. If the non-local inequalities considered here are initial inequalities, then the results obtained in this paper reduce to those from [2] and are based on results of Redheffer and Walter [6], of Szarski [7] and of the author [1].

2. Preliminaries. The notation and definitions given in this section are valid throughout the paper.

We use the following notation: $R = (-\infty, \infty)$, $R_- = (-\infty, 0]$, $N = \{1, 2, \ldots\}$, $x = (x_1, \ldots, x_n)$ $(n \in N)$.

For any vectors $z = (z^1, \ldots, z^m) \in R^m$, $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^m) \in R^m$ we write

$$z \leqslant \tilde{z} \quad \text{if} \quad z^i \leqslant \tilde{z}^i \quad (i = 1, \ldots, m).$$

Let t_0 be a real number and let $0 < T < \infty$. A set $D \subset \{(t, x)\colon t > t_0, x \in R^n\}$ (bounded or unbounded) is called a *set of type* (P) if:

1. The projection of the interior of D on the t-axis is the interval $(t_0, t_0 + T)$.
2. For every $(\tilde{t}, \tilde{x}) \in D$ there is a positive r such that

$$\Big\{(t, x)\colon (t-\tilde{t})^2 + \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 < r,\ t < \tilde{t}\Big\} \subset D.$$

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t = \begin{cases} \operatorname{int}\{x \in R^n\colon (t_0, x) \in \bar{D}\} & \text{for } t = t_0, \\ \{x \in R^n\colon (t, x) \in D\} & \text{for } t \neq t_0 \end{cases}$$

and

$$\sigma_t = \begin{cases} \operatorname{int}[\bar{D} \cap (\{t_0\} \times R^n)] & \text{for } t = t_0, \\ D \cap (\{t\} \times R^n) & \text{for } t \neq t_0. \end{cases}$$

It is obvious that S_t and σ_t are open sets in R^n and R^{n+1}, respectively.

Let $\tilde{D}$ be a set contained in $(-\infty, t_0 + T] \times R^n$ such that $\bar{D} \subset \tilde{D}$. We introduce the following sets:

$$\partial_p D := \tilde{D} \backslash D \quad \text{and} \quad \Gamma := \partial_p D \backslash \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{t}, \tilde{x}) \in D$, we denote by $S^-(\tilde{t}, \tilde{x})$ the set of points $(t, x) \in D$ that can be joined with $(\tilde{t}, \tilde{x})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (t, x) to $(\tilde{t}, \tilde{x})$.

Let $Z_m(\tilde{D})$ denote the space of mappings

$$w\colon \tilde{D} \ni (t, x) \to w(t, x) = (w^1(t, x), \ldots, w^m(t, x)) \in R^m$$

continuous in $\tilde{D}$.

In the set of mappings bounded from above in $\tilde{D}$ and belonging to $Z_m(\tilde{D})$ we define the functional

$$[w]_t = \max_{i=1,\ldots,m} \sup\{0, w^i(\tilde{t}, x)\colon (\tilde{t}, x) \in \tilde{D},\ \tilde{t} \leqslant t\}, \quad \text{where } t \leqslant t_0 + T.$$

By X we denote a fixed subset (not necessarily a linear subspace) of $Z_m(\tilde{D})$ and by $M_{n\times n}(\boldsymbol{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n\times n}$.

A mapping $u \in X$ is called *regular* in D if u_t^i, $u_x^i = \operatorname{grad}_x u^i$, $u_{xx}^i = [u_{x_j x_k}^i]_{n\times n}$ $(i = 1, \ldots, m)$ are continuous in D.

Let the mappings

$$f^i\colon D \times \boldsymbol{R}^m \times \boldsymbol{R}^n \times M_{n\times n}(\boldsymbol{R}) \times X \ni (t, x, z, q, r, w) \to f^i(t, x, z, q, r, w) \in \boldsymbol{R} \quad (i = 1, \ldots, m)$$

be given and let the operators P_i $(i = 1, \ldots, m)$ be defined by the formulae

$$P_i u(t, x) = u_t^i(t, x) - f^i(t, x, u(t, x), u_x^i(t, x), u_{xx}^i(t, x), u), \quad u \in X,\ (t, x) \in D \ (i = 1, \ldots, m).$$

A regular mapping u $[v]$ in D is called a *solution* of the system of the functional-differential inequalities

$$P_i u(t, x) \leqslant 0, \quad (t, x) \in D \ (i = 1, \ldots, m) \tag{2.1}$$

$$[P_i v(t, x) \geqslant 0, \quad (t, x) \in D \ (i = 1, \ldots, m)] \tag{2.1'}$$

in D if (2.1) [(2.1′), respectively] is satisfied.

