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The prof of the uniqueness of the solution of a mixed problem for a class of partial differentatial equations of even order

by Z. STOJEK (Kraków)

In this paper we present the proof of the uniqueness of the solution of a mixed problem in the cylinder for linear partial differential equations of even order; these equations correspond to the Euler-Lagrange equation of given variation problems. The method of the proof is similar to that applied by S. Zaremba, K. Friedrichs and others [1] to hyperbolic equations.

Let us consider the partial differential equation of the $2n$-th order in the form

$$(1)\quad \sum_{h=0}^{n} \sum_{\substack{i_1+\ldots+i_m=h \\ j_1+\ldots+j_m=h}} (-1)^h \frac{\partial^h}{\partial x_1^{i_1}\ldots \partial x_m^{i_m}} \left[a_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}}(X) \frac{\partial^h w(X,t)}{\partial x_1^{j_1}\ldots \partial x_m^{j_m}} \right] + \mu(X) \frac{\partial^2 w(X,t)}{\partial t^2} = f(X,t)$$

and the corresponding homogenous equation

$$(2)\quad \sum_{h=0}^{n} \sum_{\substack{i_1+\ldots+i_m=h \\ j_1+\ldots+j_m=h}} (-1)^h \frac{\partial^h}{\partial x_1^{i_1}\ldots \partial x_m^{i_m}} \left[a_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}}(X) \frac{\partial^h w(X,t)}{\partial x_1^{j_1}\ldots \partial x_m^{j_m}} \right] + \mu(X) \frac{\partial^2 w(X,t)}{\partial t^2} = 0 .$$

Let us suppose that $a_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}}(X)$ $(i_1+\ldots+i_m = h;\ j_1+\ldots+j_m = h;\ h = 0,1,\ldots,n)$ belongs to the class C^h in Ω and that the following relations hold:

$$(3)\quad a_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}}(X) = a_{\substack{j_1,\ldots,j_m \\ i_1,\ldots,i_m}}(X), \qquad X \in \Omega$$
$$(i_1+\ldots+i_m = h,\ j_1+\ldots+j_m = h,\ h = 1,2,\ldots,n)$$

and we assume the continuity of the function $\mu(X)$ in Ω.

Further, we assume that the domain Ω is closed, bounded and normal with respect to the system $(x_1, x_2, \ldots, x_m)$ and its boundary $F\Omega$ is an surface of the class C^1_σ (cf. [1], p. 132).

THEOREM 1. *If the quadratic form*

$$\sum_{h=0}^{n} \sum_{\substack{i_1+\ldots+i_m=h \\ j_1+\ldots+j_m=h}} a_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}}(X)\lambda_{i_1,\ldots,i_m}\lambda_{j_1,\ldots,j_m} \tag{4}$$

is positive semi-definite in Ω, and $\mu(X) > 0$ in Ω, then equation (1) *within the range of the functions of class C^3 with continuous derivatives of $2n$-th order with regard to spatial variables in the semi-cylinder Σ: $\{(X, t);\ X \in \Omega,\ t \geqslant 0\}$ can have at most one solution satisfying the initial conditions*

$$w(X, 0) = \varphi_0(X), \qquad \frac{\partial w(X, t)}{\partial t}\bigg|_{t=0} = \varphi_1(x) \quad \text{for} \quad X \in \Omega \tag{5}$$

and the boundary conditions

$$\frac{\partial^h w(X, t)}{\partial n^h} = \psi_h(X, t) \quad (h = 0, 1, \ldots, n-1) \quad \text{for} \quad X \in F\Omega \quad \text{and} \quad t \geqslant 0. \tag{6}$$

The truth of this theorem is easily demonstrated by the following

THEOREM 2. *If the quadratic form is positive semi-definite in Ω, and $\mu(X) > 0$ in Ω, then the only solution of the homogenous equation* (2) *within the range of the functions of class C^3 with continuous derivatives of $2n$-th order with respect to spatial variables in Σ, fulfilling the initial conditions*

$$w(X, 0) = 0, \qquad \frac{\partial w(X, t)}{\partial t}\bigg|_{t=0} = 0 \quad \text{for} \quad X \in \Omega \tag{7}$$

and the boundary conditions

$$\frac{\partial^h w(X, t)}{\partial n^h} = 0 \quad (h = 0, 1, \ldots, n-1) \quad \text{for} \quad X \in F\Omega \quad \text{and} \quad t \geqslant 0 \tag{8}$$

is the function identically equal to zero in the semi-cylinder Σ.

