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On the existence and uniqueness of solutions of systems of differential equations with a deviated argument

by T. Jankowski and M. Kwapisz (Gdańsk)

In this paper we consider the differential vector equation

$$(1)\qquad x'(t) = F\big(t, x(a_1(t)), \ldots, x(a_r(t)), x'(\beta_1(t)), \ldots, x'(\beta_s(t))\big),$$

where

$$x(t) = \big(x^1(t), \ldots, x^q(t)\big), \qquad F(\ ) = \big(F^1(\ ), \ldots, F^q(\ )\big).$$

It is known that, for $r = s = 1$ and $a_1(t) = t$, $\beta_1(t) \leqslant t$, for $t \in [0, a)$, equation (1) with the initial condition $x(t) = \varphi(t)$, $t \in E_0$,

$$E_0 = \{z\colon z = \beta_1(t) \leqslant 0, t \in [0, a)\},$$

has in the interval $[0, a)$ a unique solution which is a limit of ordinary successive approximation (see [2]) if among other things we suppose that the function F satisfies a Lipschitz condition with respect to the last two variables with Lipschitz constants k_1, $k_2 \geqslant 0$ and $k_2 < 1$. If $\beta_1(t) \leqslant t - a$, $a > 0$ for $t \in [0, a)$, then the condition $k_2 < 1$ is superfluous. But this is not the only case where this condition may be weakened.

Equation (1) was considered in [4] under the conditions

$$a_i(t) \leqslant t, \quad \beta_j(t) \leqslant t, \quad t \in [0, a], \quad i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, s.$$

There was established a theorem on the existence and uniqueness of solution of this equation, involving some relation between the Lipschitz constants of function F with respect to the last s variables and the functions $\beta_j(t)$.

In our paper we obtain, by the successive approximation method, a result more general than that of paper [4].

At first let $a_i(t) \leqslant 0$, $\beta_j(t) \leqslant 0$ for all $t \in [0, a]$, $i \in V_1 = \{r+1, \ldots, r_1\}$, $j \in V_2 = \{s+1, \ldots, s_1\}$, $r_1 \geqslant r$, $s_1 \geqslant s$. Further, let us have the equation

$$(1')\qquad x'(t) = f\big(t, x(a_1(t)), \ldots, x(a_{r_1}(t)), x'(\beta_1(t), \ldots, x'(\beta_{s_1}(t))\big),$$

with the initial condition $x(t) = \bar{\varphi}(t)$, $t \in E$, where E is the so-called initial set of equation (1'),

$$E = \bigcup_{i \in V_1} \{z\colon z = a_i(t) \leqslant 0, t \in [0, a]\} \cup \bigcup_{j \in V_2} \{z\colon z = \beta_j(t) \leqslant 0, t \in [0, a]\}.$$

By putting $\bar{\varphi}(a_i(t))$, $\bar{\varphi}'(\beta_j(t))$ instead of $x(a_i(t))$, $x'(\beta_j(t))$, $i\in V_1$, $j\in V_2$ we can reduce equation (1') to equation (1), which does not contain a_i's, β_j's of that sort.

If $V_1 = \{1, 2, \ldots, r_1\}$, $V_2 = \{1, 2, \ldots, s_1\}$, then equation (1') is the one considered in papers [5] and [6].

Now, we assume that

$$a_i(t),\ \beta_j(t)\in[0, a] \quad \text{for } t\in[0, a],\ i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, s,$$

and we consider equation (1) with the initial condition $x(0) = x'(0) = \Theta$, $\Theta = \underbrace{(0, \ldots, 0)}_{q}$ (by the substitution $y(t-t_0) = x(t)-c-(t-t_0)z$ the general initial condition $x(t_0) = c$, $x'(t_0) = z$ can be reduced to the condition considered here). By the substitution $y(t) = x'(t)$, equation (1) with the initial condition $x(0) = x'(0) = \Theta$ is equivalent to the equation

$$(2) \qquad y(t) = F\Big(t, \int_0^{a_1(t)} y(\tau)d\tau, \ldots, \int_0^{a_r(t)} y(\tau)d\tau, y(\beta_1(t)), \ldots, y(\beta_s(t))\Big),$$

where

$$y(t) = \big(y^1(t), \ldots, y^q(t)\big), \qquad \int_0^b y(\tau)d\tau = \Big(\int_0^b y^1(\tau)d\tau, \ldots, \int_0^b y^q(\tau)d\tau\Big).$$

We shall show that the sequence $\{y_n(t)\}$

$$y_0(t) = \Theta, \quad t\in[0, a],$$

$$(3) \quad y_{n+1}(t) = F\Big(t, \int_0^{a_1(t)} y_n(\tau)d\tau, \ldots, \int_0^{a_r(t)} y_n(\tau)d\tau, y_n(\beta_1(t)), \ldots, y_n(\beta_s(t))\Big),$$
$$\text{for } t\in[0, a],\ n = 0, 1, \ldots,$$

is uniformly convergent to a solution of equation (2).

We give theorems on the uniqueness and continuous dependence of the solution of the right-hand side of equation (2).

Moreover, we shall consider in more detail a special case for the functions $a_i(t)$, $\beta_j(t)$ satisfying the relations

$$0\leqslant a_i(t)\leqslant t, \quad 0\leqslant \beta_j(t)\leqslant \beta_j t, \quad 0\leqslant\beta_j\leqslant 1, \quad t\in[0, a],$$
$$i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, s.$$

The paper contains a generalization of some results of [3].

Finally, we shall give an example proving that our result is better than that of [4].

We do not discuss the case where the functions $a_i(t)$, $\beta_j(t)$ change their signs in the interval $[0, a]$.

1. Assumption and lemmas. We introduce

ASSUMPTION H_1. *Suppose that*

1° *the vector function* $F(t, x_1, \ldots, x_r, p_1, \ldots, p_s)$ *is defined and continuous for* $t \in [0, a]$, $x_i = (x_i^1, \ldots, x_i^q)$, $p_j = (p_j^1, \ldots, p_j^q)$, $x_i, p_j \in R^q$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$; $F(t, x_1, \ldots, x_r, p_1, \ldots, p_s) \in R^q$, $F(0, \Theta, \ldots, \Theta) = \Theta$,

2° *the functions* $a_i(t)$, $\beta_j(t)$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$ *are defined and continuous for* $t \in [0, a]$ *and* $a_i(0) = \beta_j(0) = 0$, $a_i(t), \beta_j(t) \in [0, a]$;

3° *for any* $(t, x_{1i}, \ldots, x_{ri}, p_{1i}, \ldots, p_{si}) \in [0, a] \times \underbrace{R^q \times \ldots \times R^q}_{r+s}$, $i = 1, 2$, *we have the inequality*

$$\|F(t, x_{11}, \ldots, x_{r1}, p_{11}, \ldots, p_{s1}) - F(t, x_{12}, \ldots, x_{r2}, p_{12}, \ldots, p_{s2})\| \leqslant \sum_{i=1}^{r} k_i(t)\|x_{i1} - x_{i2}\| + \sum_{i=1}^{s} l_i(t)\|p_{i1} - p_{i2}\|,$$

where the functions $k_i(t)$, $l_j(t)$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$ *are defined and continuous for* $t \in [0, a]$, *and* $k_i(t), l_j(t) \in [0, \infty)$.

ASSUMPTIONS H_2. *Suppose that*

1° *in the interval* $[0, a]$ *there exists a non-negative and continuous solution* $\bar{u}(t)$ *of the inequality*

$$\sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t)\, u\big(\beta_i(t)\big) + \sup_{0 \leqslant s \leqslant t} \|F(s, \Theta, \ldots, \Theta)\| \leqslant u(t), \tag{4}$$
$$u(0) = u(0+) = 0;$$

2° *in the class of functions satisfying the condition* $0 \leqslant u(t) \leqslant \bar{u}(t)$, $t \in [0, a]$, *the function* $u(t) \equiv 0$, $t \in [0, a]$ *is the only measurable solution of the equation*

$$u(t) = \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t)\, u\big(\beta_i(t)\big). \tag{5}$$

ASSUMPTIONS H_3. *Suppose that*

1° *in the interval* $[0, a]$ *there exists a non-negative, continuous and non-decreasing solution* $h^*(t)$ *of the inequality*

$$k(t) \int_0^{t} u(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t)\, u(\beta_i t) + \sup_{0 \leqslant s \leqslant t} \|F(s, \Theta, \ldots, \Theta)\| \leqslant u(t),$$
$$u(0) = u(0+) = 0,$$

where $k(t) = \sum_{i=1}^{r} k_i(t)$, *and* β_i *are given constants*, $0 \leqslant \beta_i \leqslant 1$, $i = 1, 2, \ldots, s$;

2° *in the class of functions satisfying the condition* $0 \leqslant u(t) \leqslant h^*(t)$, $t \in [0, a]$ *the function* $u(t) \equiv 0$, $t \in [0, a]$, *is the only measurable solution of the equation*

$$u(t) = k(t)\int_0^t u(\tau)d\tau + \sum_{i=1}^{s} l_i(t)u(\beta_i t).$$

Let us define the sequence $\{u_n(t)\}$, $t \in [0, a]$ by the relations

$$(6)\qquad \begin{aligned} u_0(t) &= \bar{u}(t), \quad t \in [0, a], \\ u_{n+1}(t) &= \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u_n(\tau)d\tau + \sum_{i=1}^{s} l_i(t)u_n(\beta_i(t)), \quad t \in [0, a], n = 0, 1, \ldots \end{aligned}$$

