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An additional note on entire functions represented by Dirichlet series (II)

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1. Introduction. This note makes out a significant point supplementary and relevent to the earlier note with the same title [3]. As in the earlier note, the entire Dirichlet series

$$f(s) = \sum_{1}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \quad 0 < \lambda_n < \lambda_{n+1} \quad (n \geqslant 1), \quad \lim \lambda_n = \infty,$$

is defined to be absolutely convergent for all finite s; and we also define for it, as in the another note [2], the following concepts plainly related to one another:

$$\mu(\sigma) = \max_{n \geqslant 1} |a_n e^{(\sigma+it)\lambda_n}| \equiv |a_\nu| e^{\sigma\lambda_\nu},$$

where ν and hence λ_ν depend on σ, so that

$$\lambda_\nu = \lambda_{\nu(\sigma)} \equiv \Lambda(\sigma),$$

is a monotonic increasing function of σ,

$$\limsup_{\sigma\to\infty} \frac{\log\log\mu(\sigma)}{\sigma} = \varrho_*;$$

$$\liminf_{\sigma\to\infty} \frac{\log\log\mu(\sigma)}{\sigma} = \lambda_*.$$

The point made out in this note has a two-fold significance.

(a) There is in general a set of results for $\mu(\sigma)$ involving the associated function $\Lambda(\sigma)$ and the associated (Sugimura) orders ϱ_* and λ_* exactly parallel to any known set of results for

$$M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma+it)|,$$

involving the associated function $w(\sigma)$ and the associated (Ritt) orders ϱ and λ. In illustration of this, the present note gives a set of results for $\mu(\sigma)$ parallel to the set for $M(\sigma)$ in the earlier note [3].

(b) While each set of results, by itself, does not require any additional condition on $\{\lambda_n\}$, such an additional condition may be needed when it is sought to connect two results, one in each set (e.g. [2], Theorem 1). The superfluity of the extra condition on $\{\lambda_n\}$, in the circumstances stated, is obscured, in the whole literature as, for instance, in two theorems given by Kamthan in a recent paper ([1], Theorems B and E). Kamthan's theorems are only Lemma 2 and Theorem II of this note supplemented by the result $\varrho_* = \varrho$ and $\lambda_* = \lambda$ which is known to hold ([2], Theorem 1) under an extra condition on $\{\lambda_n\}$, much less restrictive than that assumed by Kamthan.

2. Lemmas. In the following lemmas $\mu^j(\sigma)$, $\Lambda^j(\sigma)$, ϱ_*^j and λ_*^j $(j = 1, 2, \ldots)$ are defined for $f^j(s)$, the entire Dirichlet series obtained by j termwise differentiations of $f(s)$, exactly as $\mu(\sigma)$, $\Lambda(\sigma)$, ϱ_* and λ_* are defined for $f(s)$. Moreover, the lemmas and the theorems after them bear each the number as its analogue (exact or rough) in [3].

LEMMA 1 (cf. Lemmas 1 and 1′ of [3]). *For any* $\delta > 0$,

$$\mu(\sigma) < \text{const}\cdot\mu^j(\sigma)\,, \tag{1}$$

$$\mu^j(\sigma) \leqslant \frac{j!}{\delta^j}\,\mu(\sigma+\delta)\,. \tag{2}$$

Proof. We have (as in [2], proof of Theorem 2)

$$\Lambda(\sigma) \leqslant \frac{\mu'(\sigma)}{\mu(\sigma)} \leqslant \Lambda'(\sigma) \leqslant \frac{\mu^2(\sigma)}{\mu^1(\sigma)} \leqslant \ldots \tag{3}$$

Hence $\Lambda(\sigma)$, $\Lambda'(\sigma)$, ... being all monotonic increasing functions of σ, we get

$$\mu(\sigma) < \text{const}\cdot\mu'(\sigma),\ \ldots,\ \mu^{j-1}(\sigma) < \text{const}\cdot\mu^j(\sigma)$$

and (1) follows when we multiply together the above inequalities.

To prove (2), we have only to note that, by definition

$$\mu^j(\sigma) = \lambda_\nu^j|a_\nu|\,e^{\sigma\lambda_\nu}\,,$$

where of course ν depends on j as well as σ. Since $\lambda_\nu^j\delta^j/j! < e^{\delta\lambda_\nu}$ for any $\delta > 0$, we have at once

$$\mu^j(\sigma) \leqslant \frac{j!}{\delta^j}\,|a_\nu|\,e^{(\sigma+\delta)\lambda_\nu} \leqslant \frac{j!}{\delta^j}\,\mu(\sigma+\delta)\,.$$

LEMMA 2 (cf. Lemmas 2 and 2′ of [3]). *We have*

$$\limsup_{\sigma\to\infty}\frac{\log A(\sigma)}{\sigma}=\varrho_*\,,$$

$$\liminf_{\sigma\to\infty}\frac{\log A(\sigma)}{\sigma}=\lambda_*\,,$$

where $A(\sigma)$ *alone may be replaced by* $A^j(\sigma)$ *by Theorem* 1 *of this note.*

The main result of Lemma 2 is the same as that of Lemma 3 in [2].

