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On positive solutions of equations with two middle terms

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Abstract. In this paper we prove that for any two eventually positive and bounded solutions of the nth-order, nonlinear, forced differential equation with two middle terms, their difference is oscillatory and tends to zero. In a more special case, we prove that the equation has at most one eventually bounded positive solution.

1. Introduction. This paper continues the study of the nonlinear delay equation of the type

$$(*) \qquad x^{(n)}+p(t)x^{(n-1)}+q(t)x^{(n-2)}+H\big(t,\, x(g(t))\big) = Q(t).$$

It is proven that under certain conditions for any two eventually positive solutions $u(t)$ and $v(t)$ of $(*)$, the difference $u(t)-v(t)$ must be oscillatory and tending to zero. Under different assumptions, if $q(t) \equiv 0$ and $g(t) = t$, we prove that the equation has at most one eventually positive solution. These results, with the assumptions somewhat parallel to the ones in [6] and [10], extend [3], [4], [5] and [8], where $p(t)$ and $q(t)$ were assumed to be zero and $g(t) = t$. Examples are given throughout the paper.

2. Preliminaries. We denote by R the real line and by R_+ the interval $[0, \infty)$. Let (E) denote an nth-order differential equation or inequality. By a solution of (E) we mean a function $x(t)$, $t \in [t_x, \infty) \subset R_+$, which is n times continuously differentiable and satisfies (E) on $[t_x, \infty)$. The number $t_x \geqslant 0$ depends on the particular solution $x(t)$ under consideration. We say that a property P holds *eventually*, or *for all large* t, if there exists $T \geqslant 0$ such that P holds for all $t \geqslant T$. We denote by $C^n(I)$ the space of all n times continuously differentiable functions $f\colon I \to R$. We write $C(I)$ instead of $C^0(I)$.

In the content of this paper we consider equation $(*)$ where n is an even integer, $n \geqslant 4$, and the functions satisfy the following conditions:

(A) $p \in C^1(R_+)$, $q \in C(R_+)$ where $p(t) \leqslant 0$, $q(t) \geqslant 0$ and $q(t)-p'(t)/2 \leqslant 0$;
(B) $g \in C(R_+)$ where $g(t) \leqslant t$ with $\lim_{t\to\infty} g(t) = +\infty$, and eventually increasing;

(C) H: $R_+ \times R \to R$ is continuous, $H(t, u) > 0$ for $u > 0$ and increasing in u;

(D) $H^*(t, u) \equiv \frac{\partial}{\partial u} H(t, u)$ is continuous, nonnegative and increasing for $u > 0$;

(E) the equation $u'' + p(t)u' + q(t)u = 0$ is disconjugate on R_+ (we recall that an nth-order differential equation is said to be *disconjugate on an interval* $I \subset R_+$ if no nontrivial solution of it has more than $n-1$ zeros on I);

(F) the equation

$$S^{(n)} + p(t)S^{(n-1)} + q(t)S^{(n-2)} = Q(t)$$

has a solution $S(t)$ such that $\liminf_{t\to\infty} S(t) = 0$;

(G) for any $t_0 \geqslant 0$ and any positive constant k we have

$$\int_{t_0}^{\infty} H(t, h)dt = +\infty.$$

We can note at this point that for example the equation

$$x^{(6)} - (1/t)x^{(5)} + 1/(2t^2)x^{(4)} + (1/t)\big(x(t-3)\big)^3 = 852t^{-7}$$

with $t \geqslant 1$ satisfies all of the above conditions. We might note that here $S(t) = 1/t$.