LEMMA 1. *If Assumption* H_2 *is satisfied, then*

$$0 \leqslant u_{n+1}(t) \leqslant u_n(t) \leqslant \bar{u}(t), \quad t \in [0, a],\ n = 0, 1, \ldots,$$
$$u_n(t) \Rightarrow 0, \quad t \in [0, a],$$

where the sign $\Rightarrow$ *denotes uniform convergence.*

Proof. From relations (6) and (4) we get

$$\begin{aligned} u_1(t) &= \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u_0(\tau)d\tau + \sum_{i=1}^{s} l_i(t)u_0(\beta_i(t)) \\ &\leqslant \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} \bar{u}(\tau)d\tau + \sum_{i=1}^{s} l_i(t)\bar{u}(\beta_i(t)) + \sup_{0\leqslant s\leqslant t} \|F(s, \Theta, \ldots, \Theta)\| \\ &\leqslant \bar{u}(t) = u_0(t), \quad t \in [0, a]. \end{aligned}$$

Further, if we suppose that

$$u_n(t) \leqslant u_{n-1}(t), \quad t \in [0, a],$$

then

$$\begin{aligned} u_{n+1}(t) &= \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u_n(\tau)d\tau + \sum_{i=1}^{s} l_i(t)u_n(\beta_i(t)) \\ &\leqslant \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u_{n-1}(\tau)d\tau + \sum_{i=1}^{s} l_i(t)u_{n-1}(\beta_i(t)) = u_n(t), \quad t \in [0, a]. \end{aligned}$$

Since the sequence of continuous functions $u_n(t)$, $t \in [0, a]$, is non-increasing and bounded from below, it is convergent to a certain measurable function $\bar{\bar{u}}(t)$, $t \in [0, a]$ such that $0 \leqslant \bar{\bar{u}}(t) \leqslant \bar{u}(t)$, $t \in [0, a]$. By Lebesgue's theorem we see that the function $\bar{\bar{u}}(t)$, $t \in [0, a]$ satisfies equation (5).

Now from Assumption H_2 we have $\bar{\bar{u}}(t) \equiv 0$, $t \in [0, a]$.

The uniform convergence of the sequence $\{u_n(t)\}$, $t \in [0, a]$ follows from Dini's theorem. Thus the proof of Lemma 1 is complete.

Let us define the sequence $\tilde{u}_n(t)$, $t\in[0,a]$, by the relations

$$(7)\quad \begin{aligned} \tilde{u}_0(t) &= h^*(t), \quad t\in[0,a],\\ \tilde{u}_{n+1}(t) &= k(t)\int_0^t \tilde{u}_n(\tau)d\tau + \sum_{i=1}^{s} l_i(t)\tilde{u}_n(\beta_i t), \quad t\in[0,a],\ n=0,1,\ldots, \end{aligned}$$

and let a sequence $\{h_n(t)\}$, $t\in[0,a]$, be an arbitrary sequence satisfying the conditions

$$0\leqslant h_0(t)\leqslant h^*(t), \quad t\in[0,a],$$

$$0\leqslant h_{n+1}(t)\leqslant \sum_{i=1}^{r} k_i(t)\int_0^{a_i(t)} h_n(\tau)d\tau + \sum_{i=1}^{s} l_i(t)h_n\big(\beta_i(t)\big),$$
$$t\in[0,a],\ n=0,1,\ldots$$

We then have

LEMMA 2. *If Assumption* H_3 *is satisfied, and*

1° $0\leqslant a_i(t)\leqslant t$, $0\leqslant \beta_j(t)\leqslant \beta_j t$, $0\leqslant\beta_j\leqslant 1$, $t\in[0,a]$, $i=1,2,\ldots,r$, $j=1,2,\ldots,s$,

2° *the functions* $k(t)$, $l_i(t)$, $i=1,2,\ldots,s$, *are non-decreasing in the interval* $[0,a]$, $\big(k(t)\overset{\mathrm{df}}{=}\sum_{i=1}^{r}k_i(t)\big)$, *then*

(i) *the functions* $\tilde{u}_n(t)$, $t\in[0,a]$, $n=0,1,\ldots$, *are non-decreasing,*

(ii) $0\leqslant\tilde{u}_{n+1}(t)\leqslant\tilde{u}_n(t)\leqslant h^*(t)$, $t\in[0,a]$, $n=0,1,\ldots$, $\tilde{u}_n(t)\Rightarrow 0$, $t\in[0,a]$,

(iii) $0\leqslant h_n(t)\leqslant\tilde{u}_n(t)$, $t\in[0,a]$, $n=0,1,\ldots$, $h_n(t)\Rightarrow 0$, $t\in[0,a]$.

Proof. From condition 1° of Assumption H_3 it follows that the function $\tilde{u}_0(t)$, $t\in[0,a]$, is non-decreasing. Further, we obtain (i) by induction.

(ii) follows from Lemma 1 with $\tilde{u}_n(t)$ instead of $u_n(t)$ and Assumption H_2 instead of H_3.

Further, we see that

$$0\leqslant h_0(t)\leqslant h^*(t)=\tilde{u}_0(t), \quad t\in[0,a],$$

and

$$\begin{aligned} 0\leqslant h_1(t) &\leqslant \sum_{i=1}^{r} k_i(t)\int_0^{a_i(t)} h_0(\tau)d\tau + \sum_{i=1}^{s} l_i(t)h_0\big(\beta_i(t)\big)\\ &\leqslant \sum_{i=1}^{r} k_i(t)\int_0^{a_i(t)} \tilde{u}_0(\tau)d\tau + \sum_{i=1}^{s} l_i(t)\tilde{u}_0\big(\beta_i(t)\big)\\ &\leqslant \sum_{i=1}^{r} k_i(t)\int_0^{a_i(t)} \tilde{u}_0(\tau)d\tau + \sum_{i=1}^{s} l_i(t)\tilde{u}_0(\beta_i t)\leqslant \tilde{u}_1(t), \quad t\in[0,a]. \end{aligned}$$

Now, if we suppose that

$$h_n(t) \leqslant \tilde{u}_n(t), \quad t \in [0, a], \ n \geqslant 1,$$

then we get

$$0 \leqslant h_{n+1}(t) \leqslant \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} h_n(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t) h_n\big(\beta_i(t)\big)$$

$$\leqslant \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} \tilde{u}_n(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t)\,\tilde{u}_n\big(\beta_i(t)\big) \leqslant \tilde{u}_{n+1}(t), \quad t \in [0, a].$$

The first part of (iii) follows by induction.

Since $\tilde{u}_n(t) \Rightarrow 0$, $t \in [0, a]$; $h_n(t) \Rightarrow 0$, $t \in [0, a]$.

Thus the proof of Lemma 2 is completed.

2. The existence of a solution of equation (2). Now we can formulate the theorem on the convergence of the sequence $\{y_n(t)\}$ to a solution of equation (2).

THEOREM 1. *If Assumptions* H_1 *and* H_2 *are satisfied, then there exists in the interval* $[0, a]$ *a continuous solution* $\bar{y}(t)$ *of equation* (2) *such that* $\bar{y}(0) = \Theta$. *The sequence* $\{y_n(t)\}$ *converges uniformly on* $[0, a]$ *to* $\bar{y}(t)$ *as* $n \to \infty$; *moreover, the estimations*

(8) $$\|y_n(t) - \bar{y}(t)\| \leqslant u_n(t), \quad t \in [0, a], \ n = 0, 1, \ldots,$$

and

(9) $$\|\bar{y}(t)\| \leqslant \bar{u}(t), \quad t \in [0, a]$$

hold true.

The solution $\bar{y}(t)$, $t \in [0, a]$, *of* (2) *is unique in the class of functions satisfying relation* (9).