Let us define the following set:

$$Z = \bigcup_{i\in I} \sigma_{T_i},$$

where I is a countable set of all mutually different natural numbers such that:

(i) $\quad t_0 < T_i \leqslant t_0 + T$ for $i \in I$ and $T_i \neq T_j$ for $i, j \in I$, $i \neq j$,

(ii) $\quad T_0 := \inf_{i\in I} T_i > t_0$ if $\operatorname{card} I = \aleph_0$,

(iii) $\quad S_{T_i} \supset S_{t_0}$ for $i \in I$,

(iv) $\quad S_t \supset S_{t_0}$ for every $t \in [T_0, t_0 + T]$ if $\operatorname{card} I = \aleph_0$.

An unbounded set D of type (P) is called a *set of type* (P_{ZT}) (see Fig. 1) if:

(a) $\quad Z \neq \emptyset$,

(b) $\quad \Gamma \cap \bar{\sigma}_{t_0} \neq \emptyset$.

Let Z_* denote a non-empty subset of Z. We define the following set:

$$I_* = \{i \in I\colon \sigma_{T_i} \subset Z_*\}.$$

A bounded set D of type (P) satisfying condition (a) of the definition of a set of type (P_{ZT}) is called a *set of type* (P_{ZB}).

It is easy to see that if D is a set of type (P_{ZB}), then D satisfies condition (b) of the definition of a set of type $(P_{Z\Gamma})$. Moreover, it is obvious that if D_0 is a bounded subset [D_0 is an unbounded proper subset] of $\boldsymbol{R}^n$, then $D = (t_0, t_0+T] \times D_0$ is a set of type (P_{ZB}) [$(P_{Z\Gamma})$, respectively].

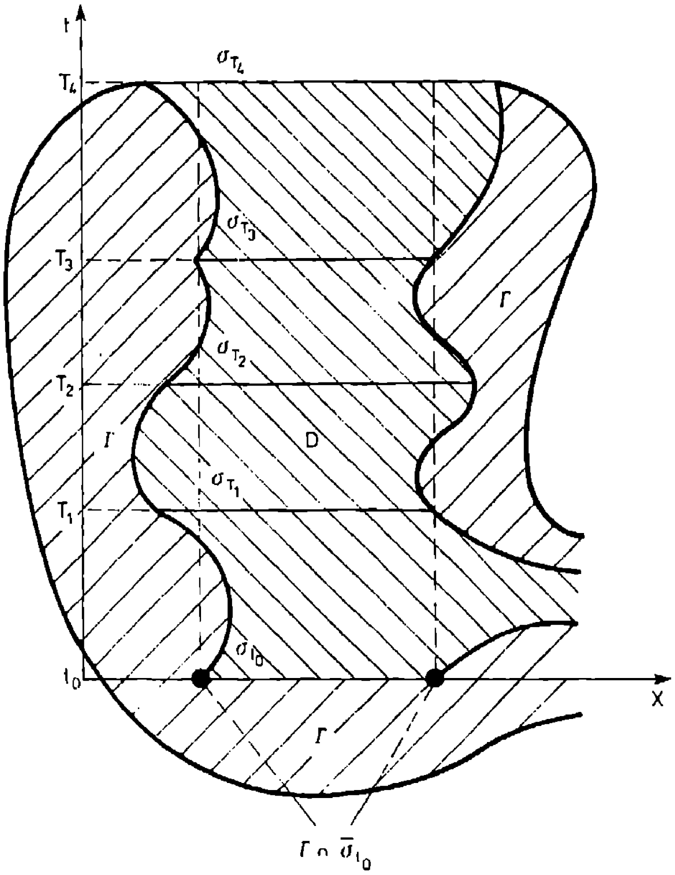

Fig. 1. The set D of type $(P_{Z\Gamma})$ if $D = (\operatorname{int} D) \cup \sigma_{t_0+T}$, $I = \{1, 2, 3, 4\}$ and $t_0 < T_1 < T_2 < T_3 < T_4 = t_0+T$

3. Strong maximum and minimum principles with non-local inequalities in sets of types $(P_{Z\Gamma})$ and (P_{ZB}). Now we shall prove the following theorem on strong maximum principles with non-local inequalities in sets of types $(P_{Z\Gamma})$ and (P_{ZB}):