From Kiguradge's paper [7] we quote

LEMMA A. *Let* $x \in C^n[t_0, \infty)$ *be given with* $t_0 \geqslant 0$. *Assume further that* $x^{(n)}(t)x(t) \leqslant 0$ *for* $t \geqslant t_0$. *Then there exist* $T \geqslant t_0$ *and an integer* m, $0 \leqslant m \leqslant n-1$, *such that for* $t \geqslant T$ *we have*

$$x^{(k)}(t)x(t) \geqslant 0, \qquad k = 0, 1, \ldots, m;$$

$$(-1)^{n+k} x^{(k)}(t)x(t) \leqslant 0, \qquad k = m+1, m+2, \ldots, n.$$

The integer m *is even if* n *is odd and odd if* n *is even.*

Next, from Kosmala's paper [9] we quote

LEMMA B. *If* $x(t)$ *is an eventually positive solution of* (∗), *then either* $\big(x(t)-S(t)\big)^{(j)} > 0$, $t \geqslant T$, $j = 0, 1, \ldots, n-1$, *for some* $T \geqslant 0$, *or* $\big(x(t) - S(t)\big)^{(n-2)} < 0$ *eventually.*

3. Main results

THEOREM 1. *If* $x(t)$ *is an eventually bounded positive solution of the equation* (∗), *then*

$$(-1)^i\big(x(t)-S(t)\big)^{(i)} < 0 \quad \textit{with} \quad \lim_{t\to\infty} \big(x(t)-S(t)\big)^{(i)} = 0$$

for $i = 0, 1, \ldots, n-2$, $t \geqslant T$ *for some* $T \geqslant 0$.

Proof. Let $x(t)$ be a bounded positive solution of $(*)$ for $t \geqslant t_0 \geqslant 0$. Set $u(t) = x(t) - S(t)$, $t \geqslant t_0$. Then $(*)$ becomes

$$(3.1) \qquad u^{(n)}(t) + p(t)u^{(n-1)}(t) + q(t)u^{(n-2)}(t) + H\big(t, u(g(t)) + S(g(t))\big) = 0.$$

By Lemma B there exists $t_1 \geqslant t_0$ such that $u^{(n-2)}(t) < 0$ or $u^{(j)}(t) > 0$ for $j = 0, 1, \ldots, n-1$. Thus $u(t)$ must be of one sign eventually. We observe that if all the derivatives of $u(t)$ are positive then $u(t) \to +\infty$ as $t \to \infty$. Since $S(t)$ is bounded, we would have $\lim_{t\to\infty} x(t) = +\infty$ which is a contradiction to the boundedness of $x(t)$. Hence we can only have the case that $u^{(n-2)}(t) < 0$ for $t \geqslant t_1$. To show $u(t)$ is negative eventually, we assume the contrary for $t \geqslant t_2 \geqslant t_1$. Since n is even, we know that $u'(t) > 0$ for $t \geqslant t_3 \geqslant t_2$, where t_3 is large enough so that $|S(t)| < \varepsilon$ for $t \geqslant t_3$ and where $0 < \varepsilon < u(t_3)/2$. Since $g(t) \to +\infty$, there exists $t_4 \geqslant t_3$ such that $g(t) > t_3$ for every $t \geqslant t_4$. Consequently we have that $u(g(t)) + S(g(t)) > u(t_3) - \varepsilon > 0$ for $t \geqslant t_4$.

Next we integrate (3.1) from t_4 to t, $t \geqslant t_4$, to obtain

$$\begin{aligned} u^{(n-1)}(t) + p(t)u^{(n-2)}(t) &= u^{(n-1)}(t_4) + p(t_4)u^{(n-2)}(t_4) \\ &\quad - \int_{t_4}^{t} (q(s) - p'(s))u^{(n-2)}(s)ds - \int_{t_4}^{t} H\big(s, u(g(s)) + S(g(s))\big)ds \\ &= M - f(t) - \int_{t_4}^{t} H\big(s, u(g(s)) + S(g(s))\big)ds, \end{aligned}$$

where M is a constant and $f(t)$ is the first integral above. Let $z(t) = u^{(n-2)}(t)$ and observe that $z(t)$ satisfies a first-order linear equation. Thus

$$\begin{aligned} z(t) = \exp\Big[-\int_{t_4}^{t} p(s)ds\Big] \Big\{z(t_4) + \int_{t_4}^{t} \Big[\exp \int_{t_4}^{s} p(r)dr\Big] \\ \Big[M - f(s) - \int_{t_4}^{s} H\big(r, u(g(r)) + S(g(r))\big)dr\Big] ds\Big\} \end{aligned}$$