Proof. We shall prove that the sequence $\{y_n(t)\}$, $t \in [0, a]$, fulfils the condition

(10) $$\|y_n(t)\| \leqslant \bar{u}(t), \quad t \in [0, a], \ n = 0, 1, \ldots$$

Evidently

$$\|y_0(t)\| = 0 \leqslant \bar{u}(t), \quad t \in [0, a].$$

Let us suppose that inequality (10) is true for $n \geqslant 0$. By the definition of $y_n(t)$, $t \in [0, a]$, and condition 3° of Assumption H_1, we have

$$\|y_{n+1}(t)\| = \Big\|F\Big(t, \int_0^{a_1(t)} y_n(\tau)\,d\tau, \ldots, \int_0^{a_r(t)} y_n(\tau)\,d\tau, y_n\big(\beta_1(t)\big), \ldots, y_n\big(\beta_s(t)\big)\Big) -$$

$$- F(t, \Theta, \ldots, \Theta) + F(t, \Theta, \ldots, \Theta)\Big\|$$

$$\leqslant \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} \|y_n(\tau)\|\,d\tau + \sum_{i=1}^{s} l_i(t) \big\|y_n\big(\beta_i(t)\big)\big\| + \sup_{0 \leqslant s \leqslant t} \|F(s, \Theta, \ldots, \Theta)\|$$

$$\leqslant \bar{u}(t),$$

for $t\in[0,a]$. Now, we obtain (10) by induction. Further, we prove that

$$\|y_{n+m}(t)-y_m(t)\|\leqslant u_m(t),\qquad t\in[0,a],\ n,m=0,1,\dots \tag{11}$$

By (10) we have

$$\|y_n(t)-y_0(t)\| = \|y_n(t)\|\leqslant \bar{u}(t) = u_0(t),\qquad t\in[0,a],\ n=0,1,\dots$$

Further, if we suppose that (11) is true for $n,m\geqslant 0$, then

$$\begin{aligned}
&\|y_{n+m+1}(t)-y_{m+1}(t)\|\\
&= \Big\|F\Big(t,\int_0^{a_1(t)} y_m(\tau)d\tau,\dots,\int_0^{a_r(t)} y_m(\tau)d\tau, y_m(\beta_1(t)),\dots,y_m(\beta_s(t))\Big)-\\
&\quad -F\Big(t,\int_0^{a_1(t)} y_{n+m}(\tau)d\tau,\dots,\int_0^{a_r(t)} y_{n+m}(\tau)d\tau, y_{n+m}(\beta_1(t)),\dots,y_{n+m}(\beta_s(t))\Big)\Big\|\\
&\leqslant \sum_{i=1}^{r} k_i(t)\int_0^{a_i(t)}\|y_{n+m}(\tau)-y_m(\tau)\|d\tau+\sum_{i=1}^{s} l_i(t)\big\|y_{n+m}(\beta_i(t))-y_m(\beta_i(t))\big\|\\
&\leqslant \sum_{i=1}^{r} k_i(t)\int_0^{a_i(t)} u_m(\tau)d\tau+\sum_{i=1}^{s} l_i(t)u_m(\beta_i(t)) = u_{m+1}(t),\qquad t\in[0,a].
\end{aligned}$$

Now, we obtain (11) by induction.

Because of Lemma 1 $u_m(t)\rightrightarrows 0$ for $t\in[0,a]$; therefore from (11) we have $y_n(t)\rightrightarrows \bar{y}(t)$, $t\in[0,a]$. The continuity of $\bar{y}(t)$ follows from the uniform convergence of the sequence $\{y_n(t)\}$ and the continuity of all functions $y_n(t)$.

If n tends to ∞, then (11) gives estimation (8). Estimation (9) is implied by (10). $\bar{y}(0)=\Theta$ follows from $y_n(0)=\Theta$, $n=0,1,\dots$

It is obvious that $\bar{y}(t)$, $t\in[0,a]$ is the solution of (2).

To prove that the solution $\bar{y}(t)$, $t\in[0,a]$, is unique let us suppose that there exists another solution $\tilde{y}(t)$, $t\in[0,a]$, such that $\bar{y}(t)\not\equiv\tilde{y}(t)$ for $t\in[0,a]$, and $\|\tilde{y}(t)\|\leqslant\bar{u}(t)$ for $t\in[0,a]$.

We get

$$\|\tilde{y}(t)-y_n(t)\|\leqslant u_n(t),\qquad t\in[0,a],\ n=0,1,\dots,$$

by induction and hence it follows that $\bar{y}(t)\equiv\tilde{y}(t)$, $t\in[0,a]$. This contradiction proves the uniqueness of $\bar{y}(t)$, $t\in[0,a]$, in the class of functions satisfying relation (9). Thus the proof of Theorem 1 is completed.

Now we can formulate an analogous theorem for equations of the delay type.

THEOREM 2. *If Assumptions* H_1 *and* H_3 *and* 1°–2° *of Lemma* 2 *are fulfilled, then the assertion of Theorem* 1 *is true with* $h^*(t)$ *instead of* $\bar{u}(t)$, *and the estimations*

$$\|\bar{y}(t)-y_n(t)\|\leqslant\tilde{u}_n(t),\qquad t\in[0,a],\ n=0,1,\dots,$$

and

$$\|\bar{y}(t)\| \leqslant h^*(t), \quad t \in [0, a],$$

hold true.

Proof. We prove that Assumption H_2 is fulfilled. We can take $h^*(t)$ instead of $\bar{u}(t)$. Let $u(t)$ be a measurable solution of (5) in the class $0 \leqslant u(t) \leqslant h^*(t)$, $t \in [0, a]$. Further, we get

$$0 \leqslant u(t) \leqslant \tilde{u}_n(t), \quad t \in [0, a], \; n = 0, 1, \ldots,$$

by induction, and since $\tilde{u}_n(t) \Rightarrow 0$ for $t \in [0, a]$, we have $u(t) \equiv 0$, $t \in [0, a]$. Hence Assumption H_2 is fulfilled.

Since all assumptions of Theorem 1 are fulfilled and $u_n(t) \leqslant \tilde{u}_n(t)$, $t \in [0, a]$, Theorem 2 is proved.

3. Uniqueness theorem. Now we give the conditions under which equation (2) has at most one solution; these conditions do not guarantee the existence of the solution. We have

THEOREM 3. *If Assumption* H_1 *is satisfied and the function* $b(t) \equiv 0$, $t \in [0, a]$, *is the only non-negative, finite and measurable solution of the inequality*

$$b(t) \leqslant \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} b(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t)\, b\big(\beta_i(t)\big), \quad t \in [0, a], \tag{12}$$

then equation (2) *has at most one solution in the interval* $[0, a]$.

Proof. Let us suppose that there exist two solutions $\tilde{y}(t)$ and $\tilde{\tilde{y}}(t)$ of equation (2) in the interval $[0, a]$, such that $\tilde{y}(t) \not\equiv \tilde{\tilde{y}}(t)$, $t \in [0, a]$. Now from condition 3° of Assumption H_1 we have

$$\begin{aligned}\|\tilde{y}(t) - \tilde{\tilde{y}}(t)\| &= \Big\| F\Big(t, \int_0^{a_1(t)} \tilde{y}(\tau)\,d\tau, \ldots, \int_0^{a_r(t)} \tilde{y}(\tau)\,d\tau, \tilde{y}\big(\beta_1(t)\big), \ldots, \tilde{y}\big(\beta_s(t)\big)\Big) - \\ &\quad - F\Big(t, \int_0^{a_1(t)} \tilde{\tilde{y}}(\tau)\,d\tau, \ldots, \int_0^{a_r(t)} \tilde{\tilde{y}}(\tau)\,d\tau, \tilde{\tilde{y}}\big(\beta_1(t)\big), \ldots, \tilde{\tilde{y}}\big(\beta_s(t)\big)\Big)\Big\| \\ &\leqslant \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} \|\tilde{y}(\tau) - \tilde{\tilde{y}}(\tau)\|\,d\tau + \sum_{i=1}^{s} l_i(t) \big\|\tilde{y}\big(\beta_i(t)\big) - \tilde{\tilde{y}}\big(\beta_i(t)\big)\big\|,\end{aligned}$$

for $t \in [0, a]$. Put

$$b(t) = \|\tilde{y}(t) - \tilde{\tilde{y}}(t)\|, \quad t \in [0, a].$$

By (12) we conclude that $b(t) \equiv 0$, $t \in [0, a]$ i.e. $\tilde{y}(t) \equiv \tilde{\tilde{y}}(t)$ for $t \in [0, a]$. This contradiction proves Theorem 3.

LEMMA 3. *If for any measurable function* $f(t) \geqslant 0$ (f: $[0, a] \to [0, \infty)$), *instead of* $\|F(t, \Theta, \ldots, \Theta)\|$ *Assumption* H_2 *is satisfied* (*with measurability* *instead of continuity of $\bar{u}(t)$), then in the interval $[0, a]$ the function $b(t) \equiv 0$ is the only non-negative, finite and measurable solution of inequality* (12).

Proof. We see that the function $b(t) = 0$ for $t \in [0, a]$ is a solution of (12).

We suppose that in the interval $[0, a]$ there exists another non-negative, finite and measurable solution $b^*(t)$ of (12). Let $b_0(t)$ for $t \in [0, a]$ be a non-negative, finite and measurable solution of the inequality

$$\sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t) u\big(\beta_i(t)\big) + \sup_{0 \leqslant s \leqslant t} b^*(s) \leqslant u(t), \qquad t \in [0, a],$$

and

$$b_{n+1}(t) \stackrel{\mathrm{df}}{=} \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} b_n(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t) b_n\big(\beta_i(t)\big), \quad t \in [0, a], \; n = 0, 1, \ldots$$

We have $b^*(t) \leqslant b_0(t)$, $t \in [0, a]$, and we get

$$b_{n+1}(t) \leqslant b_n(t), \qquad t \in [0, a], \; n = 0, 1, \ldots,$$

$$b^*(t) \leqslant b_n(t), \qquad t \in [0, a], \; n = 0, 1, \ldots, \tag{13}$$

by induction. Since the sequence $\{b_n(t)\}$, $t \in [0, a]$, is non-increasing and bounded from below, it is convergent to a non-negative measurable function $\bar{b}(t)$, $t \in [0, a]$, which satisfies the equation

$$u(t) = \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t) u\big(\beta_i(t)\big), \qquad t \in [0, a].$$

Now from Assumption H_2 we have $\bar{b}(t) \equiv 0$ for $t \in [0, a]$. Further, if $n \to \infty$, then (13) gives $b^*(t) \equiv 0$ for $t \in [0, a]$.