THEOREM 3.1. *Assume that:*

(1) *D is a set of type $(P_{Z\Gamma})$ or (P_{ZB}).*

(2) *The mappings f^i $(i = 1, \ldots, m)$ are weakly increasing with respect to $z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^m$ $(i = \ldots, m)$. Moreover, there is a positive constant L such that*

$$f^i(t, x, z, q, r, w) - f^i(t, x, \tilde{z}, \tilde{q}, \tilde{r}, \tilde{w})$$
$$\leqslant L\Big(\max_{k=1,\ldots,m} |z^k - \tilde{z}^k| + |x| \sum_{j=1}^{n} |q_j - \tilde{q}^j| + |x|^2 \sum_{j,k=1}^{n} |r_{jk} - \tilde{r}_{jk}| + [w - \tilde{w}]_i\Big)$$

for all $(t, x) \in D$, $z, \tilde{z} \in \boldsymbol{R}^m$, $q, \tilde{q} \in \boldsymbol{R}^n$, $r, \tilde{r} \in M_{n \times n}(\boldsymbol{R})$, $w, \tilde{w} \in X$, where

$$\sup_{(t,x) \in \tilde{D}} [w(t, x) - \tilde{w}(t, x)] < \infty \quad (i = 1, \ldots, m).$$

(3) *The mapping* u *belongs to* X *and the maximum of* u *on* Γ *is attained. Moreover,*

$$K := \max_{(t,x)\in\Gamma} u(t, x) \tag{3.1}$$

and $K \in X$.

(4) *The inequalities*

$$[u^j(t_0, x) - K^j] + \sum_{i\in I_*} h_i(x)[u^j(T_i, x) - K^j] \leqslant 0 \quad \text{for } x\in S_{t_0} \quad (j = 1, \ldots, m) \tag{3.2}$$

are satisfied, where $h_i\colon S_{t_0} \to \boldsymbol{R}_-$ $(i\in I_*)$ *are given functions such that* $-1 \leqslant \sum_{i\in I_*} h_i(x) \leqslant 0$ *for* $x\in S_{t_0}$ *and, additionally, if* $\operatorname{card} I_* = \aleph_0$, *then the series* $\sum_{i\in I_*} h_i(x)u^j(T_i, x)$ $(j = 1, \ldots, m)$ *are convergent for* $x\in S_{t_0}$.

(5) *The maximum of* u *in* $\tilde{D}$ *is attained. Moreover,*

$$M := \max_{(t,x)\in\tilde{D}} u(t, x) \tag{3.3}$$

and $M\in X$.

(6) $f^i(t, x, M, 0, 0, M) \leqslant 0$ *for* $(t, x)\in D$ $(i = 1, \ldots, m)$.

(7) *The mapping* u *is a solution of system* (2.1) *in* D.

(8) *The mappings* f^i $(i = 1, \ldots, m)$ *are parabolic with respect to* u *in* D *and uniformly parabolic with respect to* M *in any compact subset of* D (see [1] or [2]).

Then

$$\max_{(t,x)\in\tilde{D}} u(t, x) = \max_{(t,x)\in\Gamma} u(t, x). \tag{3.4}$$

Moreover, if there is a point $(\tilde{t}, \tilde{x})\in D$ *such that* $u(\tilde{t}, \tilde{x}) = \max_{(t,x)\in\tilde{D}} u(t, x)$, *then*

$$u(t, x) = \max_{(t,x)\in\Gamma} u(t, x) \quad \text{for } (t, x)\in S^-(\tilde{t}, \tilde{x}).$$

Proof. We shall prove Theorem 3.1 for a set of type $(P_{Z\Gamma})$ only since the proof for a set of type (P_{ZB}) is analogous.

Since each set of type $(P_{Z\Gamma})$ is a set of type (P_Γ) from [2], it follows that in the case where $\sum_{i\in I_*} h_i(x) = 0$ for $x\in S_{t_0}$, Theorem 3.1 is a consequence of Theorem 4.1 of [2]. Therefore, we shall give the proof of Theorem 3.1 only in the case where

$$-1 \leqslant \sum_{i\in I_*} h_i(x) < 0 \quad \text{for } x\in S_{t_0}. \tag{3.5}$$

Assume, so, that (3.5) holds and, since we shall argue by contradiction, suppose

(3.6) $$M \neq K.$$

From (3.1) and (3.3) we have

(3.7) $$K \leqslant M.$$

Consequently,

(3.8) $$K < M.$$

Observe, from assumption (5), that

(3.9) There is $(t^*, x^*) \in \tilde{D}$ such that $u(t^*, x^*) = M := \max\limits_{(t,x)\in\tilde{D}} u(t, x)$.