Since $u(g(t)) + S(g(t)) > u(t_3) - \varepsilon = k > 0$ and $f(t)$ is positive, we have that

$$z(t) \leqslant \int_{t_4}^{t} \exp\Big[-\int_{s}^{t} p(r)dr\Big]\Big[M - \int_{t_4}^{s} H(r, k)dr\Big]ds.$$

Note that by condition (G), there is an s_0 such that

$$\int_{t_4}^{s_0} H(r, k)dr \geqslant M + 1.$$

Thus we have that

$$z(t) \leqslant \int_{t_4}^{t} \Big[\exp\Big[-\int_{s}^{t} p(r)dr\Big]\Big](M - (M+1))ds = -\int_{t_4}^{t} \exp\Big[-\int_{s}^{t} p(r)dr\Big]ds \leqslant -\int_{t_4}^{t} 1 ds.$$

Therefore we have that $\lim_{t\to\infty} z(t) = -\infty$ and so $u^{(n-2)}(t) \to -\infty$. This is a contradiction to the positiveness of $u(t)$. Thus we can conclude that there exists $t_5 \geqslant t_4$ so that $u^{(n-2)}(t) < 0$ and $u(t) < 0$ for $t \geqslant t_5$. Therefore, there exists an even integer m, $0 \leqslant m < n-2$, such that

$$u^{(i)}(t) < 0, \qquad i = 0, 1, \ldots, m,$$

$$(-1)^i u^{(i)}(t) < 0, \qquad i = m+1, \ldots, n-2$$

for $t \geqslant$ (some) T. If $m \neq 0$, then $u''(t) < 0$ and $u'(t) < 0$ for $t \geqslant T$. Therefore

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \big(x(t) - S(t)\big) = -\infty.$$

Thus $\lim_{t\to\infty} S(t) = +\infty$ which is a contradiction to assumption (F). So $m = 0$, which implies that $\lim_{t\to\infty} (x(t)-S(t))^{(i)} = 0$ for $i = 1, 2, \ldots, n-2$. To show that $\lim_{t\to\infty} u(t) = 0$, we first observe that $\lim_{t\to\infty} u(t) \leqslant 0$ since $u(t) < 0$. Now, if we suppose that $\lim_{t\to\infty} u(t) < 0$ and recall that $\lim_{t\to\infty} \inf S(t) = 0$, we can find a sequence $\{\bar{t}_n\}_{n=1}^{\infty}$ such that $\bar{t}_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} S(\bar{t}_n) = 0$, and so $\lim_{n\to\infty} u(\bar{t}_n) = \lambda < 0$ which is a contradiction to the fact that $x(t)$ is positive. Hence $\lim_{t\to\infty} u(t) = 0$, which completes the proof.

We would like to note that Lemma B as well as Theorem 1 hold true even if the equation in condition (E) is replaced by

$$u'' + \big(q(t) - p'(t)/2\big)u = 0.$$

For more on this subject we refer the reader to p. 246 in [9].

THEOREM 2. *If $u(t)$ and $v(t)$ are two eventually bounded positive solutions of equation $(*)$, then the difference $u(t)-v(t)$ must be oscillatory and tending to zero, provided that for $t_0 \geqslant 0$ we have*

$$\int_{t_0}^{\infty} H^*(t, 0)\,dt = +\infty. \tag{3.2}$$

By oscillatory function we mean a function which has an unbounded set of zeros.

Proof. Let $u(t)$ and $v(t)$ be two bounded and positive solutions of the equation $(*)$ for $t \geqslant t_0 \geqslant 0$, and let $w(t) = u(t) - v(t)$ for $t \geqslant t_0$. Therefore $w(t)$ is bounded. From $(*)$ now we can obtain

$$w^{(n)}(t) + p(t)\,w^{(n-1)}(t) + q(t)\,w^{(n-2)}(t) + H\big(t, u(g(t))\big) - H\big(t, v(g(t))\big) = 0. \tag{3.3}$$

To show that $w(t)$ is oscillatory we will assume that $w(t)$ is eventually positive. The negative case follows the similar steps.