Remark 1. If Assumption H_2 is satisfied, then the function $b(t) \equiv 0$ for $t \in [0, a]$, is the only measurable solution of (12) in the class of functions $0 \leqslant b(t) \leqslant \bar{u}(t)$, $t \in [0, a]$.

Indeed, we can prove by induction that

$$0 \leqslant b(t) \leqslant u_n(t), \qquad t \in [0, a], \; n = 0, 1, \ldots,$$

and if $n \to \infty$, then we have, in view of Lemma 1, $b(t) \equiv 0$ for $t \in [0, a]$.

Remark 2. Equation (2) has at most one solution in the class of continuous functions satisfying the condition

$$\|y(t)\| \leqslant Lt, \qquad t \in [0, a], \; L \geqslant 0,$$

if the function $b(t) \equiv 0$ is the only non-negative and measurable solution of (12) fulfilling the condition $b(t) \leqslant 2Lt$, $t \in [0, a]$, $L \geqslant 0$. In consequence, we can suppose that the assumptions of Lemma 3 are fulfilled only for $f(t) \leqslant 2Lt$, $t \in [0, a]$, $L \geqslant 0$.

Remark 3. If Assumption H_1 and 1°–2° of Lemma 2 are satisfied, and the function $u(t) \equiv 0$, $t \in [0, a]$, is the only non-negative, non-decreasing and finite solution of the inequality

$$(14)\qquad u(t) \leqslant k(t) \int_0^t u(\tau)d\tau + \sum_{i=1}^{s} l_i(t) u(\beta_i t), \qquad t \in [0, a],$$

where $k(t) = \sum_{i=1}^{r} k_i(t)$, then equation (2) has at most one solution.

Remark 4. We can formulate a lemma analogous to Lemma 3 for an equation of the delay type.

4. Continuous dependence of solutions on the right-hand side of equation (2). We ask: what is the influence of the form of an equation on the solution of that equation.

Let us consider the equation

$$(15)\qquad w(t) = W\Big(t, \int_0^{\gamma_1(t)} w(\tau)d\tau, \ldots, \int_0^{\gamma_r(t)} w(\tau)d\tau, w\big(\delta_1(t)\big), \ldots, w\big(\delta_s(t)\big)\Big),$$
$$w(0) = \Theta,$$

where the vector function W and the functions $\gamma_i(t)$, $\delta_j(t)$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$, have the same properties as F and $a_i(t)$, $\beta_j(t)$, as given in Assumption H_1.

Now we have

THEOREM 4. *If Assumption* H_1 *is satisfied, and*

1° $y^*(t)$ *and* $w^*(t)$ *for* $t \in [0, a]$ *are solutions of equations* (2) *and* (15),

2° *the sequence* $\{z_n(t)\}$, $t \in [0, a]$, *defined by the relations*

$$z_0(t) = \|y^*(t)\| + \|w^*(t)\|, \qquad t \in [0, a],$$

$$z_{n+1}(t) = \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} z_n(\tau)d\tau + \sum_{i=1}^{s} l_i(t) z_n\big(\beta_i(t)\big) + v^*(t), \qquad t \in [0, a],$$
$$n = 0, 1, \ldots,$$

$$v^*(t) \overset{\text{df}}{=} \Big\|F\Big(t, \int_0^{a_1(t)} w^*(\tau)d\tau, \ldots, \int_0^{a_r(t)} w^*(\tau)d\tau, w^*\big(\beta_1(t)\big), \ldots, w^*\big(\beta_s(t)\big)\Big) -$$
$$- w^*(t)\Big\|, \qquad t \in [0, a]$$

has the limit $\bar{z}(t)$ *for* $t \in [0, a]$,
then

$$(16)\qquad \|y^*(t) - w^*(t)\| \leqslant \bar{z}(t), \qquad t \in [0, a].$$

Proof. Let

$$v(t) = \|y^*(t) - w^*(t)\|, \qquad t \in [0, a].$$

Thus for $t \in [0, a]$ we have

$$v(t) = \Big\| F\Big(t, \int_0^{a_1(t)} y^*(\tau)d\tau, \ldots, \int_0^{a_r(t)} y^*(\tau)d\tau, y^*\big(\beta_1(t)\big), \ldots, y^*\big(\beta_s(t)\big)\Big) -$$
$$- F\Big(t, \int_0^{a_1(t)} w^*(\tau)d\tau, \ldots, \int_0^{a_r(t)} w^*(\tau)d\tau, w^*\big(\beta_1(t)\big), \ldots, w^*\big(\beta_s(t)\big)\Big) +$$
$$+ F\Big(t, \int_0^{a_1(t)} w^*(\tau)d\tau, \ldots, \int_0^{a_r(t)} w^*(\tau)d\tau, w^*\big(\beta_1(t)\big), \ldots, w^*\big(\beta_s(t)\big)\Big) - w^*(t)\Big\|$$
$$\leqslant \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} \|y^*(\tau) - w^*(\tau)\| d\tau + \sum_{i=1}^{s} l_i(t) \big\|y^*\big(\beta_i(t)\big) - w^*\big(\beta_i(t)\big)\big\| + v^*(t)$$
$$= \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} v(\tau)d\tau + \sum_{i=1}^{s} l_i(t) v\big(\beta_i(t)\big) + v^*(t).$$

Since

$$v(t) \leqslant \|y^*(t)\| + \|w^*(t)\| = z_0(t), \quad t \in [0, a],$$

this and the last inequality give

$$v(t) \leqslant z_n(t), \quad t \in [0, a],\ n = 0, 1, \ldots,$$

by induction.

Inequality (16) is implied by the last one as $n \to \infty$.

Remark 5. If the functions $z_n(t)$, $t \in [0, a]$, $n = 0, 1, \ldots$, are finite and measurable and there exists a Lebesgue-integrable function T: $[0, a] \to [0, \infty)$ such that

$$z_n(t) \leqslant T(t), \quad t \in [0, a],\ n = 0, 1, \ldots,$$

then the limit function $\bar{z}(t)$, $t \in [0, a]$ (see 2° of Theorem 4) is a finite and measurable solution of the equation

$$z(t) = \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} z(\tau)d\tau + \sum_{i=1}^{s} l_i(t) z\big(\beta_i(t)\big) + v^*(t), \quad t \in [0, a].$$

Remark 6. From the proof of Theorem 4 it follows that this theorem is true if in the interval $[0, a]$ there exists a non-negative and continuous function $m_0(t)$ satisfying the inequality

$$\sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} m_0(\tau)d\tau + \sum_{i=1}^{s} l_i(t) m_0\big(\beta_i(t)\big) + \max[v^*(t), z_0(t)] \leqslant m_0(t),$$
$$t \in [0, a].$$

Now, in the class of measurable functions satisfying the condition $0 \leqslant u(t) \leqslant m_0(t)$, $t \in [0, a]$, there exists a function $\overline{m}(t)$, $t \in [0, a]$, being a solution of the equation

$$\sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} u(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t) u\big(\beta_i(t)\big) + v^*(t) = u(t), \qquad t \in [0, a].$$

Put

$$m_{n+1}(t) = \sum_{i=1}^{r} k_i(t) \int_0^{a_i(t)} m_n(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t) m_n\big(\beta_i(t)\big) + v^*(t),$$

for $t \in [0, a]$, $n = 0, 1, \ldots$

We see that

$$z_n(t) \leqslant m_n(t), \qquad m_{n+1}(t) \leqslant m_n(t), \qquad t \in [0, a], \qquad n = 0, 1, \ldots,$$

and hence $v(t) \leqslant m_n(t)$, $t \in [0, a]$, $n = 0, 1, \ldots$ ($v(t)$ is defined in the proof of Theorem 4). From the last inequality we get $m_n(t) \to \overline{m}(t)$, $t \in [0, a]$, and $v(t) \leqslant \bar{z}(t) \leqslant \overline{m}(t)$, $t \in [0, a]$.

Remark 7. If the $\bar{z}(t)$ depends continuously on the $v^*(t)$, then from Theorem 4 we get a theorem on the continuous dependence of the solutions of (2) on the right-hand side and on the initial conditions. This take place, for example, if the condition of Remark 6 and inequality (12) hold.

