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Existence of invariant measures for piecewise continuous transformations

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Abstract. We show the existence of absolutely continuous invariant measures for a class of piecewise continuous transformations from the unit interval into itself.

1. Introduction. Let τ be a transformation from the unit interval into itself, piecewise smooth. Lasota and Yorke [2] discovered an interesting result for such maps. It says that "general instability" implies trajectories wander so irregularly that their long term behavior can be described statistically in terms of invariant measures. The instability assumption is $|\tau'(x)| \geqslant \lambda > 1$, whenever $\tau'(x)$ is defined. In particular no fixed points or periodic orbits are stable. Yet there is a contradictory aspect of their result. The larger τ', the more unstable the process is, and yet their result requires τ' to be bounded. We remove this restriction, we do not require τ to have piecewise continuous second derivative. An example of τ which has an unbounded derivative was investigated by Lorenz [3].

In Theorem 2 we consider the case where the process is unstable but the condition $|\tau'(x)| > 1$ is not satisfied. The result of Bunimovič [1] is a corollary.

2. Notations and definitions. In what follows m is the Lebesgue measure on $[0, 1]$. $L^1 = (L^1, m)$ is the space of all integrable functions on $[0, 1]$ with the usual norm $\|\cdot\|$. A transformation τ is said to be *piecewise* C^1 if:

(i) there exists a finite partition of $[0, 1]$, $0 = a_0 < a_1 < \ldots < a_n = 1$ such that the restriction τ_i of τ to the open interval (a_{i-1}, a_i) is a C^1-function;

(ii) the function $1/\tau_i'(x)$ is absolutely continuous on $(\tau_i(a_{i-1}), \tau_i(a_i))$.

The Frobenius–Perron operator P_τ is defined by

$$P_\tau(f(x)) = \frac{d}{dx} \int_{\tau^{-1}(0,x)} f(s)\,dm.$$

It is well known that P_τ is a linear continuous operator from L^1 into itself; P_τ is positive and preserves the integral, i.e. $f(x) \geqslant 0$ implies $P_\tau(f(x)) \geqslant 0$ and $\int_0^1 P_\tau(f(x))\,dm = \int_0^1 f(x)\,dm$. The condition $P_\tau(f(x)) = f(x)$ holds if and only if the measure μ defined by $d\mu = f dm$ is τ-invariant.

3. The existence theorems.

THEOREM 1. *Let τ be a piecewise C^1-transformation from the unit interval into itself. Let*

$$\inf |\tau'(x)| \geqslant \lambda > 1,$$

where the infimum is over those x at which τ' is defined. Then there exists an absolutely continuous invariant measure.

Proof. Let $0 = a_0 < a_1 < \ldots < a_n = 1$, be the partition of $[0, 1]$ relative to τ. The Frobenius–Perron operator takes the form

$$P_\tau(f(x)) = \sum_{i=1}^{n} |\varphi_i'(x)| f(\varphi_i(x)) \chi_{I_i}(x), \tag{3.1}$$

where $\varphi_i = (\tau_i)^{-1}$ and χ_{I_i} is the characteristic function of the interval $I_i = \tau(a_{i-1}, a_i)$.

Let q be an integer such that $\varrho = \lambda^{-q} < 1/3$, set $\tau^n(x) = \tau(\tau^{n-1}(x))$, $P_\tau^n(f(x)) = P_\tau(P_\tau^{n-1}(f(x)))$. Let $0 = b_0 < b_1 < \ldots < b_m = 1$ be the partition of $[0, 1]$ relative to the transformation $\gamma = \tau^q$ and let $\psi_i = (\tau_i^q)^{-1} = \gamma_i^{-1}$. We have

$$P_\gamma(f(x)) = P_\tau^q(f(x)) = \sum_{i=1}^{N} |\psi_i'(x)| f(\psi_i(x)) \chi_{J_i}(x), \tag{3.2}$$

where $0 \leqslant |\psi_i'(x)| \leqslant \lambda^{-q} = \varrho$ and χ_{J_i} is the characteristic function of the interval $J_i = \gamma(b_{i-1}, b_i)$.