By (3.9), by assumption (3) and by (3.8) we have

(3.10) $$(t^*, x^*) \in \tilde{D} \backslash \Gamma = D \cup \sigma_{t_0}.$$

An argument analogous to the proof of Theorem 4.1 from [2] yields

$$(t^*, x^*) \notin D.$$

Hence

(3.11) $$(t^*, x^*) \in \sigma_{t_0}.$$

On the other hand, by the definition of the sets I and I_*, we must consider the following cases:

(A) I_* is a finite set, i.e., without loss generality there is a number $p \in N$ such that $I_* = \{1, \ldots, p\}$.

(B) $\operatorname{card} I_* = \aleph_0$.

First we shall consider case (A). By (3.2) and by the inequalities $u(T_i, x^*) < u(t_0, x^*)$ $(i = 1, \ldots, p)$, which are consequences of (3.9) and (3.11) and of (a) (i) and (a) (iii) of the definition of a set of type (P_{Zr}), we have

$$0 \geqslant [u^j(t_0, x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*)\,[u^j(T_i, x^*) - K^j]$$

$$\geqslant [u^j(t_0, x^*) - K^j] \cdot [1 + \sum_{i=1}^{p} h_i(x^*)] \quad (j = 1, \ldots, m).$$

Hence

(3.12) $$u(t_0, x^*) \leqslant K \quad \text{if} \quad 1 + \sum_{i=1}^{p} h_i(x^*) > 0.$$

Then, from (3.8) and (3.11), we obtain a contradiction of (3.12) with (3.9). Assume now $\sum_{i=1}^{p} h_i(x^*) = -1$. Since for every $j \in \{1, \ldots, m\}$ there is $l_j \in \{1, \ldots, p\}$ such that

$$u_j(T_{l_j}, x^*) = \max_{i=1,\ldots,p} u^j(T_i, x^*),$$

we obtain by (3.2)

$$\begin{aligned} u^j(t_0, x^*) - u^j(T_{l_j}, x^*) &= [u^j(t_0, x^*) - K^j] - [u^j(T_{l_j}, x^*) - K^j] \\ &= [u^j(t_0, x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*) [u^j(T_{l_j}, x^*) - K^j] \\ &\leqslant [u^j(t_0, x^*) - K^j] + \sum_{i=1}^{p} h_i(x^*) [u^j(T_i, x^*) - K^j] \leqslant 0 \\ &\qquad (j = 1, \ldots, m). \end{aligned}$$

Hence

$$(3.13) \quad u^j(t_0, x^*) \leqslant u^j(T_{l_j}, x^*) \quad (j = 1, \ldots, m) \quad \text{if} \quad \sum_{i=1}^{p} h_i(x^*) = -1.$$

Since, by (a) (i) of the definition of a set of type $(P_{Z\Gamma})$, $T_{l_j} > t_0$ $(j = 1, \ldots, m)$, we see from (3.11) that (3.13) contradicts (3.9). This completes the proof of (3.4) if I_* is a finite set.

It remains to investigate case (B). Analogously to the proof of (3.4) in case (A), by assumption (4) and by the inequalities $u(T_i, x^*) < u(t_0, x^*)$ $(i \in I_*)$ we have

$$\begin{aligned} 0 &\geqslant [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*) [u^j(T_i, x^*) - K^j] \\ &\geqslant [u^j(t_0, x^*) - K^j] \cdot [1 + \sum_{i \in I_*} h_i(x^*)] \quad (j = 1, \ldots, m). \end{aligned}$$

Hence

$$(3.14) \qquad u(t_0, x^*) \leqslant K \quad \text{if} \quad 1 + \sum_{i \in I_*} h_i(x^*) > 0.$$

