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# On a proof of the boundedness and nuclearity of pseudodifferential operators in $R^n$

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**Abstract.** A simple proof is given of the boundedness and nuclearity of pseudodifferential operators in $R^n$ with symbol bounded with derivatives up to order $2n+2$. The proof is based on an identity for symbols which replaces the group representation approach of Howe [5].

In a very interesting paper, R. Howe gave a proof of the boundedness on $L^2(R^n)$ of a pseudodifferential operator with symbol bounded with derivatives up to order $2n+1$ ([5], Theorem 3.1.3). This theorem was first proved by Calderón and Vaillancourt [2] in a slightly stronger form and is known as the Calderón–Vaillancourt $(0,0)$ $L^2$-boundedness theorem. Now it is a classical result in the theory of $\Psi DOs$, and a starting point for various generalizations ([3]). The novelty of Howe's proof lies in some group representation theory arguments. In this paper we show that in fact representation theory is not needed in Howe's approach (to the above-mentioned theorem!). It turns out that the main estimate in Howe's proof follows from an identity for symbols (Theorem 1). Also, this identity leads to an easy proof of nuclearity of an operator with an integrable symbol. By standard interpolation arguments one can deduce that an operator with symbol in $L^p$, $1 \leqslant p < \infty$, is $p$-nuclear ([1], [4]).

We shall use the following notation and terminology: $R^n$ is the standard $n$-dimensional Euclidean space and $R_n$ is its dual. The value of a functional $\xi \in R_n$ on a vector $x \in R^n$ is denoted by $x\xi$. The Lebesgue measures in $R^n$ and $R_n$ are denoted by $dx$ and $d\xi$. We also write $đ\xi = (2\pi)^{-n} d\xi$.

For any finite-dimensional Hilbert space $V$, $\mathcal{S}(V)$ denotes the Schwartz space of rapidly decreasing functions and $\mathcal{S}'(V)$ the dual of $\mathcal{S}(V)$ with the weak topology.

The Fourier and inverse Fourier transforms are defined by

$$\mathcal{F}\colon\ \mathcal{S}(R^n) \to \mathcal{S}(R_n), \qquad \mathcal{F}(f)(\xi) := \int e^{-ix\xi} f(x)\, dx =: \hat{f}(\xi),$$

$$\mathcal{F}^{-1}\colon\ \mathcal{S}(R_n) \to \mathcal{S}(R^n), \qquad \mathcal{F}^{-1}(g)(x) := \int e^{ix\xi} g(\xi)\, đ\xi.$$

We shall also use the symplectic Fourier transform:

$$^s\mathscr{F}\colon \mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n)\to\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n),$$
$$^s\mathscr{F}(a)(x,\xi):=\iint e^{i(x\eta-y\xi)}a(y,\eta)\,dy\,d\!\!\!{}^-\eta$$
$$=\mathscr{F}_{y\to\xi}\otimes\mathscr{F}^{-1}_{\eta\to x}(a(y,\eta))=:\hat{a}(x,\xi).$$

The following properties of $^s\mathscr{F}$ are obvious:

$$^s\mathscr{F}(^s\mathscr{F}(a))=a,$$

(1) $$\mathscr{F}^{-1}_{\xi\to y}(\hat{a}(x,\xi))=\mathscr{F}^{-1}_{\eta\to x}(a(y,\eta)),$$

(2) $$\mathscr{F}_{x\to\eta}(\hat{a}(x,\xi))=\mathscr{F}_{y\to\xi}(a(y,\eta)).$$

The Fourier transforms are extended by duality to $\mathscr{S}'$.

We define the injection $\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n)\hookrightarrow\mathscr{S}'(\boldsymbol{R}^n\times\boldsymbol{R}_n)$ by

$$a[b]=\iint a(x,\xi)\,b(x,\xi)\,dx\,d\!\!\!{}^-\xi,\qquad a,b\in\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n).$$

As usual we denote by $C_0^\infty$ the space of compactly supported smooth functions and by $\mathscr{E}'$ the space of compactly supported distributions.