Proof. We multiply both sides of equation (2) by $\partial w/\partial t$ and integrate over the area $D(X \in \Omega,\ 0 \leqslant t \leqslant T$, where T is an arbitrary number $0 < T < \infty)$; we obtain the equality

$$\int_0^T dt \iiint_{\Omega} \frac{\partial w}{\partial t}\Bigg\{\sum_{h=0}^{n} \sum_{\substack{i_1+\ldots+i_m=h \\ j_1+\ldots+j_m=h}} (-1)^h \frac{\partial^h}{\partial x_1^{i_1}\ldots\partial x_m^{i_m}}\left[a_{\substack{i_1,\ldots,i_m \\ j_1,\ldots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{j_1}\ldots\partial x_m^{j_m}}\right] + \mu(X)\frac{\partial^2 w}{\partial t^2}\Bigg\} dX = 0. \tag{9}$$

From the formula for integration by parts the m-fold integrals [1]

$$\text{(10)}\quad \underset{\Omega}{\int\int\int} P(X)\frac{\partial Q(X)}{\partial x_k}dX = -\underset{F\Omega}{\int\int} P(X)Q(X)\cos(nx_k)d\sigma - \underset{\Omega}{\int\int\int} Q(X)\frac{\partial P(X)}{\partial x_k}dX \quad (k = 1, 2, \dots, m)$$

resulting from Green's theorem, we arrive by induction, for an arbitrary $t \in [0, T]$, at the equality

$$\text{(11)}\quad \underset{\Omega}{\int\int\int} \frac{\partial w}{\partial t}\cdot\frac{\partial^h}{\partial x_1^{i_1}\dots\partial x_k^{i_k}\dots\partial x_m^{i_m}}\left[a_{\substack{i_1,\dots,i_m\\ j_1,\dots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{j_1}\dots\partial x_m^{j_m}}\right]dX$$

$$= -\underset{F\Omega}{\int\int} \frac{\partial w}{\partial t}\cdot\frac{\partial^{h-1}}{\partial x_1^{i_1}\dots\partial x_k^{i_k-1}\dots\partial x_m^{i_m}}\left[a_{\substack{i_1,\dots,i_m\\ j_1,\dots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{j_1}\dots\partial x_m^{j_m}}\right]\times$$

$$\times\cos(n, x_k)d\sigma +$$

$$+\underset{F\Omega}{\int\int} \frac{\partial^2 w}{\partial x_1\cdot\partial t}\cdot\frac{\partial^{h-2}}{\partial x_1^{i_1-1}\dots\partial x_k^{i_k-1}\dots\partial x_m^{i_m}}\left[a_{\substack{i_1,\dots,i_m\\ j_1,\dots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{j_1}\dots\partial x_m^{j_m}}\right]\times$$

$$\times\cos(n, x_1)d\sigma + \dots +$$

$$+(-1)^h\underset{F\Omega}{\int\int} \frac{\partial^h w}{\partial x_1^{i_1}\dots\partial x_k^{i_k-1}\dots\partial x_m^{i_m}\cdot\partial t}\, a_{\substack{i_1,\dots,i_m\\ j_1,\dots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{j_1}\dots\partial x_m^{j_m}}\times$$

$$\times\cos(n, x_m)d\sigma +$$

$$+(-1)^h\underset{\Omega}{\int\int\int} \frac{\partial^{h+1} w}{\partial x_1^{i_1}\dots\partial x_k^{i_k}\dots\partial x_m^{i_m}\cdot\partial t}\, a_{\substack{i_1,\dots,i_m\\ j_1,\dots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{j_1}\dots\partial x_m^{j_m}}dX$$

$$(h = 1, 2, \dots, n;\ i_k \geqslant 1;\ i_1 + \dots + i_k + \dots + i_m = h;$$

$$j_1 + \dots + j_m = h;\ k = 1, 2, \dots, m).$$

From the boundary conditions (8) it follows that the derivatives of the functions $w(X, t)$ on the boundary $F\Omega$ equal zero in an arbitrary tangential direction as far as the range $n-1$ for each $t \geqslant 0$, and the derivatives in the direction of the normal as far as $n-1$ equal zero: hence, for $X \in F\Omega$ and $t \geqslant 0$, we have

$$\text{(12)}\quad \frac{\partial^h w}{\partial x_1^{i_1}\dots\partial x_m^{i_m}} = 0 \quad (h = 0, 1, \dots, n-1;\ i_1 + \dots + i_m = h)$$