From Theorem 4 for the equation of the delay type follows

THEOREM 5. *If the assumptions of Theorem* 4 (except 2°) *and* 1°–2° *of Lemma* 2 *are satisfied, and the sequence* $\{\tilde{z}_n(t)\}$,

$$\tilde{z}_0(t) = \sup_{0 \leqslant s \leqslant t} \{\|y^*(s)\| + \|w^*(s)\|\}, \qquad t \in [0, a],$$

$$\tilde{z}_{n+1}(t) = k(t) \int_0^t \tilde{z}_n(\tau)\,d\tau + \sum_{i=1}^{s} l_i(t) \tilde{z}_n(\beta_i t) + \sup_{0 \leqslant s \leqslant t} v^*(s),$$

for $t \in [0, a]$, $n = 0, 1, \ldots$, *has the limit* $z^*(t)$, $t \in [0, a]$, *then*

$$\|y^*(t) - w^*(t)\| \leqslant z^*(t), \qquad t \in [0, a], \tag{17}$$

and the functions $\tilde{z}_n(t)$, $t \in [0, a]$, *are non-decreasing.*

Proof. It is easy to prove that functions $\tilde{z}_n(t)$ are non-decreasing for $t \in [0, a]$, $n = 0, 1, \ldots$ Further, we get

$$z_n(t) \leqslant \tilde{z}_n(t), \qquad t \in [0, a], \ n = 0, 1, \ldots,$$

by induction, where the sequence $\{z_n(t)\}$, $t \in [0, a]$, is defined in 2° of Theorem 4. Hence $v(t) \leqslant \tilde{z}_n(t)$, $t \in [0, a]$, $n = 0, 1, \ldots$ ($v(t)$ is defined in the proof of Theorem 4), and if $n \to \infty$, then we have (17).

5. Discussion of equation (2) for the case $k_i(t) = 0$. In this section we consider equation (2) for the case $k_i(t) = 0$, $t \in [0, a]$, $i = 1, 2, \ldots, r$, i.e. when the right-hand side of equation (2) is independent of $\int\limits_0^{a_i(t)} y(\tau)d\tau$, $i = 1, 2, \ldots, r$.

For $t \in [0, a]$, $i_n = 1, 2, \ldots, s$, under $n = 0, 1, \ldots,$ let

$$\tag{18} \begin{aligned} &\beta_0^{i_0}(t) = t, \qquad \beta_{n+1}^{i_0,\ldots,i_{n+1}}(t) = \beta_n^{i_1,\ldots,i_{n+1}}\big(\beta_{i_0}(t)\big), \\ &l_0^{i_0}(t) = 1/s, \qquad l_{n+1}^{i_0,\ldots,i_{n+1}}(t) = l_{i_0}(t)\, l_n^{i_1,\ldots,i_{n+1}}\big(\beta_{i_0}(t)\big), \end{aligned}$$

where $\beta_i(t)$, $l_i(t)$, $t \in [0, a]$, $i = 1, 2, \ldots, s$, are with Assumption H_1.

It is obvious that $\beta_n^{i_0,\ldots,i_n}(t) \in [0, a]$ for $t \in [0, a]$, $i_n = 1, 2, \ldots, s$, $n = 0, 1, \ldots$

Now we formulate lemmas by which Assumption H_2 is fulfilled.

LEMMA 4 (cf. [1]). *For any function $v(u) \geqslant 0$, $u \in [0, a]$, the condition*

$$\tag{19} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \ldots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t)\, v\big(\beta_n^{i_0,\ldots,i_n}(t)\big) < \infty, \qquad t \in [0, a],$$

is necessary and sufficient for the equation

$$\tag{20} u(t) = \sum_{i=1}^{s} l_i(t)\, u\big(\beta_i(t)\big) + v(t), \qquad t \in [0, a],$$

to have a non-negative solution $u^*(t)$, $t \in [0, a]$.

If condition (19) *is fulfilled, then the function*

$$\tag{21} \bar{u}(t) = \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \ldots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t)\, v\big(\beta_n^{i_0,\ldots,i_n}(t)\big), \qquad t \in [0, a],$$

is a solution of equation (20), *and*

$$\tag{22} \lim_{n\to\infty} \sum_{i_0=1}^{s} \ldots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t)\, \bar{u}\big(\beta_n^{i_0,\ldots,i_n}(t)\big) = 0, \qquad t \in [0, a].$$

There is no other solution of equation (20) *in the class of functions $0 \leqslant u(t) \leqslant \bar{u}(t)$, $t \in [0, a]$.*

Remark 8. If $s = 1$, $\beta(t) \overset{\text{df}}{=} \beta_1(t)$, $l(t) \overset{\text{df}}{=} l_1(t)$, $t \in [0, a]$, then the sequences $\{\beta_n(t)\}$, $\{l_n(t)\}$ defined by (18) are of the form

$$\beta_0(t) = t, \qquad \beta_{n+1}(t) = \beta\big(\beta_n(t)\big), \qquad t \in [0, a],\ n = 0, 1, \ldots,$$

$$l_0(t) = 1, \qquad l_{n+1}(t) = \prod_{i=0}^{n} l\big(\beta_i(t)\big), \qquad t \in [0, a],\ n = 0, 1, \ldots$$

Now (21) and (22) are of the form

$$\bar{u}(t) = \sum_{n=0}^{\infty} l_n(t) v(\beta_n(t)), \quad t \in [0, a], \tag{21'}$$

and

$$\lim_{n\to\infty} l_n(t)\bar{u}(\beta_n(t)) = 0; \quad t \in [0, a]. \tag{22'}$$

LEMMA 5. *If*

1° $0 \leqslant \varphi_1(t) \leqslant \varphi_2(t), \quad t \in [0, a],$

2° $\sum\limits_{n=0}^{\infty} \sum\limits_{i_0=1}^{s} \cdots \sum\limits_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t)\varphi_2(\beta_n^{i_0,\ldots,i_n}(t)) < \infty, \quad t \in [0, a],$

then the functions

$$\bar{v}_i(t) = \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t)\varphi_i(\beta_n^{i_0,\ldots,i_n}(t)) \quad \textit{for } t \in [0, a],\ i = 1, 2,$$

are non-negative solutions of the equations

$$v(t) = \sum_{i=1}^{s} l_i(t) v(\beta_i(t)) + \varphi_j(t), \quad t \in [0, a],\ j = 1, 2, \tag{23}$$

respectively, and

$$\lim_{n\to\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t)\bar{v}_i(\beta_n^{i_0,\ldots,i_n}(t)) = 0, \quad t \in [0, a],\ i = 1, 2. \tag{24}$$

Moreover, the functions $\bar{v}_i(t)$, $t \in [0, a]$, $i = 1, 2$, are the unique solutions of (23) *in the class of functions satisfying* $0 \leqslant v(t) \leqslant \bar{v}_2(t)$, $t \in [0, a]$.

Proof. From Lemma 4 it follows that for $i = 1$ the function $\bar{v}_1(t)$, $t \in [0, a]$ is the unique solution of (23) in the class $0 \leqslant v(t) \leqslant \bar{v}_1(t)$, $t \in [0, a]$, and, for $i = 2$ the function $\bar{v}_2(t)$, $t \in [0, a]$ is the unique solution of (23) in the class $0 \leqslant v(t) \leqslant \bar{v}_2(t)$, $t \in [0, a]$, and (24) is true. Further, we prove that for $i = 1$ the function $\bar{v}_1(t)$, $t \in [0, a]$, is the unique solution of (23) in the class $\bar{v}_1(t) \leqslant v(t) \leqslant \bar{v}_2(t)$, $t \in [0, a]$. We assume that for $i = 1$ there exists another solution $z(t)$, $t \in [0, a]$ of (23) in this class, such that $z(t) \not\equiv \bar{v}_1(t)$, $t \in [0, a]$. Since any solutions $r_i(t)$, $t \in [0, a]$, $i = 1, 2$, of (23) satisfy the conditions

$$\begin{aligned} r_i(t) = {} & \sum_{n=0}^{m} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t)\varphi_i(\beta_n^{i_0,\ldots,i_n}(t)) + \\ & + \sum_{i_0=1}^{s} \cdots \sum_{i_{m+1}=1}^{s} l_{m+1}^{i_0,\ldots,i_{m+1}}(t) r_i(\beta_{m+1}^{i_0,\ldots,i_{m+1}}(t)), \quad m = 0, 1, \ldots, \end{aligned} \tag{25}$$

then for $t \in [0, a]$ we have

$$0 \leqslant z(t) - \bar{v}_1(t) = \sum_{i_0=1}^{s} \cdots \sum_{i_{m+1}=1}^{s} l_{m+1}^{i_0,\ldots,i_{m+1}}(t) z\big(\beta_{m+1}^{i_0,\ldots,i_{m+1}}(t)\big) - \sum_{i_0=1}^{s} \cdots \sum_{i_{m+1}=1}^{s} l_{m+1}^{i_0,\ldots,i_{m+1}}(t) \bar{v}_1\big(\beta_{m+1}^{i_0,\ldots,i_{m+1}}(t)\big)$$
$$\leqslant \sum_{i_0=1}^{s} \cdots \sum_{i_{m+1}=1}^{s} l_{m+1}^{i_0,\ldots,i_{m+1}}(t) \bar{v}_2\big(\beta_{m+1}^{i_0,\ldots,i_{m+1}}(t)\big).$$

Now, if $m \to \infty$ then we have $z(t) \equiv \bar{v}_1(t)$, $t \in [0, a]$. The resulting contradiction proves the uniqueness of the solution $\bar{v}_1(t)$, $t \in [0, a]$ of equation (23) in the class of functions $0 \leqslant v(t) \leqslant \bar{v}_2(t)$, $t \in [0, a]$.