LEMMA 3 (cf. Lemmas 3 and 3 of [3]). *For* $j=1,2,\ldots$

$$\limsup_{\sigma\to\infty}\frac{\log\mu^j(\sigma)/\mu(\sigma)}{\sigma}\geqslant j\varrho_*\,,$$

$$\liminf_{\sigma\to\infty}\frac{\log\mu^j(\sigma)/\mu(\sigma)}{\sigma}\geqslant j\lambda_*\,.$$

Proof. From (3) we obtain

$$\frac{\mu^j(\sigma)}{\mu(\sigma)}=\frac{\mu^j(\sigma)}{\mu^{j-1}(\sigma)}\cdot\frac{\mu^{j-1}(\sigma)}{\mu^{j-2}(\sigma)}\cdots\frac{\mu'(\sigma)}{\mu(\sigma)}\geqslant A^{j-1}(\sigma)\cdot A^{j-2}(\sigma)\ldots A(\sigma)\geqslant\{A(\sigma)\}^j\,.$$

Hence we get, taking logarithms, and using Lemma 2

$$\limsup_{\sigma\to\infty}\frac{\log[\mu^j(\sigma)/\mu(\sigma)]}{\sigma}\geqslant j\limsup_{\sigma\to\infty}\frac{\log A(\sigma)}{\sigma}=j\varrho_*\,,$$

$$\liminf_{\sigma\to\infty}\frac{\log[\mu^j(\sigma)/\mu(\sigma)]}{\sigma}\geqslant j\liminf_{\sigma\to\infty}\frac{\log A(\sigma)}{\sigma}=j\lambda_*\,.$$

3. Theorems. The proofs of all the results which follow, being exactly similar to the proofs of the analogous results of [3], are omitted.

THEOREM I. $\varrho_*=\varrho'_*$, $\lambda_*=\lambda'_*$.

THEOREM II. *For* $j=1,2,\ldots,$

$$\limsup_{\sigma\to\infty}\frac{\log[\mu^j(\sigma)/\mu(\sigma)]^{1/j}}{\sigma}=\varrho_*\,,$$

$$\liminf_{\sigma\to\infty}\frac{\log[\mu^j(\sigma)/\mu(\sigma)]^{1/j}}{\sigma}=\lambda_*\,.$$

COROLLARY II. *We have*

$$\log\mu'(\sigma)\approx\log\mu(\sigma)\qquad(0\leqslant\varrho_*<\infty)\,,$$

$$\tau_*=\tau'_*\,,\qquad\omega_*=\omega'_*\qquad(0<\varrho_*<\infty)\,,$$

where

$$\tau_* = \limsup_{\sigma\to\infty} \frac{\log\mu(\sigma)}{e^{\sigma\varrho_*}}, \qquad \omega_* = \liminf_{\sigma\to\infty} \frac{\log\mu(\sigma)}{e^{\sigma\varrho_*}},$$

and τ'_* *and* ω'_* *are defined similarly.*

Remark. The proof indicated above for Theorem II can be more shortly presented by using the result of Lemma 3 in conjunction with the formulae

$$\limsup_{\sigma\to\infty} \frac{\log[\mu^j(\sigma)/\mu(\sigma)]}{\sigma} \leqslant j\varrho_*, \qquad \liminf_{\sigma\to\infty} \frac{\log[\mu^j(\sigma)/\mu(\sigma)]}{\sigma} \leqslant j\lambda_*,$$

which may be readily obtained from (3) and an appeal to the part of Lemma 2 for $\Lambda^j(\sigma)$.

Added in proof.

THEOREM III. *For* $j = 1, 2, \ldots,$

$$\varrho_* = \limsup_{\sigma\to\infty} \frac{\log[\mu^j(\sigma) + \ldots + \mu^1(\sigma)/\mu^{j-1}(\sigma) + \ldots + \mu(\sigma)]}{\sigma},$$

$$\lambda_* = \liminf_{\sigma\to\infty} \frac{\log[\mu^j(\sigma) + \ldots + \mu^1(\sigma)/\mu^{j-1}(\sigma) + \ldots + \mu(\sigma)]}{\sigma}.$$

The proof cf this follows easili from (3) and Theorem I, hence omitted.

References

[1] P. K. Kamthan, *On entire functions represented by Dirichlet series*, Ann. Inst. Fourier 16 (1966), pp. 202-224.

[2] C. T. Rajagopal and A. R. Reddy, *A note on entire functions represented by Dirichlet series*, Ann. Polon. Math. 17 (1965), pp. 199-208.

[3] A. R. Reddy, *An additional note on entire functions represented by Dirichlet series*, ibidem 19 (1967), pp. 147-152.

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