We claim that for every function f of bounded variation

$$\limsup_k \bigvee_0^1 P_\tau^k(f(x)) < \infty.$$

Let f be piecewise absolutely continuous. Without loss of generality we may suppose f to be non-negative. We have:

$$\bigvee_0^1 P_\gamma(f(x)) \leqslant \sum_{i=1}^{N} \bigvee_{J_i} |\psi_i'(x)| f(\psi_i(x)) + \varrho \sum_{i=1}^{N} (f(b_{i-1}) + f(b_i)), \tag{3.3}$$

$$\bigvee_{J_i} |\psi_i'(x)| f(\psi_i(x)) = \int_{J_i} \left|(\psi_i'(x) f(\psi_i(x)))'\right| dm + \varrho \sum |f(x+) - f(x-)|, \tag{3.4}$$

where the sum is over all discontinuity points of f in $[b_{i-1}, b_i]$.

$$(3.5)\qquad \int_{J_i} \left|\left(\psi_i'(x) f(\psi_i(x))\right)'\right| dm \leqslant \int_{J_i} |\psi_i''(x)| f(\psi_i(x)) dm + \int_{J_i} |\psi_i'(x)| \left|\left(f(\psi_i(x))\right)'\right| dm$$
$$\leqslant \varrho \int_{J_i} \left|\left(f(\psi_i(x))\right)'\right| dm + \int_{J_i} |\psi_i''(x)| f(\psi_i(x)) dm$$
$$\leqslant \varrho \int_{b_{i-1}}^{b_i} |f'(x)| dm + \int_{J_i} |\psi_i''(x)| f(\psi_i(x)) dm.$$

Then from (3.4) and (3.5) we obtain

$$(3.6)\qquad \bigvee_{J_i} |\psi_i'(x)| f(\psi_i(x)) \leqslant \varrho \bigvee_{b_{i-1}}^{b_i} f(x) + \int_{J_i} |\psi_i''(x)| f(\psi_i(x)) dm.$$

In order to evaluate the last term of (3.6) let $t_0^i, t_1^i, \ldots, t_{n_i}^i$ be such that:

$$\gamma(b_{i-1}) = t_0^i < t_1^i < \ldots < t_{n_i}^i = \gamma(b_i) \quad \text{and} \quad \int_{t_{k-1}^i}^{t_k^i} |\psi_i''(x)| dm < \varrho.$$

Such t_k^i exist since $1/\tau_i'$ is absolutely continuous, and so φ_i' and ψ_i' are absolutely continuous,

$$(3.7)\qquad \int_{J_i} |\psi_i''(x)| f(\psi_i(x)) dm \leqslant \sum_{k=1}^{n_i} \Big(\sup_{[t_{k-1},t_k]} f(\psi_i(x)) \cdot \int_{t_{k-1}}^{t_k} |\psi_i''(x)| dm \Big)$$
$$\leqslant \varrho \sum_{k=1}^{n_i} \sup_{I_k^i} f(x),$$

where $I_k^i = [\psi_i(t_{k-1}^i), \psi_i(t_k^i)]$.

Set $\delta_i = \min_k m(I_k^i)$. Since ψ_i are strictly monotonic, it follows that δ_i is positive. Observe that δ_i is independent of f, it depends only on τ^q,

$$\sup_{I_k^i} f(x) \leqslant \bigvee_{I_k^i} f(x) + \inf_{I_k^i} f(x) \quad \text{and} \quad \inf_{I_k^i} f(x) \leqslant (1/\delta_i) \int_{I_k^i} f(x) dm.$$

From (3.7) we obtain

$$(3.8)\qquad \int_{J_i} |\psi_i''(x)| f(\psi_i(x)) dm \leqslant \varrho \sum_{k=1}^{n_i} \Big(\bigvee_{I_k^i} f(x) + (1/\delta_i) \int_{I_k^i} f(x) dm \Big)$$
$$\leqslant \varrho \bigvee_{b_{i-1}}^{b_i} f(x) + (\varrho/\delta_i) \int_{b_{i-1}}^{b_i} f(x) dm.$$

Then (3.6) and (3.8) imply

$$(3.9)\qquad \bigvee_{J_i} |\psi_i'(x)| f(\psi_i(x)) \leqslant 2\varrho \bigvee_{b_{i-1}}^{b_i} f(x) + (\varrho/\delta_i) \int_{b_{i-1}}^{b_i} f(x) dm.$$

And from (3.3) we obtain

$$\bigvee_0^1 P_\gamma(f(x)) \leqslant 2\varrho \bigvee_0^1 f(x) + (\varrho/\delta)\,\|f\| + \varrho \sum_{i=1}^{N} (f(b_{i-1}) + f(b_i)), \tag{3.10}$$

where $\delta = \min_i \delta_i$ is positive and independent of f,

$$\begin{aligned}\sum_{i=1}^{N} (f(b_{i-1})+f(b_i)) &\leqslant \sum_{i=1}^{N} \bigvee_{b_{i-1}}^{b_i} f(x) + 2 \inf_{[b_{i-1}, b_i]} f(x) \\ &\leqslant \bigvee_0^1 f(x) + \sum_{i=1}^{N} (2/|b_i - b_{i-1}|) \int_{b_{i-1}}^{b_i} f(x)\,dm \\ &\leqslant \bigvee_0^1 f(x) + (2/\delta)\,\|f\|.\end{aligned}$$