These considerations and Theorem 1 imply

THEOREM 6. *If Assumption* H_1 *is satisfied, and*

1° $k_i(t) = 0, \quad t \in [0, a],\ i = 1, 2, \ldots, r,$

2° $\displaystyle\sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t) v\big(\beta_n^{i_0,\ldots,i_n}(t)\big) < \infty, \quad t \in [0, a],$

where

$$v(t) = \sup_{0 \leqslant s \leqslant t} \|F(s, \Theta, \ldots, \Theta)\|, \quad t \in [0, a],$$

then there exists a unique solution $\bar{y}(t)$, $\bar{y}(0) = \Theta$, *of equation* (2) *in the interval* $[0, a]$ *with the following properties*:

$$\|\bar{y}(t)\| \leqslant \bar{u}(t), \quad t \in [0, a],$$

$$\|\bar{y}(t) - y_n(t)\| \leqslant u_n(t), \quad t \in [0, a],\ n = 0, 1, \ldots,$$

where

$$u_0(t) = \bar{u}(t) = \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t) v\big(\beta_n^{i_0,\ldots,i_n}(t)\big), \quad t \in [0, a],$$

$$u_{n+1}(t) = \sum_{m=n}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_m=1}^{s} l_m^{i_0,\ldots,i_m}(t) v\big(\beta_m^{i_0,\ldots,i_m}(t)\big), \quad t \in [0, a],\ n = 0, 1, \ldots$$

Theorem 4 implies the following

THEOREM 7. *If Assumption* H_1 *is satisfied, and*

1° $k_i(t) = 0, \quad t \in [0, a],\ i = 1, 2, \ldots, r,$

2° *the functions* $y^*(t)$ *and* $w^*(t)$, $t \in [0, a]$, *are solutions of equations* (2) *and* (15),

3° $\displaystyle\sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t) c\big(\beta_n^{i_0,\ldots,i_n}(t)\big) < +\infty, \quad t \in [0, a],$

where

$$c(t) = \max\{\|y^*(t)\| + \|w^*(t)\|,\ v^*(t)\}, \quad t \in [0, a],$$

and $v^*(t)$ *is defined by condition* 2° *of Theorem* 4, *then*

$$\|y^*(t)-w^*(t)\| \leqslant \sum_{n=0}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t)\, v^*\big(\beta_n^{i_0,\ldots,i_n}(t)\big), \quad t\in[0,a].$$

6. Discussion of the equation of the delay type for the case $k_i(t)$, $l_j(t)$, $t\in[0,a]$, being non-negative constants. Let for $t\in[0,a]$,

(26) $$k_i(t) = k_i, \quad l_j(t) = l_j, \quad k_i, l_j \geqslant 0, \quad i = 1,2,\ldots,r,\ j = 1,2,\ldots,s.$$

In this section we consider equation (2) when the functions $a_i(t)$, $\beta_j(t)$ satisfy the conditions

(27) $$0\leqslant a_i(t)\leqslant t, \quad 0\leqslant \beta_j(t)\leqslant \beta_j t, \quad 0\leqslant\beta_j\leqslant 1, \quad t\in[0,a], \quad i = 1,2,\ldots,r, \quad j = 1,2,\ldots,s.$$

Now the sequences $\{\beta_n^{i_0,\ldots,i_n}(t)\}$, $\{l_n^{i_0,\ldots,i_n}(t)\}$, $t\in[0,a]$, $i_n = 1,2,\ldots,s$, $n = 0,1,\ldots,$ defined by (18) satisfy the relations

(28) $$\beta_n^{i_0,\ldots,i_n}(t)\leqslant t\prod_{r=0}^{n-1}\beta_{i_r}, \quad l_n^{i_0,\ldots,i_n}(t) = \frac{1}{s}\prod_{r=0}^{n-1} l_{i_r}, \quad t\in[0,a],$$

where

$$\prod_{r=0}^{n-1} C_r \stackrel{\mathrm{df}}{=} \begin{cases} 1 & \text{if } n = 0,\\ \prod_{r=0}^{n-1} C_r & \text{if } n\geqslant 1.\end{cases}$$

We have

LEMMA 6. *If* $t\in[0,\infty)$ *and* $a\in[0,1]$, *then*

(29) $$e^{t(a-1)}\leqslant a(1-e^{-t})+e^{-t} \quad (e^t \equiv \exp t).$$

Proof. Put

$$f(a,t) = e^{t(a-1)}-a(1-e^{-t})-e^{-t} \quad \text{for } a\in[0,1],\ t\in[0,\infty).$$

Now for $a\in[0,1]$, $t\in[0,\infty)$ we have

$$\frac{df(a,t)}{dt} = (a-1)e^{-t}(e^{ta}-1)\leqslant 0$$

and therefore

$$f(a,t)\leqslant f(a,0) = 0.$$

LEMMA 7. *If*

1° $H(t) \stackrel{\mathrm{df}}{=} \sum_{n=0}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\big(\prod_{r=0}^{n-1} l_{i_r}\big) g\big(t\prod_{r=0}^{n-1}\beta_{i_r}\big) < \infty$ *is continuous for* $t\in[0,a]$,

2° $0 \leqslant \sum_{i=1}^{s} l_i \beta_i < 1$,

3° $0 \leqslant \beta_i \leqslant 1$, $i = 1, 2, \ldots, s$,

4° *the function* $g(t)$ *is continuous, non-negative and non-decreasing in the interval* $[0, a]$, *and* $g(0) = 0$, *then*

(a) *there exists a unique solution* $h^*(t)$, $h^*(0) = h^*(0+) = 0$ *of the equation*

$$(30)\qquad u(t) = \frac{k}{s} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r}\Big) \int_0^{t \prod_{r=0}^{n-1} \beta_{i_r}} u(\tau)\,d\tau + \\ + \frac{1}{s} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r}\Big) g\Big(t \prod_{r=0}^{n-1} \beta_{i_r}\Big), \qquad t \in [0, a],\ k \geqslant 0;$$

this solution is continuous, non-negative and non-decreasing in the interval $[0, a]$,

(b) *in the class of measurable functions satisfying the condition* $0 \leqslant u(t) \leqslant h^*(t)$, $t \in [0, a]$, *the function* $h^*(t)$ *is the unique, continuous, non-negative and non-decreasing solution of the equation*

$$(31)\qquad u(t) = \sum_{i=1}^{s} l_i u(\beta_i t) + k \int_0^t u(\tau)\,d\tau + g(t), \qquad t \in [0, a],\ k \geqslant 0,$$

(c) *in the class of measurable functions satisfying the condition* $0 \leqslant u(t) \leqslant h^*(t)$, $t \in [0, a]$, *the function* $u(t) \equiv 0$, $t \in [0, a]$, *is the unique solution of the inequality*

$$(32)\qquad u(t) \leqslant \sum_{i=1}^{s} l_i u(\beta_i t) + k \int_0^t u(\tau)\,d\tau, \qquad t \in [0, a],\ k \geqslant 0.$$

Proof. Let A be the operator defined by the right-hand side of equation (30), and

$$\|u\|_* = \max_{0 \leqslant t \leqslant a} e^{-Lt} |u(t)| \qquad \text{for } u \in C[0, a],$$

where $L \geqslant k\big(1 - \sum_{i=1}^{s} l_i \beta_i\big)^{-1}$, and $C[0, a]$ denotes the class of continuous functions in $[0, a]$.

We get

$$\frac{1}{s} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r} \beta_{i_r}\Big) = \Big(\sum_{i=1}^{s} l_i \beta_i\Big)^n, \qquad n = 0, 1, \ldots,$$

by induction.

Now from Lemma 6, for $u, z \in C[0, a]$, we have

$$\|Au - Az\|_*$$

$$= \frac{k}{s} \max_{0 \leqslant t \leqslant a} \left| e^{-Lt} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \left(\prod_{r=0}^{n-1} l_{i_r}\right) \int_0^{t \prod_{r=0}^{n-1} \beta_{i_r}} [u(\tau) - z(\tau)] e^{-L\tau} e^{L\tau} d\tau \right|$$

$$\leqslant \frac{k}{s} \|u - z\|_* \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \left(\prod_{r=0}^{n-1} l_{i_r}\right) \max_{0 \leqslant t \leqslant a} e^{-Lt} \int_0^{t \prod_{r=0}^{n-1} \beta_{i_r}} e^{L\tau} d\tau$$

$$= \frac{k}{sL} \|u - z\|_* \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \left(\prod_{r=0}^{n-1} l_{i_r}\right) \max_{0 \leqslant t \leqslant a} \{e^{Lt[\prod_{r=0}^{n-1} \beta_{i_r} - 1]} - e^{-Lt}\}$$

$$\leqslant \frac{k}{sL} \|u - z\|_* \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \left(\prod_{r=0}^{n-1} l_{i_r}\right) \max_{0 \leqslant t \leqslant a} \left[\left(\prod_{r=0}^{n-1} \beta_{i_r}\right)(1 - e^{-Lt})\right]$$

$$= \frac{k}{L}(1 - e^{-La}) \|u - z\|_* \sum_{n=0}^{\infty} \left(\sum_{i=1}^{\infty} l_i \beta_i\right)^n \leqslant (1 - e^{-La}) \|u - z\|_* .$$

Since $1 - e^{-La} < 1$, then by the well-known Banach theorem we infer that equation (30) has a unique solution $h^*(t)$ in the interval $[0, a]$. This solution is the limit of the uniformly convergent sequence $\{z_n(t)\}$ of the continuous functions of the form

$$z_0(t) = 0, \quad t \in [0, a],$$

$$z_{n+1}(t) = Az_n(t), \quad t \in [0, a], \ n = 0, 1, \ldots,$$

and therefore it is continuous, non-negative and non-decreasing because $z_n(t)$ are such. This completes the proof of part (a).