So, let us suppose that $w(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Since $g(t) \to +\infty$ as $t \to \infty$, there exists $t_2 \geqslant t_1$ such that $g(t) \geqslant t_1$ and is increasing for every $t \geqslant t_2$. Thus from condition (C) we know that

$$H(t, u(g(t))) - H(t, v(g(t))) > 0, \qquad t \geqslant t_2,$$

and so from (3.3) we have

$$z''(t) + p(t)z'(t) + q(t)z(t) < 0 \tag{3.4}$$

where $z(t) = w^{(n-2)}(t)$, $t \geqslant t_2$. We are going to show that $z(t) < 0$ eventually.

First, we will suppose that $z(t)$ is oscillatory. Let $z(t_3) = z(t_4) = 0$ for $t_4 > t_3 \geqslant t_2$ and $z(t) \neq 0$ for $t \in (t_3, t_4)$. To show that in this situation $z(t)$ must be positive, we assume the contrary is true. So, letting $y(t) = -z(t) > 0$, we obtain from (3.4) that

$$y''(t) + p(t)y'(t) + q(t)y(t) > 0, \qquad t \in (t_3, t_4).$$

Now, applying Theorem 3.1 of Jackson and Schrader [2], we find that there exists a solution $r(t)$ of

$$r''(t) + p(t)r' + q(t)r = 0$$

with $r(t_3) = r(t_4) = 0$ and $0 < y(t) \leqslant r(t)$, $t \in (t_3, t_4)$. This is a contradiction to condition (E). Thus we can conclude that $w^{(n-2)}(t) < 0$ or $w^{(n-2)}(t) \geqslant 0$ eventually.

Next we observe that $w^{(n-2)}(t) \geqslant 0$ implies that $w^{(n-1)}(t)$ must be positive. To verify this we proceed as follows. Let $w^{(n-1)}(t_5) = 0$ and $w^{(n-2)}(t) \geqslant 0$ for $t \geqslant t_5 \geqslant t_2$. Then (3.3) gives

$$w^{(n)}(t_5) = -q(t_5)w^{(n-2)}(t_5) - \big(H(t_5, u(g(t_5))) - H(t_5, v(g(t_5)))\big) < 0.$$

It follows that $w^{(n-1)}(t)$ is decreasing at each one of its zeros. This implies that $w^{(n-1)}(t) < 0$ for all $t > t_5$. So, if $w^{(n-1)}(t)$ has one zero t_5, it must be negative to the right of t_5. Thus it cannot oscillate. Naturally, $w^{(n-1)}(t)$ cannot be eventually negative, because, if it were, (3.3) would give $w^{(n)}(t) < 0$ eventually. This contradicts the positiveness of $w(t)$. Therefore we must have that $w^{(n-1)}(t) > 0$ eventually.

We can now observe that $w^{(n-2)}(t) \geqslant 0$ together with $w^{(n-1)}(t) > 0$ for all large t will imply that $w(t)$ is unbounded which is a contradiction. Hence, we conclude that $w^{(n-2)}(t)$ must be negative for $t \geqslant t_6 \geqslant t_2$.

Since n is even, from Lemma A we know that $w'(t) > 0$ for $t \geqslant t_7 \geqslant t_6$. By the Mean Value Theorem we know that

$$\big(H(t, u(g(t))) - H(t, v(g(t)))\big)/\big(u(g(t)) - v(g(t))\big) = H^*(t, \lambda(t)),$$