Hence, from 3.10, by setting $H = 3\varrho/\delta$, we get

$$\bigvee_0^1 P_\gamma(f(x)) \leqslant 3\varrho \bigvee_0^1 f(x) + H\,\|f\|. \tag{3.11}$$

Since $3\varrho < 1$ and H is independent on f, by iterating (3.11) we obtain

$$\limsup_n \bigvee_0^1 P_\gamma^n(f(x)) < (H/(1-3\varrho))\,\|f\| = K\,\|f\|. \tag{3.12}$$

Inequality (3.12) is valid for any function f piecewise absolutely continuous and is independent of the variation of f. Hence for every function f of bounded variation the set $\{P_\gamma^n(f(x))\}_{n=1}^{\infty}$ is uniformly bounded. Since $\gamma = \tau^q$, the set $\{P_\tau^n(f(x))\}_{n=1}^{\infty}$ is uniformly bounded, too; in fact,

$$P_\tau^n(f(x)) \leqslant \max_{i=1,\dots,q-1} P_\tau^i(f(x)), \quad n = 1, 2, \dots$$

Hence the sequence $\{(1/n) \sum_{i=0}^{n-1} P_\tau^i(f(x))\}$ is relatively compact in L^1 and from the Kakutani–Yoshida Theorem it follows that it converges strongly to a function $\bar{f}(x)$ which is a fixed point of P_τ.

Following [3] we could prove that for every $f \in L^1$ the sequence $\{(1/n) \sum_{i=0}^{n-1} P_\tau^i(f(x))\}$ is convergent. The limit function, say $\hat{f}$, satisfies the condition $\bigvee_0^1 \hat{f}(x) \leqslant c\,\|f\|$ with the constant c independent of f. This finishes the proof.

THEOREM 2. *Let τ be a piecewise C^1-transformation from the unit interval into itself. If there exists a real function h such that:*

(i) $h(x) > 0$ *a.e. and* $\int h(x)dm = 1$,

(ii) $\inf|\tau'(x)|\, h(\tau(x))/h(x) > 1$,

(iii) $h(x)/\tau'(x)h(\tau(x))$ *is piecewise absolutely continuous,*

then there exists an absolutely continuous invariant measure.

Proof. Set $g(x) = \int_0^x h(t)dm$ and consider the new transformation T defined by $T = g\circ\tau\circ g^{-1}$. We have

$$T'(g(x)) = g'(\tau(x))\tau'(x)/g'(x) = \tau'(x)h(\tau(x))/h(x).$$

Hence T satisfies the hypothesis of Theorem 1; thus there exists a measure μ_T invariant under T. The measure $\mu = \mu_T\circ g$ is invariant under τ and is absolutely continuous. Sometimes it is useful to have conditions (ii) and (iii) in Theorem 2 in terms of the inverse functions φ_i of τ. The corresponding conditions are: $\sup|\varphi_i'(x)|\, h(\varphi_i(x))/h(x) < 1$ and $\varphi_i'(x)h(\varphi_i(x))/h(x)$ piecewise absolutely continuous.

4. Remarks. Theorem 1 is an extension of the Lasota–Yorke Theorem. Moreover, the Lasota–Yorke Theorem seems to privilege the Lebesgue measure with respect to the other absolutely continuous measures; in fact, by the result of their theorem the density of the invariant measure is a function of bounded variation. Hence, if the invariant measure is "near" the Lebesgue measure (in the sense that the density is of bounded variation), then the Lasota–Yorke Theorem might solve the problem, otherwise not. In the last case, we can replace the Lebesgue measure by another measure μ_h, and if the invariant measure is "near" μ_h, our Theorem 2 can solve the problem.

Suppose that the hypotheses of Theorem 2 hold; then if we start with $f(x) = 1$, the sequence $\{(1/n)\sum_{i=0}^{n-1} P_\tau^i(f(x))\}$ converges to $\bar{f}$ which is invariant, so we have an indication of how to change the measure. We will use this fact in the applications.

