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On regular $\mathscr{K}'\{M_p\}$-distributions

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Abstract. The characterizations of regular tempered distributions given by Z. Szmydt [6] are generalized to the characterizations of regular $\mathscr{K}'\{M_p\}$-distributions, where M_p is a sequence of functions which satisfies suitable conditions. For a class of these spaces a new characterization of regular $\mathscr{K}'\{M_p\}$-distributions is given which implies, in particular, a new characterization of regular tempered distributions.

Basic notions. Spaces of $\mathscr{K}'\{M_p\}$-type are introduced and investigated in [1]. Here we shall quote the definition of these spaces and some of their properties.

Let $x \mapsto M_p(x)$, $p \in \boldsymbol{N}$, be real-valued functions defined on $\boldsymbol{R}$ ($\boldsymbol{N}$ is the set of natural numbers and $\boldsymbol{R}$ the set of real numbers) which satisfy the following conditions:

(1) $$1 \leqslant M_p(x) \leqslant M_{p+1}(x), \quad p \in \boldsymbol{N},\ x \in \boldsymbol{R};$$

(2) For any fixed $x \in \boldsymbol{R}$ there are only two possible cases:
$$M_p(x) = \infty \quad \text{for all } p \quad \text{or} \quad M_p(x) < \infty \quad \text{for all } p;$$

(3) M_p is continuous with respect to x at any x, where the function is finite. The set of points x for which $M_p(x) = \infty$ is contained in the interval $(-\alpha, \alpha)$, $\alpha < \infty$;

$(\mathrm{N})_1$ (¹) For every $p \in \boldsymbol{N}$, there is $p' \in \boldsymbol{N}$, $p' > p$, such that $M_p M_{p'}^{-1} \in L^1$ (²).

Given such a system of functions M_p, we denote by $\mathscr{K}\{M_p\}$ the set of all infinitely differentiable (smooth) functions $x \mapsto \varphi(x)$, $x \in \boldsymbol{R}$, for which the countably many norms
$$\|\varphi\|_p := \sup\{M_p(x)|\varphi^{(q)}(x)|;\ q \leqslant p,\ x \in \boldsymbol{R}\}$$
are all finite.

(¹) This condition is a part of condition (N) in [1], p. 111.

(²) L^1 is the space of Lebesgue integrable function on $\boldsymbol{R}$. If $M_p(x) = M_{p'}(x) = \infty$ we set $M_p(x) M_{p'}^{-1}(x) = 0$.

It was shown in [1], p. 28, that $\mathcal{K}\{M_p\}$ is a complete countably normed space.

The space of all continuous linear functionals on $\mathcal{K}\{M_p\}$ is denoted by $\mathcal{K}'\{M_p\}$.

If all the functions $M_p(x)$, $p \in \boldsymbol{N}$, are finite on $\boldsymbol{R}$ and if the following condition holds:

(P) For every $p \in \boldsymbol{N}$ and $\varepsilon > 0$ there are $p' \in \boldsymbol{N}$, $p' > p$, and $k > 0$ such that if $|x| > k$ or $M_p(x) > k$, then $M_p(x) < \varepsilon M_{p'}(x)$(3),

then ([1], Chapter II 2.5) the Schwartz space $\mathcal{S}$ is a dense subspace of $\mathcal{K}\{M_p\}$ and the identity mapping $i\colon \mathcal{S} \to \mathcal{K}\{M_p\}$ is continuous. In this case the space $\mathcal{K}'\{M_p\}$ can be identified with a subspace of the space $\mathcal{S}'$ of Schwartz distributions.

Elements of $\mathcal{K}'\{M_p\}$ will be called *$\mathcal{K}'\{M_p\}$-distributions.*

Clearly, any subsequence of a sequence M_p generates the same space of test functions.