$t \geqslant t_7$ where $\lambda(t)$ is a continuous function lying between $u(g(t))$ and $v(g(t))$, which are both positive. Thus for $t \geqslant t_7$ we can write

$$\begin{aligned} H(t, u(g(t))) - H(t, v(g(t))) &= H^*(t, \lambda(t))\,w(g(t)) \\ &\geqslant H^*(t, u(g(t)))\,w(t_1) \geqslant kH^*(t, 0), \end{aligned}$$

where $k = w(t_1)$ is a constant. Therefore, integrating (3.3) from t_7 to $t \geqslant t_7$, we obtain

$$w^{(n-1)}(t)+p(t)w^{(n-2)}(t) = w^{(n-1)}(t_7)+p(t_7)w^{(n-2)}(t_7)$$
$$-\int_{t_7}^{t}(q(s)-p'(s))w^{(n-2)}(s)ds-\int_{t_7}^{t}\big(H(s, u(g(s)))-H(s, v(g(s)))\big)ds$$
$$= M-f(t)-\int_{t_7}^{t}\big(H(s, u(g(s)))-H(s, v(g(s)))\big)ds,$$

where M is a constant and $f(t)$ is the first integral. If $z(t) = w^{(n-2)}(t)$, we have as in Theorem 1 that

(3.5) $z(t)$

$$= e^{-\int_{t_7}^{t} p(s)ds}\Big[z(t_7)+\int_{t_7}^{t} e^{\int_{t_7}^{t} p(r)dr}\Big[M-f(s)-\int_{t_7}^{s}\big(H(r, u(g(r)))-H(r, v(g(r)))\big)dr\Big]\, ds\Big]$$
$$\leqslant \int_{t_7}^{t}\big[\exp-\int_{t_7}^{t} p(s)ds\big]\big[M-\int_{t_7}^{s} kH^*(r, 0)dr\big]ds.$$

From condition (3.2) we know that there exists s_0 large enough so that

$$M-\int_{t_7}^{s_0} kH^*(r, 0)dr < 0,$$

and since $\exp\big(-\int_{t_7}^{t} p(s)ds\big) \geqslant 1$ we conclude that the expression in line (3.5) tends to $-\infty$ and thus $\lim_{t\to\infty} z(t) = -\infty$. This implies that $\lim_{t\to\infty} w^{(n-2)}(t) = -\infty$ which is a contradiction to the positiveness of $w(t)$. It follows that $w(t)$ is oscillatory. Moreover, since from Theorem 1 we have that $0 < u(t) < S(t)$ and $0 < v(t) < S(t)$ with $\lim_{t\to\infty}(u(t)-S(t)) = 0 = \lim_{t\to\infty}(v(t)-S(t))$, it is clear that $\lim_{t\to\infty} w(t) = 0$, and so the proof is complete.

At this point we would like to note that assumption (G) is not equivalent to condition (3.2). For example, the function

$$H(t, u) = (e^{tu}-1)/(t^3+1)$$

satisfies assumptions (C), (D), and (G) but not (3.2). However,

$$H(t, u) = e^{ue^{-t}}-1+e^{-t}(1-1/(1+ue^{t}))$$

satisfies (3.2), assumptions (C), (D) but not (G).

THEOREM 3. *If we assume that* $S(t)$ *is bounded and that*

$$\int_{t_0}^{\infty} t^{n-1}H^*(t, k)dt < +\infty \tag{3.6}$$

for any $k > 0$ and $t_0 \geqslant 0$ constants, then equation $(*)$ has at most one eventually bounded positive solution, provided that $q(t) \equiv 0$ and $g(t) = t$.

Proof. Let $u(t)$ and $v(t)$ be two bounded positive solutions of $(*)$ for $t \geqslant t_0 \geqslant 0$, and let $w(t) = u(t) - v(t)$. Therefore, $w(t)$ is bounded and tends to zero (follows from Theorem 1). Since $w(t)$ and $S(t)$ are bounded and (3.6) is satisfied, we can find $t_1 \geqslant t_0$ such that

$$|w(t_1)| = \sup_{t\in[t_1,\infty)} |w(t)|$$

and

$$\int_{t_1}^{\infty} (t - t_1 + 1)^{n-1} H^*(t, k)dt \leqslant 1 \tag{3.7}$$

for any $k > 0$ constant, in particular the one where $|S(t)| \leqslant k$ when $t \geqslant t_1$.