Then, from (3.8) and (3.11), we obtain a contradiction of (3.14) with (3.9). Assume now $\sum_{i \in I_*} h_i(x^*) = -1$ and let

$$T_0^* := \inf_{i \in I_*} T_i.$$

Since $u \in C(\bar{D})$ and since, by (a) (iv) of the definition of a set of type $(P_{Z\Gamma})$, $x^* \in S_t$ for every $t \in [T_0, t_0 + T]$ if $\operatorname{card} I = \aleph_0$, it follows that for every $j \in \{1, \ldots, m\}$ there is $\hat{t}_j \in [T_0^*, t_0 + T]$ such that

$$u^j(\hat{t}_j, x^*) = \max_{t \in [T_0^*, t_0 + T]} u^j(t, x^*).$$

Consequently, by assumption (4), we obtain

$$\begin{aligned} u^j(t_0, x^*) - u^j(\hat{t}_j, x^*) &= [u^j(t_0, x^*) - K^j] - [u^j(\hat{t}_j, x^*) - K^j] \\ &= [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*)\,[u^j(\hat{t}_j, x^*) - K^j] \\ &\leqslant [u^j(t_0, x^*) - K^j] + \sum_{i \in I_*} h_i(x^*)\,[u^j(T_i, x^*) - K^j] \leqslant 0 \\ &\qquad\qquad (j = 1, \ldots, m). \end{aligned}$$

Hence

$$u^j(t_0, x^*) \leqslant u^j(\hat{t}_j, x^*) \quad (j = 1, \ldots, m) \quad \text{if} \quad \sum_{i \in I_*} h_i(x^*) = -1. \tag{3.15}$$

Since, by (a) (ii) of the definition of a set of type $(P_{Z\Gamma})$, $\hat{t}_j > t_0$ $(j = 1, \ldots, m)$, we see from (3.11) that (3.15) contradicts (3.9). This completes the proof of (3.4).

The second part of Theorem 3.1 is a consequence of (3.4) and of Lemma 3.1 from [2]. Therefore, the proof of Theorem 3.1 is complete.

Arguing analogously to the proof of Theorem 3.1, we obtain the following theorem on strong minimum principles with non-local inequalities in sets of types $(P_{Z\Gamma})$ and (P_{ZB}):

THEOREM 3.2. *Assume that:*

1. *Assumptions* (1) *and* (2) *of Theorem* 3.1 *hold.*

2. *The mapping v belongs to X and the minimum of v on Γ is attained. Moreover,*

$$k := \min_{(t,x) \in \Gamma} v(t, x)$$

and $k \in X$.

3. $[v^j(t_0, x) - k^j] + \sum_{i \in I_*} h_i(x)[v^j(T_i, x) - k^j] \geqslant 0$ *for* $x \in S_{t_0}$ $(j = 1, \ldots, m)$, *where $h_i\colon S_{t_0} \to \mathbf{R}_-$ $(i \in I_*)$ are given functions such that $-1 \leqslant \sum_{i \in I_*} h_i(x) \leqslant 0$ for $x \in S_{t_0}$ and, additionally, if* $\operatorname{card} I_* = \aleph_0$, *then the series $\sum_{i \in I_*} h_i(x) v^j(T_i, x)$ $(j = 1, \ldots, m)$ are convergent for $x \in S_{t_0}$.*

4. *The minimum of v in $\tilde{D}$ is attained. Moreover,*

$$\mathfrak{m} := \min_{(t,x) \in \tilde{D}} v(t, x)$$

and $\mathfrak{m} \in X$.

5. $f^i(t, x, \mathfrak{m}, 0, 0, \mathfrak{m}) \geqslant 0$ *for* $(t, x) \in D$ $(i = 1, \ldots, m)$.

6. *The mapping v is a solution of system* (2.1′) *in D.*

7. *The mappings f^i $(i = 1, \ldots, m)$ are parabolic with respect to* m *in D and uniformly parabolic with respect to v in any compact subset of D.*

Then

$$\min_{(t,x)\in\tilde{D}} v(t, x) = \min_{(t,x)\in\Gamma} v(t, x).$$

Moreover, if there is a point $(\tilde{t}, \tilde{x})\in D$ such that $v(\tilde{t}, \tilde{x}) = \min_{(t,x)\in\tilde{D}} v(t, x)$, then

$$v(t, x) = \min_{(t,x)\in\Gamma} v(t, x) \quad \textit{for } (t, x)\in S^-(\tilde{t}, \tilde{x}).$$