We prove that the function $h^*(t)$, $t \in [0, a]$, satisfies equation (31). Indeed, we have

$$R(t) \stackrel{\mathrm{df}}{=} h^*(t) - \sum_{i=1}^{s} l_i h^*(\beta_i t) - k \int_0^t h^*(p)\, dp - g(t)$$

$$= h^*(t) - \frac{k}{s} \sum_{i=1}^{s} l_i \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \left(\prod_{r=0}^{n-1} l_{i_r}\right) \int_0^{t\beta_i \prod_{r=0}^{n-1} \beta_{i_r}} h^*(\tau)\, d\tau -$$

$$- \frac{1}{s} \sum_{i=1}^{s} l_i \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \left(\prod_{r=0}^{n-1} l_{i_r}\right) g\left(t\beta_i \prod_{r=0}^{n-1} \beta_{i_r}\right) -$$

$$-\frac{k^2}{s}\int_0^t \sum_{n=0}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)\int_0^{p\prod_{r=0}^{n-1}\beta_{i_r}} h^*(\tau)\,d\tau\,dp-$$

$$-\frac{k}{s}\int_0^t \sum_{n=0}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)g\Big(p\prod_{r=0}^{n-1}\beta_{i_r}\Big)dp-g(t).$$

Because $h^*(t)$, $t\in[0,a]$, is the unique solution of (30) we have

$$R(t) = h^*(t)-\frac{k}{s}\sum_{n=1}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)\int_0^{t\prod_{r=0}^{n-1}\beta_{i_r}} h^*(\tau)\,d\tau-$$

$$-\frac{k}{s}\sum_{i_0=1}^{n}\int_0^t h^*(\tau)\,d\tau+k\int_0^t h^*(\tau)\,d\tau-$$

$$-\frac{1}{s}\sum_{n=1}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)g\Big(t\prod_{r=0}^{n-1}\beta_{i_r}\Big)-\frac{1}{s}\sum_{i_0=1}^{s}g(t)-$$

$$-\frac{k^2}{s}\int_0^t \sum_{n=0}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)\int_0^{p\prod_{r=0}^{n-1}\beta_{i_r}} h^*(\tau)\,d\tau\,dp-$$

$$-\frac{k}{s}\int_0^t \sum_{n=0}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)g\Big(p\prod_{r=0}^{n-1}\beta_{i_r}\Big)dp.$$

Further, by changing the sum index, we get

$$R(t) = k\int_0^t\Big[h^*(p)-\frac{k}{s}\sum_{n=0}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)\int_0^{p\prod_{r=0}^{n-1}\beta_{i_r}} h^*(\tau)\,d\tau-$$

$$-\frac{1}{s}\sum_{n=0}^{\infty}\sum_{i_0=1}^{s}\cdots\sum_{i_n=1}^{s}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)g\Big(p\prod_{r=0}^{n-1}\beta_{i_r}\Big)\Big]dp\equiv 0;$$

thus $h^*(t)$, $t\in[0,a]$, is a solution of equation (31).

We prove that any measurable solution $u(t)$, $t\in[0,a]$, of equation (31) satisfying the condition $0\leqslant u(t)\leqslant h^*(t)$, $t\in[0,a]$, is a solution of equation (30).

Let $u_0(t)$, $t\in[0,a]$, be a measurable solution of equation (31) satisfying the condition $0\leqslant u_0(t)\leqslant h^*(t)$, $t\in[0,a]$. Put

$$\varphi_1(t) = k\int_0^t u_0(\tau)\,d\tau+g(t).$$

Now for $t \in [0, a]$ we have

$$(33) \qquad v(t) \overset{\mathrm{df}}{=} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} l_n^{i_0,\ldots,i_n}(t) \varphi_1\big(\beta_n^{i_0,\ldots,i_n}(t)\big)$$

$$= \frac{k}{s} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r}\Big) \int_0^{t \prod_{r=0}^{n-1} \beta_{i_r}} u_0(\tau)\, d\tau +$$

$$+ \frac{1}{s} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r}\Big) g\Big(t \prod_{r=0}^{n-1} \beta_{i_r}\Big) = Au_0$$

$$\leqslant \frac{ka}{s} \max_{0 \leqslant t \leqslant a} |u_0(t)| \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r} \beta_{i_r}\Big) + H(t)$$

$$= ka \max_{0 \leqslant t \leqslant a} |u_0(t)| \sum_{n=0}^{\infty} \Big(\sum_{i=1}^{s} l_i \beta_i\Big)^n + H(t) < \infty,$$

and from Lemma 4 it follows that the equation

$$(34) \quad u(t) = \sum_{i=1}^{s} l_i u(\beta_i t) + \varphi_1(t), \quad t \in [0, a], \text{ with } \varphi_1(t) = k \int_0^t u_0(\tau)\, d\tau + g(t)$$

has a unique solution in the class $0 \leqslant u(t) \leqslant Au_0$, $Au_0 \leqslant h^*(t)$, and this solution is the function $v(t) = Au_0$.

Further, we put

$$\varphi_2(t) = k \int_0^t h^*(\tau)\, d\tau + g(t).$$

It is obvious that equation (34) with $\varphi_2(t)$ instead of $\varphi_1(t)$ has also a unique solution in the class $0 \leqslant u(t) \leqslant Ah^* = h^*$.

Now from Lemma 5 it follows that the function $v(t) = Au_0$ is the unique solution of (34) in the class $0 \leqslant u(t) \leqslant h^*(t)$, $t \in [0, a]$.

Since $u_0(t)$ is also solution of (34) in the class $0 \leqslant u(t) \leqslant h^*(t)$, $t \in [0, a]$, we have $v(t) = u_0(t)$, $t \in [0, a]$. Hence $u_0(t)$, $t \in [0, a]$, is as olution of (30) and therefore it is continuous.

Since each measurable solution of (31) in the class $0 \leqslant u(t) \leqslant h^*(t)$, $t \in [0, a]$, is a solution of (30), the function $h^*(t)$, $t \in [0, a]$, is the unique solution of (30), and $h^*(t)$, $t \in [0, a]$, satisfies equation (31), then the function $h^*(t)$, $t \in [0, a]$, is the unique solution of (31). This completes the proof of part (b).

Now we prove that the function $u(t) \equiv 0$, $t \in [0, a]$, is the unique solution of the equation

$$(35) \qquad u(t) = \sum_{i=1}^{s} l_i u(\beta_i t) + k \int_0^t u(\tau)\, d\tau, \qquad t \in [0, a],$$

satisfying the condition $0 \leqslant u(t) \leqslant h^*(t)$, $t \in [0, a]$.

Let $u_0(t)$ be a measurable solution of (35) fulfilling this condition. Putting in Lemma 5

$$\varphi_1(t) = k\int_0^t u_0(\tau)d\tau, \qquad \varphi_2(t) = k\int_0^t u_0(\tau)d\tau + g(t), \qquad t\in[0, a],$$

we see that the equation

$$u(t) = \sum_{i=1}^{s} l_i u(\beta_i t) + \varphi_1(t), \qquad t\in[0, a]$$

has a unique solution in the class of functions $0 \leqslant u(t) \leqslant h^*(t)$, $t\in[0, a]$.

The uniqueness of the solution of equation (35) in the class of functions $0 \leqslant u(t) \leqslant h^*(t)$, $t\in[0, a]$, can be obtained by a similar argument to that used in proving (b). Hence we get $u_0(t) = 0$, $t\in[0, a]$.

Now (c) is implied by Remark 1.

Thus the proof of Lemma 7 is completed.

These considerations and Theorem 2 imply

THEOREM 8. *If Assumption* H_1 *is satisfied, and*

1° *conditions* (26) *and* (27) *are satisfied,*

2° $H(t) \overset{\mathrm{df}}{=} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \dots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r}\Big) g\Big(t\prod_{r=0}^{n-1}\beta_{i_r}\Big) < \infty, \qquad t\in[0, a],$

where $g(t) = \sup_{0\leqslant s\leqslant t} \|F(s, \Theta, \dots, \Theta)\|$, *and* $H(t)$ *is continuous for* $t\in[0, a]$,

3° $0 \leqslant \sum_{i=1}^{s} l_i\beta_i < 1,$

4° $0 \leqslant \beta_i \leqslant 1,\ i = 1, 2, \dots, s,$

5° $k = \sum_{i=1}^{r} k_i,$

hen there exists a unique and continuous solution $\bar{y}(t)$, $\bar{y}(0) = \Theta$, *of equation 2) in th e interval* $[0, a]$ *with the following properties:*

$$\|\bar{y}(t)\| \leqslant h^*(t), \qquad t\in[0, a],$$

$$\|\bar{y}(t) - y_n(t)\| \leqslant h_n(t), \qquad t\in[0, a],\ n = 0, 1, \dots,$$

where $h_0(t) = h^*(t)$, $t\in[0, a]$, $h^*(t)$ *is defined in Lemma* 7,

$$h_{n+1}(t) = k\int_0^t h_n(\tau)d\tau + \sum_{i=1}^{s} l_i h_n(\beta_i t), \qquad t\in[0, a],\ n = 0, 1, \dots$$

Further, Theorem 3, Lemma 3 and Remark 2 imply

THEOREM 9. *If the assumptions of Theorem* 8 (*except* 2°) *are satisfied, then equation* (2) *has at most one solution* $y(t)$, $t\in[0, a]$, *in the class* $\|y(t)\| \leqslant Lt$, $t\in[0, a]$, $L \geqslant 0$.