Next we write $(*)$ as

$$w^{(n)}(t) + p(t)w^{(n-1)}(t) + H\big(t, u(t)\big) - H\big(t, v(t)\big) = 0. \tag{3.8}$$

But, by the Mean Value Theorem, as in the proof of Theorem 2, there exists $\lambda(t)$, a continuous function lying between $u(t)$ and $v(t)$ such that

$$H\big(t, u(t)\big) - H\big(t, v(t)\big) = H^*\big(t, \lambda(t)\big)w(t).$$

Thus (3.8) becomes

$$w^{(n)}(t) + p(t)w^{(n-1)}(t) + H^*\big(t, \lambda(t)\big)w(t) = 0,$$

which can be written as

$$\frac{d}{dt}\Big[w^{(n-1)}(t)\big(\exp\int_{t_1}^{t} p(s)ds\big)\Big] = -H^*\big(t, \lambda(t)\big)w(t)\exp\int_{t_1}^{t} p(s)ds$$

which yields

$$w^{(n-1)}(t)\exp\int_{t_1}^{t} p(s)ds - w^{(n-1)}(t_1) = -\int_{t_1}^{t} H^*\big(u, \lambda(u)\big)w(u)\big(\exp\int_{t_1}^{u} p(s)ds\big)du.$$

Next we observe that $\lim_{t\to\infty} w^{(n-1)}(t) = 0$, for otherwise we get a contradiction to the fact that $\lim_{t\to\infty} w(t) = 0$. Hence, by taking limits of the above expression we obtain

$$w^{(n-1)}(t_1) = \int_{t_1}^{\infty} H^*\big(u, \lambda(u)\big)w(u)\big(\exp\int_{t_1}^{u} p(s)ds\big)du. \tag{3.9}$$

Since we can replace t_1 by any t where $t \geqslant t_1$, we have

$$w^{(n-1)}(t) = \int_{t}^{\infty} R(u)du, \qquad t \geqslant t_1,$$

where $R(u)$ is the integrand in (3.9). Next we integrate above from t_1 to t, $t \geqslant t_1$, using the integration by parts, to get

$$\begin{aligned} w^{(n-2)}(t) &= w^{(n-2)}(t_1) + \int_{t_1}^{t}\int_{s}^{\infty} R(u)\,du\,ds \\ &= w^{(n-2)}(t_1) + t\int_{t}^{\infty} R(u)\,du - t_1\int_{t_1}^{\infty} R(u)\,du + \int_{t_1}^{t} sR(s)\,ds \\ &= w^{(n-2)}(t_1) + \int_{t}^{\infty} (t-s)\,R(s)\,ds + \int_{t_1}^{\infty} (s-t_1)\,R(s)\,ds. \end{aligned}$$

Again we take limits as $t \to \infty$ to obtain

$$0 = w^{(n-2)}(t_1) + \int_{t_1}^{\infty} (s-t_1)\,R(s)\,ds.$$

As before we replace t_1 by t, $t \geqslant t_1$, to get

$$w^{(n-2)}(t) = \int_{t}^{\infty} (t-s)\,R(s)\,ds, \qquad t \geqslant t_1.$$

Thus, we can obtain expressions

$$w^{(m)}(t) = \int_{t}^{\infty} \left((t-s)^{n-m-1}/(n-m-1)!\right) R(s)\,ds, \qquad m = 0, 1, 2, \ldots, n-1,$$

for $t \geqslant t_1$.