For a given $a\in\mathscr{S}'(\boldsymbol{R}^n\times\boldsymbol{R}_n)$, we define the operator $A=:Op_E(a)$: $\mathscr{S}(\boldsymbol{R}^n)\to\mathscr{S}'(\boldsymbol{R}^n)$ with (exotic) symbol $a:=\sigma_E(A)$ by

$$A\varphi[\psi]=a[J_E(\psi\otimes\varphi)],\qquad \varphi,\psi\in\mathscr{S}(\boldsymbol{R}^n)$$

with

$$J_E(\psi\otimes\varphi)(x,\xi)=\int e^{iy\xi}\psi(y)\,\varphi(-x+y)\,dy$$
$$=(2\pi)^n\mathscr{F}^{-1}(\psi\tau_x(\varphi))(\xi),$$

where $\tau_x(\varphi)(y)=\varphi(-x+y)$.

If $a\in\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}_n)$ we have

$$A\varphi(x)=\iint e^{ix\eta}a(x-y,\eta)\,\varphi(y)\,dy\,d\!\!\!{}^-\eta,\qquad \varphi\in\mathscr{S}(\boldsymbol{R}^n).$$

The usual (Mikhlin–Giraud) symbol of $A$ is then equal to $\hat{a}$ (see [6] for the definition of various symbols).

If a continuous linear operator $A$: $\mathscr{S}(\boldsymbol{R}^n)\to\mathscr{S}'(\boldsymbol{R}^n)$ is given and $\mathscr{A}\in\mathscr{S}'(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ is its Schwartz kernel, then

(3) $$\sigma_E(A)(x,\xi)=\mathscr{F}_{y\to\xi}(\mathscr{A}(y,-x+y)).$$

Let $\varkappa$, $\lambda\in\mathscr{S}(\boldsymbol{R}^n)$. We denote by $\varkappa\otimes\bar{\lambda}$ the operator

$$\varkappa\otimes\bar{\lambda}(\varphi):=(\varphi|\lambda)\,\varkappa,\qquad \varphi\in\mathscr{S}(\boldsymbol{R}^n),$$

where $(\varphi|\lambda)$ is the $L^2$-scalar product.

By (3) it follows that

$$\sigma_E(\varkappa\otimes\bar{\lambda})(x,\xi)=\int e^{-iy\xi}\varkappa(y)\,\bar{\lambda}(-x+y)\,dy$$
$$=\mathscr{F}_{y\to\xi}(\varkappa\tau_x(\bar{\lambda})(y))=\overline{J_E(\varkappa\otimes\lambda)(x,\xi)}.$$

PROPOSITION 1. *For any* $\varkappa$, $\lambda\in\mathscr{S}(\boldsymbol{R}^n)$

$$\|J_E(\varkappa\otimes\bar{\lambda})\|_{L^2(dx\,d\!\!\!{}^-\xi)}=\|\varkappa\|_{L^2}\|\lambda\|_{L^2}.$$

**Proof.** This follows easily from the Parseval equality.

The next result permits any group-theoretic considerations of Howe's paper to be avoided.

THEOREM 1. *Let* $a \in \mathscr{S}'(\boldsymbol{R}^n \times \boldsymbol{R}_n)$ *and* $\varkappa$, $\lambda$, $\varphi \in \mathscr{S}(\boldsymbol{R}^n)$. *Then*

$$Op_E(a\sigma_E(\varkappa \otimes \bar{\lambda}))\varphi = Op_E(\hat{a}\,\sigma_E(\varphi \otimes \bar{\lambda}))\varkappa.$$

**Proof.** We shall use the well-known properties of the Fourier transform and convolution:

For $\psi \in \mathscr{S}(\boldsymbol{R}^n)$ we have

$$\begin{aligned}(Op_E(a\sigma_E(\varkappa \otimes \bar{\lambda}))\varphi)[\psi] &= a\sigma_E(\varkappa \otimes \bar{\lambda})[J_E(\psi \otimes \varphi)]\\ &= (2\pi)^n a(x,\xi)\mathscr{F}_{z\to\xi}((\varkappa\tau_x(\bar{\lambda}))(z))[\mathscr{F}^{-1}_{y\to\xi}(\psi\tau_x(\varphi))]\\ &= \mathscr{F}^{-1}_{\xi\to y}(a(x,\xi)\mathscr{F}_{z\to\xi}((\varkappa\tau_x(\bar{\lambda}))(z)))[(\psi\tau_x(\varphi))(y)]\\ &= \mathscr{F}^{-1}_{\xi\to y}(a(x,\xi) * ((\varkappa\tau_x(\bar{\lambda}))(y))[(\psi\tau_x(\varphi))(y)]\\ &= \mathscr{F}^{-1}_{\xi\to y}(a(x,\xi))[(\varkappa\tau_x(\bar{\lambda}))^{\vee} * (\psi\tau_x(\varphi))(y)].\end{aligned}$$