and consequently

$$\text{(13)}\quad \frac{\partial^{h+1} w}{\partial x_1^{i_1}\dots\partial x_m^{i_m}\cdot\partial t} = 0 \quad \text{for} \quad X \in F\Omega, \quad t \geqslant 0$$

$$(h = 0, 1, \dots, n-1;\ i_1 + \dots + i_m = h).$$

From (13) it follows that the surface integrals in inequality (11) equal zero, and thus we obtain:

$$(14)\qquad \mathop{\mathrm{SSS}}_{\Omega} \frac{\partial w}{\partial t}\cdot\frac{\partial^h}{\partial x_1^{i_1}\ldots\partial x_m^{i_m}}\left[a_{\substack{i_1,\ldots,i_m\\ j_1,\ldots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{j_1}\ldots\partial x_m^{j_m}}\right]dX$$

$$=(-1)^h \mathop{\mathrm{SSS}}_{\Omega} \frac{\partial^{h+1} w}{\partial x_1^{i_1}\ldots\partial x_m^{i_m}\cdot\partial t}\,a_{\substack{i_1,\ldots,i_m\\ j_1,\ldots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{j_1}\ldots\partial x_m^{j_m}}\,dX$$

$$(h=1,2,\ldots,n;\ i_1+\ldots+i_m=h;\ j_1+\ldots+j_m=h).$$

Substituting equality (14) in (9) we arrive at

$$(15)\qquad \int_0^T dt \mathop{\mathrm{SSS}}_{\Omega}\left\{\sum_{h=0}^{n}\sum_{\substack{i_1+\ldots+i_m=h\\ j_1+\ldots+j_m=h}}\left[a_{\substack{i_1,\ldots,i_m\\ j_1,\ldots,j_m}}(X)\frac{\partial^{h+1} w}{\partial x_1^{i_1}\ldots\partial x_m^{i_m}\cdot\partial t}\cdot\frac{\partial^h w}{\partial x_1^{j_1}\ldots\partial x_m^{j_m}}\right]+\right.$$

$$\left.+\mu(X)\frac{\partial^2 w}{\partial t^2}\cdot\frac{\partial w}{\partial t}\right\}dX=0\,.$$

Considering relations (3), after having integrated the left side of equality (15) with respect to the parameter t, we obtain the following equality:

$$(16)\qquad \mathop{\mathrm{SSS}}_{\Omega}\left\{\frac{1}{2}\sum_{h=0}^{n}\sum_{\substack{i_1+\ldots+i_m=h\\ j_1+\ldots+j_m=h}}\left[a_{\substack{i_1,\ldots,i_m\\ j_1,\ldots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{i_1}\ldots\partial x_m^{i_m}}\cdot\frac{\partial^h w}{\partial x_1^{j_1}\ldots\partial x_m^{j_m}}\right]+\right.$$

$$\left.+\frac{1}{2}\mu(X)\left(\frac{\partial w}{\partial t}\right)^2\right\}\Bigg|_0^T dX=0\,.$$

Since the initial conditions are (7) by $t=0$, the subintegral expression equals zero; and since T has been an arbitrary positive number, hence for each $t>0$ the equality

$$(17)\qquad \mathop{\mathrm{SSS}}_{\Omega}\left\{\frac{1}{2}\sum_{h=0}^{n}\sum_{\substack{i_1+\ldots+i_m=0\\ j_1+\ldots+j_m=0}}\left[a_{\substack{i_1,\ldots,i_m\\ j_1,\ldots,j_m}}(X)\frac{\partial^h w}{\partial x_1^{i_1}\ldots\partial x_m^{i_m}}\cdot\frac{\partial^h w}{\partial x_1^{j_1}\ldots\partial x_m^{j_m}}\right]+\right.$$

$$\left.+\frac{1}{2}\mu(X)\left(\frac{\partial w}{\partial t}\right)^2\right\}dX=0$$

is true.

The quadratic form (4) being positive semi-definite in Ω, $\mu(X)>0$ in Ω and the continuous derivative of the function $w(X,t)$ being $\partial w/\partial t$, it follows from the equality (17) that in the semi-cylinder Σ we have

$$(18)\qquad \frac{\partial w(X,t)}{\partial t}\equiv 0\,,$$

and that means that the function $w(X, t)$ is independent of the parameter t; being continuous in Σ, this function must also satisfy the homogeneous initial conditions (7); hence $w(X, t) \equiv 0$ in Σ. Q.e.d.

Reference

[1] M. Krzyżański, *Równania różniczkowe cząstkowe rzędu drugiego*, Cz. 1., Warszawa 1957.

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