Suppose, once more, that there exists an invariant measure with density $\bar{f}(x) \neq 0$ a.e. The Frobenius–Perron operator is

$$\bar{f}(x) = P_\tau(\bar{f}(x)) = \sum_{i=1}^{n} |\varphi_i'(x)|\,\bar{f}(\varphi_i(x))\chi_{I_i}(x).$$

By dividing by $\bar{f}(x)$ we obtain

$$\sum_{i=1}^{n} |\varphi_i'(x)|\,\bar{f}(\varphi_i(x))\chi_{I_i}(x)/\bar{f}(x) = 1.$$

Then if $\chi_{I_i}(x) \neq 0$, we have $|\varphi_i'(x)| \bar{f}(\varphi_i(x))/\bar{f}(x) \leqslant 1$, $i = 1, \ldots, n$. This last inequality shows that our assumptions are "nearly" necessary.

5. Applications. In this section we show some applications.

Consider, for each natural number $n \geqslant 1$, the transformation τ_n: $[0, 1] \to [0, 1]$ defined by

$$\tau_n(x) = n \sin(\pi x) \pmod 1.$$

COROLLARY (**Bunimovič**). *For every $n \geqslant 1$ there exists an absolutely continuous invariant measure (with respect to τ_n).*

Proof. The Frobenius–Perron operator relative to τ_n is

$$P_{\tau_n}(f(x)) = \sum_{i=0}^{n-1} |\varphi_i'(x)| \{f(\varphi_i(x)) + f(1-\varphi_i(x))\},$$

where $\varphi_i(x) = (1/\pi) \arcsin((x+i)/n)$, $i = 0, \ldots, n-1$. Let us consider the sequence $P^k_{\tau_n}(1)$. It is easy to see that, for $k \geqslant 2$, $P^k_{\tau_n}(1)$ is infinite of order 1/2 at zero and 1. Hence, let us set $h(x) = 1/\sqrt{x(1-x)}$. We will prove that $\alpha_i(x) = |\varphi_i'(x)| h(\varphi_i(x))/h(x) \leqslant \beta < 1$, where φ_i are the inverses of τ_n,

$$\alpha_i(x) = \frac{\sqrt{x(1-x)}}{\sqrt{(n^2-(x+i)^2)}\sqrt{\left(\arcsin((x+i)/n)\right)\left(\pi - \arcsin((x+i)/n)\right)}},$$

$i = 0, \ldots, n-1$. For $i = 0$ we obtain

$$\alpha_0(x) = \frac{\sqrt{x(1-x)}}{\sqrt{n^2-x^2}\sqrt{\arcsin(x/n)(\pi - \arcsin(x/n))}}$$

$$= \frac{\sqrt{x/n}}{\sqrt{\arcsin(x/n)}} \cdot \frac{\sqrt{1-x}}{\sqrt{n^2-x^2}} \cdot \frac{\sqrt{n}}{\sqrt{\pi - \arcsin(x/n)}} \leqslant \sqrt{2/\pi} < 1.$$

For $i = n-1$ we obtain

$$\alpha_{n-1}(x) = \frac{\sqrt{x(1-x)}}{\sqrt{(1-x)(2n-1+x)}\sqrt{\arcsin((x+n-1)/n)(\pi - \arcsin((x+n-1)/n))}}$$

$$\leqslant \frac{1}{\sqrt{2n-1}} < 1.$$

For $1 \leqslant i \leqslant n-2$ we obtain

$$\alpha_i(x) = \frac{\sqrt{x(1-x)}}{\sqrt{n^2-(x+i)^2}\sqrt{\arcsin((x+i)/n)(\pi - \arcsin((x+i)/n))}}$$

$$\leqslant \frac{1}{\sqrt{n^2-(n-1)^2}\sqrt{\arcsin(i/n)\big(\pi-\arcsin(1+i)/n\big)}}$$

$$\leqslant \frac{1}{\sqrt{2n-1}\sqrt{i/n}\sqrt{\pi/2}} < 1.$$

This completes the proof.

Let us now consider, for every integer $n \geqslant 1$, the n-parabola defined by: $\tau_n(x) = n \cdot 4x(1-x) \pmod 1$. In a completely analogous way we can prove that for every n there exists an absolutely continuous τ_n-invariant measure.

Consider, finally, the transformations τ_α defined by $\tau_\alpha(x) = \big(4x(1-x)\big)^{1/\alpha}$, $\alpha \geqslant 1$.

By similar arguments to those in Theorem 2 we could prove the existence of an absolutely continuous τ_α-invariant measure for $1 \leqslant \alpha \leqslant 5$. For $\alpha = 2$ we can exactly compute this measure, obtaining $d\mu = dm/2\sqrt{1-x}$. For large values of α ($\alpha > 5, 8, \ldots$) the diagonal crosses τ_α at a point at which the slope is less than 1, and so in this case an absolutely continuous invariant measure cannot exist.

References

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