Next we have

$$\begin{aligned}((\varkappa\tau_x(\bar{\lambda}))^{\vee} * \psi\tau_x(\varphi))(y) &= \int \varkappa(z-y)\bar{\lambda}(-x+z-y)\psi(z)\varphi(-x+z)\,dz\\ &= ((\varphi\tau_y(\bar{\lambda}))^{\vee} * \psi\tau_y(\varkappa))(x).\end{aligned}$$

Thus using (1) we obtain

$$\begin{aligned}&(Op_E(a\sigma_E(\varkappa \otimes \bar{\lambda}))\varphi)[\psi]\\ &\quad = \mathscr{F}^{-1}_{\eta\to x}(\hat{a}(y,\eta))[(\varphi\tau_y(\bar{\lambda}))^{\vee} * (\psi\tau_y(\varkappa))(x)]\\ &\quad = (2\pi)^{2n}\hat{a}(y,\eta)[\mathscr{F}^{-1}_{x\to\eta}((\varphi\tau_y(\bar{\lambda}))^{\vee}(x))\mathscr{F}^{-1}_{x\to\eta}((\psi\tau_y(\varkappa))(x))]\\ &\quad = \hat{a}(y,\eta)[\mathscr{F}_{x\to\eta}((\varphi\tau_y(\bar{\lambda}))(x))J_E(\psi \otimes \varkappa)(y,\eta)]\\ &\quad = \hat{a}\,\sigma_E(\varphi \otimes \bar{\lambda})[J_E(\psi \otimes \varkappa)] = (Op_E(\hat{a}\,\sigma_E(\varphi \otimes \bar{\lambda}))\varkappa)[\psi].\end{aligned}$$

It is easy to generalize Theorem 1 to $k \times l$ matrices.

THEOREM 1′. *Let* $a \in \mathscr{S}(\boldsymbol{R}^n \times \boldsymbol{R}_n) \otimes L(\boldsymbol{C}^l, \boldsymbol{C}^k)$, $\varkappa, \lambda \in \mathscr{S}(\boldsymbol{R}^n)$, $\varphi \in \mathscr{S}(\boldsymbol{R}^n) \times \boldsymbol{C}^l$. *Then*

$$Op_E(a\sigma_E(\varkappa \otimes \bar{\lambda}))\varphi = Op_E(\hat{a}\,\sigma_E(\varphi \otimes \bar{\lambda}))\varkappa.$$

Write $\lambda_0(x) = \pi^{n/2} e^{x^2/2}$; obviously $\lambda_0 \in \mathscr{S}(\boldsymbol{R}^n)$.

LEMMA 1. $\|\lambda_0\|_{L^2} = 1$ *and* $\sigma_E(\lambda_0 \otimes \bar{\lambda}_0)(x,\xi) = e^{x^2/4 - \xi^2/4 - ix\xi/2}$ *does not vanish for all* $(x,\xi) \in \boldsymbol{R}^n \times \boldsymbol{R}_n$.

**Proof.** The formula for $\sigma_E(\lambda_0 \otimes \bar{\lambda}_0)$ follows easily from the well-known fact that $\int_{\operatorname{Im} z = \mathrm{const}} e^{-z^2}\,dz = \pi^{n/2}$.

Moreover, $\|\lambda_0\|^2_{L^2} = \sigma_E(\lambda_0 \otimes \bar{\lambda}_0)(0,0) = 1$.

From now on we fix $\lambda \in \mathscr{S}(\boldsymbol{R}^n)$ with $\|\lambda\|_{L^2} = 1$ and $\sigma_E(\lambda \otimes \bar{\lambda}) \neq 0$ for all $(x, \xi) \in \boldsymbol{R}^n \times \boldsymbol{R}_n$.

The following observation results easily from Theorem 1.