4. Remarks.

Remark 4.1. It is easy to see, by the proof of Theorem 3.1 from this paper and by the proof of Theorem 4.1 from [2], that if the functions h_i $(i\in I_*)$ from assumptions (4) and 3 of Theorems 3.1 and 3.2, respectively, satisfy the condition

$$[\sum_{i\in I_*} h_i(x) = 0] \qquad -1 < \sum_{i\in I_*} h_i(x) \leqslant 0 \quad \text{ for } x\in S_{t_0},$$

then it is sufficient to assume in these theorems that [D is an unbounded set of type (P) satisfying condition (b) of the definition of a set of type $(P_{Z\Gamma})$ or D is a bounded set of type (P), i.e., according to the terminology of [2], D is a set of type (P_Γ) or (P_B), respectively] D is an unbounded set of type (P) satisfying conditions (a) (i), (a) (iii) and (b) of the definition of a set of type $(P_{Z\Gamma})$ or D is a bounded set of type (P) satisfying conditions (a) (i) and (a) (iii) of the definition of a set of type $(P_{Z\Gamma})$. Moreover, if I_* is a finite set and $-1 \leqslant \sum_{i\in I_*} h_i(x) \leqslant 0$ for $x\in S_{t_0}$, then it is sufficient to assume in Theorems 3.1 and 3.2 that D is an unbounded set of type (P) satisfying conditions (a) (i), (a) (iii) and (b), or D is a bounded set of type (P) satisfying conditions (a) (i) and (a) (iii).

Remark 4.2. If D is a set of type (P_{ZB}) and if $\tilde{D} = \bar{D}$, then the first parts of assumptions (3) and 2 of Theorems 3.1 and 3.2 relative to the maximum of u and the minimum of v and the first parts of assumptions (5) and 4 of these theorems are trivially satisfied since $u, v\in C(\bar{D})$ and Γ is a bounded and closed set in this case.

Remark 4.3. If the mappings f^i $(i = 1, \ldots, m)$ do not depend on the functional argument w, then Theorems 3.1 and 3.2 reduce to theorems on parabolic differential inequalities of the form

$$u_t^i(t, x) \underset{(\geqslant)}{\leqslant} f^i(t, x, u(t, x), u_x^i(t, x), u_{xx}^i(t, x)) \quad (i = 1, \ldots, m)$$

and in this case we can put $\tilde{D} = \bar{D}$.

Remark 4.4. Non-local conditions have very interesting physical applications. Namely, non-local conditions together with boundary conditions often give better descriptions of diffusion phenomena than the initial conditions together with the boundary conditions from the non-local problems. For example, in gaseous diffusion the measurement of amount of a gas in the form of the following sum

$$u(t_0, x)+\sum_{i\in I_*} h_i(x)u(T_i, x)$$

(h_i ($i\in I_*$) are known functions) is usually more precise than the measurement of amount of this gas at the initial instant t_0.

If $I_*=\{1\}$, $T_1=t_0+T$ and $h_1(x)=-1$ for $x\in S_{t_0}$, then the non-local condition

$$u(t_0, x)+\sum_{i\in I_*} h_i(x)u(T_i, x)=0 \quad \text{for } x\in S_{t_0}$$

reduces to the following periodic condition:

$$u(t_0, x)=u(t_0+T, x) \quad \text{for } x\in S_{t_0}$$

and this condition can be used to the description of heat effects in atomic reactors.

References

[1] L. Byszewski, *Strong maximum principle for implicit non-linear parabolic functional-differential inequalities in arbitrary domains*, Univ. Iagell. Acta Math. 24 (1984), 327–339.

[2] —, *Strong maximum and minimum principles for parabolic functional-differential problems with initial inequalities* $u(t_0, x) \underset{(\geqslant)}{\leqslant} K$, this fasc., 187-194.

[3] J. Chabrowski, *On non-local problems for parabolic equations*, Nagoya Math. J. 93 (1984), 109–131.

[4] P. Hess, *Spatial homogeneity of stable solutions of some periodic-parabolic problems with Neumann boundary conditions*, J. Differential Equations 68 (1987), 320–331.

[5] A. A. Kerefov, *Non-local boundary value problems for parabolic equation* (in Russian), Diff. Uravn. 15 (1979), 74–78.

[6] R. Redheffer, W. Walter, *Das Maximumprinzip in unbeschränkten Gebieten für parabolische Ungleichungen mit Funktionalen*, Math. Ann. 226 (1977), 155–170.

[7] J. Szarski, *Strong maxinum principle for non-linear parabolic differential-functional inequalities in arbitrary domains*, Ann. Polon. Math. 29 (1974), 207–217.

[8] P. N. Vabishchevich, *Non-local parabolic problems and the inverse heat-conduction problem* (in Russian), Diff. Uravn. 17 (1981), 1193–1199.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF CRACOW
KRAKÓW, POLAND

Reçu par la Rédaction le 5.08.1988