From the above expression we have that

$$\begin{aligned} |w(t)| &\leqslant \int_{t}^{\infty} |(t-s)^{n-1} R(s)|\,ds \\ &\leqslant \int_{t_1}^{\infty} |(s-t)^{n-1} R(s)|\,ds \leqslant \int_{t_1}^{\infty} (s-t_1+1)^{n-1} |R(s)|\,ds \\ &= \int_{t_1}^{\infty} (s-t_1+1)^{n-1} \left(\exp\int_{t_1}^{s} p(v)\,dv\right) |H^*(s, \lambda(s))|\,|w(s)|\,ds. \end{aligned}$$

Hence, we have

$$\begin{aligned} |w(t_1)| &\leqslant |w(t_1)| \int_{t_1}^{\infty} (t-t_1+1)^{n-1} \left(\exp\int_{t_1}^{t} p(s)\,ds\right) |H^*(t, \lambda(t))|\,dt \\ &\leqslant |w(t_1)| \int_{t_1}^{\infty} (t-t_1+1)^{n-1} |H^*(t, k)|\,dt, \end{aligned}$$

which is true because $\exp\int_{t_1}^{t} p(s)\,ds \leqslant 1$ and $\lambda(t) < S(t) \leqslant k$ since $\lambda(t)$ is between $u(t)$ and $v(t)$ and they both are less than $S(t)$ by Theorem 1. Therefore in view of

(3.7) we have that $|w(t_1)| \leqslant |w(t_1)|$, which is true only if $\sup_{t\in[t_1,\infty)} |w(t)| = 0$. This proves the result.

We can observe that assumption (G) is not equivalent to condition (3.6). For example the function $H(t, u) = (e^{tu}-1)/(t^3+1)$ satisfies assumptions (C), (D) and (G) but not (3.6), whereas the function $H(t, u) = -1+\exp(ue^{-t})$ satisfies (3.6), assumptions (C) and (D) but not (G). Furthermore observe that the integral in (3.2) is that of (3.6) with $n = 1$ and $k = 0$, it is clear that these conditions are distinct. Further, since $H^*(t, u)$ is increasing in u, it follows that $H^*(t, 0) < H^*(t, k)$ for $k > 0$. Hence, if (3.2) is satisfied, then (3.6) cannot hold, and conversely, if (3.6) is satisfied, then (3.2) must fail. As respective examples we have the functions $H(t, u) = \exp(tu)-1$ and $H(t, u) = \exp(ue^{-t})$. Note also that the function $H(t, u) = \exp(ue^{-t})$ satisfies (3.6) and assumptions (C), (D) and (G).

In contrast, if $H(t, u) = f(t)g(u)$, then assumption (G) and (3.2) are equivalent since assumption (G) becomes

$$g(k)\int_{t_0}^{\infty} f(u)du = +\infty,$$

and (3.2) is changed to

$$g'(0)\int_{t_0}^{\infty} f(u)du = +\infty.$$

Here we should note that $H^*(t, u)$ being nonnegative and increasing with respect to u $(u > 0)$ forces $g'(0) \neq 0$. We also see that (3.6) is reduced to

$$g'(k)\int_{t_0}^{\infty} t^{n-1} f(u)du < +\infty.$$

Thus we see that in this situation assumption (G) precludes (3.6).

To conclude, we would like to point out that one can verify the result which states that: if the equation

$$x^{(n)}+p(t)x^{(n-1)}+H(t, x(g(t))) < 0 \tag{3.10}$$

has an eventually positive solution, then so does the equality

$$x^{(n)}+p(t)x^{(n-1)}+H(t, x(g(t))) = 0. \tag{3.11}$$

(Papers [1], [3] and [8] can help the reader in verifying this statement.) Now, if in the above theorems we were to assume that equation (3.11) is oscillatory, the results would be true for all the solutions, not just the bounded ones. For observe that, for example, in Theorem 1 in the case where $u^{(n-2)}(t)$ is positive we had a contradiction due to the boundedness. However, if we were to note that

$$u^{(n)}(t)+p(t)u^{(n-1)}(t)+H(t, u(g(t))+S(g(t))) \leqslant -q(t)u^{(n-2)}(t) \leqslant 0,$$

and thus

$$u^{(n)}(t)+p(t)u^{(n-1)}(t)+H\big(t,\ u(g(t))-\varepsilon\big) < 0,$$

we can come up with the positive solution $v(t) = u(t)-\varepsilon$ to inequality (3.10). Thus (3.11) has a positive solution which yields a contradiction. In order to obtain the conditions for the oscillation of (3.11), we refer the reader to [10].

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