COROLLARY 1. *For $\varphi \in \mathscr{S}(\boldsymbol{R}^n)$ define*

$$I_\varphi(x, \xi) := \int e^{-iy\xi} \varphi(y) \bar{\lambda}(-x+y)\, dy = \sigma_E(\varphi \otimes \bar{\lambda})(x, \xi) = \overline{J_E(\varphi \otimes \lambda)(x, \xi)}.$$

*Then for any $a \in \mathscr{E}'(\boldsymbol{R}^n \times \boldsymbol{R}_n)$*

$$Op_E(a)\varphi = Op_E((a I_\lambda^{-1})^\frown I_\varphi)\lambda.$$

Remarks. (1) We take $a \in \mathscr{E}'$ instead of $a \in \mathscr{S}'$ since $I_\lambda^{-1}$ is in $C^\infty$ but need not be a multiplier in $\mathscr{S}'$.

(2) By Proposition 1, $\|I_\varphi\|_{L^2} = \|\lambda\|_{L^2} \|\varphi\|_{L^2} = \|\varphi\|_{L^2}$.

Now we can prove the results which were the aim of the paper.

THEOREM 2 ([5], 3.1.8). *Let $a \in \mathscr{E}'(\boldsymbol{R}^n \times \boldsymbol{R}_n)$. Then*

$$|||Op_E(a)|||_\infty \leqslant \sup_{x \in \boldsymbol{R}^n, \xi \in \boldsymbol{R}_n} |(a I_\lambda^{-1})^\frown (x, \xi)| =: \|(a I^{-1})^\frown\|,$$

*where $|||\cdot|||_\infty$ denotes the norm in $L(L^2(\boldsymbol{R}^n))$.*

Proof. We use Theorem 1 and Proposition 1:

$$\begin{aligned}
|||Op_E(a)|||_\infty &= \sup_{\|\varphi\|_{L^2}, \|\psi\|_{L^2} \leqslant 1} |(Op_E(a)\varphi | \psi)| \\
&= \sup \left|\left(Op_E((a I_\lambda^{-1})^\frown I_\varphi)\lambda | \psi\right)\right| \\
&= \sup \left|Op_E((a I_\lambda^{-1})^\frown I_\varphi)\lambda[\bar{\psi}]\right| \\
&= \sup |(a I_\lambda^{-1})^\frown I_\varphi [J_E(\bar{\psi} \otimes \lambda)]| = \sup |(a I_\lambda^{-1})^\frown [I_\varphi \bar{I}_\psi]| \\
&\leqslant \|(a I_\lambda^{-1})^\frown\|_\infty \sup \|I_\varphi\|_{L^2} \|I_\psi\|_{L^2} = \|(a I^{-1})^\frown\|_\infty.
\end{aligned}$$

For an operator $A$ in $L^2(\boldsymbol{R}^n)$ the $p$-nuclear norm ($1 \leqslant p < \infty$) may be defined by

$$|||A|||_p = \sup \{ \sum_{k=1}^{\infty} |A\varphi_k[\psi_k]|^p \}^{1/p},$$

where the supremum is taken over all orthonormal bases $\{\varphi_k\}$, $\{\psi_k\}$ in $L^2$.

For an operator $A$ with $|||A|||_1 < \infty$ the trace

$$\operatorname{Tr} A := \sum A\varphi_k[\bar{\varphi}_k].$$

is defined.

Theorem 1 allows us to estimate the trace norm of an operator.

THEOREM 3 ([4], 3.5). *Let* $a \in \mathscr{E}'(\mathbf{R}^n \times \mathbf{R}_n)$. *Then*

$$|||Op_E(a)|||_1 \leqslant \int |(a I_\lambda^{-1})^\frown (x, \xi)| dx đ\xi =: \|(a I_\lambda^{-1})^\frown\|_1.$$