We denote by L^1_{loc} the space of locally Lebesgue integrable functions and identify every complex-valued function $u \in L^1_{\mathrm{loc}}$ with the distribution defined by

$$\langle u, \varphi \rangle = \int_{\boldsymbol{R}} u(x)\varphi(x)\,dx, \qquad \varphi \in C_0^\infty.$$

Following [6], we say that a $\mathcal{K}'\{M_p\}$-distribution u is regular if there exists a function $u \in L^1_{\mathrm{loc}}$ such that $u\varphi \in L^1$ for every $\varphi \in \mathcal{K}\{M_p\}$ and

$$\langle u, \varphi \rangle = \int_{\boldsymbol{R}} u(x)\varphi(x)\,dx.$$

The set of all regular $\mathcal{K}'\{M_p\}$-distributions will be denoted by $(\mathcal{K}'\{M_p\})_r$.

Remark. In this paper, functions and distributions are defined on $\boldsymbol{R}$. With suitable technical changes all results of this paper can be formulated with $\boldsymbol{R}^n$ instead of $\boldsymbol{R}$.

Function which generates a non-regular $\mathcal{K}'\{M_p\}$-distribution. We assume that a sequence M_p satisfies conditions (1)–(3) and $(\mathrm{N})_1$.

Observe that those conditions imply that constant functions define regular $\mathcal{K}'\{M_p\}$-distributions.

LEMMA 1. *Let r_k, $k \in \boldsymbol{N}$, be a sequence of positive integers such that $r_1 > \alpha + 2$(4), $r_{k+1} > r_k + 3$, $k \in \boldsymbol{N}$, and let $a_k = \max\{M_k(x);\ x \in [r_k - 1, r_k + 2]\}$. If $\Omega \in C_0^\infty$, $0 \leqslant \Omega \leqslant 1$, $\operatorname{supp} \Omega \subset [-1, 1]$, and $\omega \in C_0^\infty$, $0 \leqslant \omega \leqslant 1$,*

(3) If M_p is finite on $\boldsymbol{R}$, then (P) is equivalent to $\lim_{|x| \to \infty} M_p(x)/M_{p'}(x) = 0$. The last condition is stated in Theorem 3 as $(\mathrm{N})_2$.

(4) See (3).

$\operatorname{supp}\omega \subset [0, 1]$, $\int_0^1 \omega(t)\,dt = 1$, *then the function*

$$\psi(x) = \sum_{k=1}^{\infty} \frac{1}{a_k}\big(\Omega(t-r_k) * \omega(t)\big)(x)$$

is a non-negative function from $\mathscr{K}\{M_p\}$ *with unbounded support, where*

$$\big(\Omega(t-r_k) * \omega(t)\big)(x) = \int_0^1 \Omega(x-t-r_k)\,\omega(t)\,dt.$$

Proof. We only have to prove that $\psi \in \mathscr{K}\{M_p\}$. For given $p_0 \in N$ and $j \leqslant p_0$

$$\sup_{x\in \boldsymbol{R}} \{|\psi^{(j)}(x)|\ M_{p_0}(x)\}$$
$$\leqslant \sup_{k\in N}\left\{\frac{1}{a_k}\cdot \max\{M_{p_0}(x);\ x\in [r_k-1,\ r_k+2]\} \int_0^1 |\omega^{(j)}(t)|\,dt\right\}.$$

Since $\max\{M_{p_0}(x);\ x\in[r_k-1,\ r_k+2]\} \leqslant a_k$ for $k \geqslant p_0$, we obtain

$$\sup_{x\in \boldsymbol{R}} \{|\psi^{(j)}(x)|\, M_{p_0}(x)\} < \infty, \quad j \leqslant p_0.$$

Let f be a non-negative continuous function equal to zero for $x \leqslant \alpha$ such that

$$f(x)\psi(x) \notin L^1,$$

where ψ is from Lemma 1. We define a function F by

$$F(x) = \int_0^x f(t)\,dt, \quad x \in \boldsymbol{R}.$$

THEOREM 1. *The function*

$$x \mapsto A(x) = \frac{d}{dx} e^{iF(x)}, \quad x \in \boldsymbol{R},$$

defines a $\mathscr{K}'\{M_p\}$*-distribution by*

$$\varphi \mapsto -\int e^{iF(x)}\varphi'(x)\,dx, \quad \varphi \in \mathscr{K}\{M_p\}.$$

This $\mathscr{K}'\{M_p\}$*-distribution is not regular.*

Proof. Let ψ be a function from Lemma 1. Since $|A(x)| = f(x)$ we have $A\psi \notin L^1$.