Proof. Since for any measurable function $0 \leqslant f(t) \leqslant 2Lt$, $t \in [0, a]$, $L \geqslant 0$, we have

$$\sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r}\Big) f\Big(t \prod_{r=0}^{n-1} \beta_{i_r}\Big) \leqslant 2Lt \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r}\beta_{i_r}\Big)$$
$$= 2Lt \sum_{n=0}^{\infty} \Big(\sum_{i=1}^{s} l_i \beta_i\Big)^n < \infty,$$

it follows from Lemma 7 that Assumption H_3 is fulfilled with $\|F(t, \Theta, \ldots, \Theta)\|$ replaced by $f(t)$. Now the assertion of Theorem 9 implies, by Remark 2, Lemma 3 with H_2 instead of H_3 and Theorem 3.

Theorem 5 implies the following

THEOREM 10. *If the assumptions of Theorem* 8 (*except* 2°), *are satisfied and if*

1° *the functions* $y^*(t)$ *and* $w^*(t)$, $t \in [0, a]$, *are solutions of equations* (2) *and* (15),

2° $$H(t) \overset{\text{df}}{=} \sum_{n=0}^{\infty} \sum_{i_0=1}^{s} \cdots \sum_{i_n=1}^{s} \Big(\prod_{r=0}^{n-1} l_{i_r}\Big) \psi\Big(t \prod_{r=0}^{n-1} \beta_{i_r}\Big) < \infty, \quad t \in [0, a],$$

where

$$\psi(t) = \max\{\sup_{0 \leqslant s \leqslant t} [\|y^*(s)\| + \|w^*(s)\|], \sup_{0 \leqslant s \leqslant t} v^*(s)\}, \quad t \in [0, a],$$

and $v^*(t)$ *is defined by condition* 2° *of Theorem* 4, *and* $H(t)$ *is continuous in* $[0, a]$,

then

(a) *there exists a continuous, non-negative and non-decreasing solution* $\tilde{z}(t)$, $t \in [0, a]$ *of the equation*

$$z(t) = k\int_0^t z(\tau)\,d\tau + \sum_{i=1}^{s} l_i z(\beta_i t) + \psi(t), \quad t \in [0, a],$$

(b) *the sequence* $\{\tilde{z}_n(t)\}$, $t \in [0, a]$,

$$\tilde{z}_0(t) = \tilde{z}(t), \quad t \in [0, a],$$

$$\tilde{z}_{n+1}(t) = k\int_0^t \tilde{z}_n(\tau)\,d\tau + \sum_{i=1}^{s} l_i \tilde{z}_n(\beta_i t) + \sup_{0 \leqslant s \leqslant t} v^*(s), \quad t \in [0, a],\ n = 0, 1, \ldots,$$

has the limit function $z^*(t)$ *in the interval* $[0, a]$, *and the function* $z^*(t)$ *is continuous, non-negative and non-decreasing*, $z^*(t) \leqslant \tilde{z}(t)$, $t \in [0, a]$,

(c) *the estimation*

$$\|y^*(t) - w^*(t)\| \leqslant z^*(t), \quad t \in [0, a]$$

holds true.

Remark 9. Condition 2° of Theorem 8 is satisfied if

(36) $$\|F(t, \Theta, \ldots, \Theta)\| \leqslant L_1 t, \quad t \in [0, a],\ L_1 \geqslant 0.$$

If we assume that the function F satisfies a Lipschitz condition with respect to t, then (36) is fulfilled.

Remark 10. If

$$\max_{1\leqslant i\leqslant s} l_i\beta_i < \frac{1}{s},$$

then condition 3° of Theorem 8 holds.

Remark 11. Equation (1) was considered in paper [4]. In this paper it is assumed that the function F satisfies a Lipschitz condition with respect to all variables. The functions $a_i(t)$ and $\beta_j(t)$ for $t\in[0, a]$, are of the form

$$a_1(t) = t, \quad a_{i+1}(t) = t - \Delta_i(t), \quad \Delta_i(t) \geqslant 0, \quad i = 1, 2, \ldots, s,$$
$$\beta_j(t) = t - \Delta_j(t), \quad \Delta_j(t) \geqslant 0, \quad j = 1, 2, \ldots, s.$$

The sufficient condition for the existence of a solution of (1) in some interval $[0, \varepsilon)$, $\varepsilon < a$ given in [4] is of the form

$$\max_{1\leqslant i\leqslant s} l_i(1-\mu_\xi^i) < \frac{1}{s}, \tag{37}$$

where

$$\mu_\xi^i = \inf_{0\leqslant x\leqslant \xi} \varliminf_{z\to x} \frac{\Delta_i(z)-\Delta_i(x)}{z-x} > 0, \quad \xi\in(0, a], \; i = 1, 2, \ldots, s.$$

We prove that under the assumptions of Theorem 1 [4] the assumptions of Theorem 8 are satisfied. From Remark 9 it follows that condition 2° is satisfied if (36) holds. Since

$$\Delta_i(t) \geqslant 0, \quad t\in[0, a], \quad i = 1, 2, \ldots, s,$$

we have $a_i(t) \leqslant t$, $t\in[0, a]$, $i = 1, 2, \ldots, s+1$. Defining

$$\lambda_i(t) = \begin{cases} 1 & \text{for } t = 0, \\ \dfrac{\Delta_i(t)}{t} & \text{for } 0 < t \leqslant a, \end{cases} \qquad i = 1, 2, \ldots, s$$

and putting

$$\gamma_i = \inf_{0\leqslant t\leqslant a} \lambda_i(t), \quad i = 1, 2, \ldots, s,$$

we see that

$$0 \leqslant \beta_i(t) = t - \Delta_i(t) \leqslant t(1-\gamma_i) \overset{\text{df}}{=} \beta_i t, \quad t\in[0, a], \; i = 1, 2, \ldots, s.$$

Now our condition has the form

$$\sum_{i=1}^{s} l_i(1-\gamma_i) < 1 \quad \text{or} \quad \max_{1\leqslant i\leqslant s} l_i(1-\gamma_i) < 1/s. \tag{38}$$

Since $\gamma_i \geqslant \mu_\xi^i$, $i = 1, 2, \ldots, s$, it follows that condition (37) implies (38). The following example proves that condition (38) is weaker than (37).

Example. We take in Remark 11

$$\Delta_i(t) = \sin t, \quad t \in [0, \tfrac{1}{3}\pi],\ i = 1, 2, \ldots, s.$$

By Remark 11 we have

$$\mu^i_{\pi/3} = \inf_{0 \leqslant x \leqslant \pi/3} \cos x = \tfrac{1}{2}, \quad i = 1, 2, \ldots, s,$$

$$\gamma_i = \inf_{0 \leqslant t \leqslant \pi/3} \lambda_i(t) = \frac{3\sqrt{3}}{2\pi}, \quad i = 1, 2, \ldots, s,$$

$$\beta_i = (1-\gamma_i) = \frac{2\pi - 3\sqrt{3}}{2\pi}, \quad i = 1, 2, \ldots, s.$$

Now conditions (37) and (38) are of the form

$$(37') \qquad \tfrac{1}{2} \max_{1 \leqslant i \leqslant s} l_i < 1/s,$$

and

$$(38') \qquad \frac{2\pi - 3\sqrt{3}}{2\pi} \sum_{i=1}^{s} l_i < 1 \quad \text{or} \quad \frac{2\pi - 3\sqrt{3}}{2\pi} \max_{1 \leqslant i \leqslant s} l_i < \frac{1}{s}.$$

Since

$$\frac{2\pi - 3\sqrt{3}}{2\pi} < \frac{1}{2},$$

Theorem 8 gives a better result than that given by Theorem 1 [4].

Remark 12. Note that our result can be applied to the equations with

$$\Delta_j(t) = \begin{cases} 0 & \text{for } t = 0, \\ t\left|\sin\frac{1}{t}\right| & \text{for } 0 < t \leqslant a, \end{cases} \quad j = 1, 2, \ldots, s,$$

but the result of paper [4] does not hold because in this case

$$\mu_\xi^i = -\infty$$

for any $\xi \in (0, a]$.

Remark 13. Assumption H_2 is fulfilled if

$$\sum_{i=1}^{r} \bar{k}_i \bar{a}_i + \sum_{j=1}^{s} \bar{l}_j < 1,$$

where $\bar{k}_i = \max k_i(t)$, $\bar{a}_i = \max a_i(t)$, $l_j = \max l_j(t)$, $t \in [0, a]$, $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$, and from Theorem 1 we get the result contained in Theorem 4 of paper [3].

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