Proof. Observe that for any orthonormal basis $\{\varphi_k\}$ in $L^2(\mathbf{R}^n)$ we have

$$\sum_k | I_{\varphi_k}(x, \xi)|^2 = \sum |(\varphi_k | e_\xi \tau_x(\lambda))|^2 = \|e_\xi \tau_x(\lambda)\|_{L^2} = \|\lambda\|_{L^2} = 1,$$

where $e_\xi(y) = e^{iy\xi}$. Now

$$\begin{aligned} |||Op(a)|||_1 &= \sup_{\{\varphi_k\},\{\psi_k\}} \sum |Op_E(a)\varphi_k[\psi_k]| \\ &= \sup \sum \left|Op_E\big((a I_\lambda^{-1})^\frown I_{\varphi_k}\big)\lambda[\psi_k]\right| \\ &= \sup \sum |(a I_\lambda^{-1})^\frown I_{\varphi_k}[I_{\psi_k}]| \leqslant \sup \sum \iint |(a I_\lambda^{-1})^\frown| |I_{\varphi_k} \cdot I_{\psi_k}| \, dx đ\xi \\ &\leqslant \sup \iint |(a I_\lambda^{-1})^\frown (x, \xi)| \big(\sum |I_{\varphi_k}(x, \xi)|^2\big)^{1/2} \big(\sum | I_{\psi_k}(x, \xi)|^2\big)^{1/2} \, dx đ\xi \\ &= \iint |(a I_\lambda^{-1})^\frown (x, \xi)| \, dx đ\xi. \end{aligned}$$

Remark 3. Arguing as above one gets

$$\operatorname{Tr} Op_E(a) = \sum_k Op_E(a)\varphi_k[\bar{\varphi}_k] = (a I_\lambda^{-1})^\frown \left[\sum |I_{\varphi_k}|^2\right] = \iint (a I_\lambda^{-1})^\frown (x, \xi) \, dx đ\xi.$$

Remark 4. By the standard interpolation argument it follows from Theorem 2 and 3 that

$$|||Op_E(a)|||_p \leqslant \|(a I_\lambda^{-1})^\frown\|_{L^p}$$

(see [4], 3.7).

To make the paper self-contained we give a sketch of Howe's proof of the Calderón–Vaillancourt theorem (cf. [5], Theorems 3.1.1–3.1.3; [4], 3.6–3.8).

PROPOSITION 2. *For any compact set* $K \subset \mathbf{R}^n \times \mathbf{R}_n$, *there exists a constant* $c_K > 0$ *such that for* $a \in \mathscr{S}'(\mathbf{R}_n \times \mathbf{R}^n)$ *with* $\operatorname{supp} a \subset K$

$$|||Op_E(a)|||_p \leqslant C_k \|\hat{a}\|_p, \quad p = 1, \infty.$$

Proof. We choose $f \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}_n)$ with $f = I_\lambda^{-1}$ on $K$.

Then $a I_\lambda^{-1} = fa$ and $(fa)^\frown = \hat{f} * \hat{a}$.

By Theorem 2 and the Young inequality we get

$$|||Op_E(a)|||_\infty \leqslant \|(a I_\lambda^{-1})\|_\infty = \|\hat{f} * \hat{a}\|_\infty \leqslant \|\hat{f}\|_{L_1} \|\hat{a}\|_\infty$$

and we may take $C_k = \|f\|_{L^1}$.

Similarly for $p = 1$ (or $1 \leqslant p < \infty$ according to Remark 4) we have

$$|||Op_E(a)|||_p \leqslant \|\hat{f} * \hat{a}\|_p \leqslant \|\hat{f}\|_{L^1} \|\hat{a}\|_p$$

and as before we may take $C_K = \|f\|_{L^1}$.

We also need the following easy result

LEMMA 2. *For* $(x,\xi)\in \boldsymbol{R}^n\times \boldsymbol{R}_n$, $a\in \mathscr{S}'(\boldsymbol{R}^n\times \boldsymbol{R}_n)$

$$Op_E(\tau_{x,\xi}(a)) = e_\xi \circ Op_E(a)\circ \tau_x,$$

*where* $\tau_{x,\xi}(a)(y,\eta) = a(y-x,\eta-\xi)$, $(e_\xi u)(y) = e^{iy\xi}u(y)$.

Since $\tau_x$ and $e_\xi$ are isometries in $L^2$ we have the following fact:

COROLLARY 2. $|||Op_E\tau_{x,\xi}(a)|||_p = |||Op_E(a)|||_p$, $1\leqslant p\leqslant \infty$.

LEMMA 3. *Let* $a\in \mathscr{S}'(\boldsymbol{R}^n\times \boldsymbol{R}_n)$ *with* $\hat{a}^{(\alpha)}_{(\beta)} := \partial_y^\beta \partial_\eta^\alpha \hat{a}(y,\eta)$ *in* $L^p$, $1\leqslant p\leqslant\infty$, *for* $|\alpha|+|\beta|\leqslant 2m$, *where m is a positive integer. For any* $f\in C_0^\infty(\boldsymbol{R}^n\times \boldsymbol{R}_n)$ *there is a constant C such that for all* $(x,\xi)\in \boldsymbol{R}^n\times \boldsymbol{R}_n$

$$\|(\tau_{x,\xi}(f)a)^\wedge\|_p \leqslant C(1+|x|^2+|\xi|^2)^{-m} \max_{|\alpha|+|\beta|\leqslant 2m} \|\hat{a}^{(\alpha)}_{(\beta)}\|_p.$$

Proof. We have

$$(\tau_{x,\xi}(f)\,a)^\wedge(y,\eta) = \int e^{-ix\zeta+iz\xi}\hat{f}(z,\zeta)\,\hat{a}(y-z,\eta-\zeta)\,dz\,d\zeta$$

and

$$e^{-ix\zeta+iz\xi} = (1+|x|^2+|\xi|^2)^{-m}(1-\Delta_\zeta-\Delta_z)^m e^{-ix\zeta+iz\xi}.$$

Integrating by parts and using the Leibniz formula we get the assertion.

Now we can prove the most interesting result.

THEOREM 4. *Let* $a\in \mathscr{S}'(\boldsymbol{R}^n\times \boldsymbol{R}_n)$ *with* $\hat{a}^{(\alpha)}_{(\beta)}\in L^p(\boldsymbol{R}^n\times \boldsymbol{R}_n)$ *for* $|\alpha|+|\beta|\leqslant 2n+2$, $1\leqslant p\leqslant\infty$. *Then the operator* $Op_E(a)$ *on* $L^2(\boldsymbol{R}^n)$ *is p-nuclear if* $1\leqslant p<\infty$ *and bounded if* $p=\infty$.

Proof. We give a complete proof for $p=1$ and $p=\infty$; the case $1<p<\infty$ follows from the unproved Remark 4 (see [4] for details or [1] for another proof).

Let $f\in C_0^\infty(\boldsymbol{R}^n\times \boldsymbol{R}_n)$ with $\int f\,dx\,d\xi = 1$ and

$$K_0 = \{(x,\xi)\in \boldsymbol{R}^n\times \boldsymbol{R}_n:\ 0\leqslant x_i,\xi_i<1,\ i=1,2,\ldots,n\}.$$

For $k,l < \boldsymbol{Z}^n$ define

$$f_{k,l}(x,\xi) = \iint_{(y,\eta)\in K_0} f(x-k-y,\xi-l-\eta)\,dy\,d\eta.$$

Then $f_{k,l}\in C_0^\infty$, $f_{k,l} = \tau_{k,l}f_{0,0}$, $\sum_{k,l\in \boldsymbol{Z}^n} f_{k,l} = 1$, and thus

$$a = \sum_{k,l}\tau_{k,l}(f_{0,0})a = \sum_{k,l}\tau_{k,l}(f_{0,0}\,\tau_{-k,-l}(a)).$$

In view of the previous results we have for any integer $m>n$

$$\begin{aligned}
|||Op_E(a)|||_p &\leq \sum_{k,l} |||Op_E \tau_{k,l}(f_{0,0}\tau_{-k,-l}(a))|||_p \\
&= \sum_{k,l} |||Op_E(f_{0,0}\tau_{-k,-l}(a))||| \leq \sum_{k,l} C_{K_0} \|(f_{0,0}\tau_{-k,-l}(a))^{\frown}\|_p \\
&= C_{K_0} \sum_{k,l} \|(\tau_{k,l}(f_{0,0})a)^{\frown}\|_p \leq C_m \sum_{k,l} (1+|k|^2+|l|^2)^{-m} \max_{|\alpha|+|\beta|\leq 2m} \|\hat{a}^{(\alpha)}_{(\beta)}\|_p \\
&= \tilde{C}_m \max_{|\alpha|+|\beta|\leq 2m} \|\hat{a}^{(\alpha)}_{(\beta)}\|_p.
\end{aligned}$$

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