If we suppose that $\lim_{|x|\to\infty} M_p(x) = \infty$, $p \in N$(5), then one can prove that

$$\lim_{k\to\infty} \int_{-k}^{k} if(x)e^{iF(x)}\varphi(x)\,dx = -\int_{\boldsymbol{R}} e^{iF(x)}\varphi'(x)\,dx.$$

(5) In this case if $\varphi \in \mathscr{K}'\{M_p\}$ and $|x| \to \infty$, then $\varphi(x) \to 0$.

Characterizations of the space $(\mathscr{K}'\{M_p\})_r$. We shall give characterizations of the space $(\mathscr{K}'\{M_p\})_r$ when the sequence M_p satisfies, besides conditions (1)–(3) and $(\mathrm{N})_1$, the following ones:

(4) For every $p \in N$ there is Y_p such that M_p is non-decreasing on (Y_p, ∞) and non-increasing on $(-\infty, -Y_p)$;

(5) For every $p \in N$ there are $p' \in N$ and $X_p > 0$ such that $M_p(x+1) \leqslant M_{p'}(x)$ for $x > X_p$ and $M_p(x-1) \leqslant M_{p'}(x)$ for $x < -X_p$.

We shall always assume that $X_p \geqslant Y_p$, $p \in N$.

First, we shall prove Lemmas 2 and 3 which are analogous to Lemmas 1 and 2 from [6]. From these lemmas characterizations of the space $(\mathscr{K}'\{M_p\})_r$ will directly follow. These characterizations will be given in Theorem 2 and this theorem is analogous to Theorem 1 from [6] (and a generalization of this theorem).

Observe that condition (5) implies the existence of a sequence p_k and a sequence X_{p_k} such that

$$(5') \qquad \begin{aligned} &M_{p_k}(x+1) \leqslant M_{p_{k+1}}(x) \quad \text{for } x \in (X_{p_k}, \infty) \quad \text{and} \\ &M_{p_k}(x-1) \leqslant M_{p_{k+1}}(x) \quad \text{for } x \in (-\infty, -X_{p_k}). \end{aligned}$$

LEMMA 2. *Let M_p be a sequence of functions which satisfies the above conditions and let r_k be a sequence of positive numbers such that $r_1 > \alpha + 1$, $r_{k+1} > r_k + 1$ and $r_k > X_{p_k}$, $k \in N$* (6).

There exists a smooth non-negative function $x \mapsto \gamma(x)$, $x \in \boldsymbol{R}$, such that:

(a) $\gamma \in \mathscr{K}\{M_p\}$;

(b) $\gamma(x) \geqslant M_{p_k}^{-1}(x)$ (7) *for* $x \in \{x;\ r_k + 1 \leqslant |x| \leqslant r_{k+1}\}$, $k \in N$.

Proof. Let ω be the function used in Lemma 1 and let $\varphi \in C_0^\infty$ be such that $\varphi(0) = 1$, $\varphi(1) = 0$, $0 \leqslant \varphi(t) \leqslant 1$ for $t \in [0, 1]$, $\varphi^{(k)}(0) = \varphi^{(k)}(1) = 0$ for $k \in N$ and φ decreases on $[0, 1]$ (8).

The functions

$$x \mapsto (M_{p_k}^{-1} * \omega)(x) = \int_0^1 M_{p_k}^{-1}(x-t)\,\omega(t)\,dt,$$

$$x \in [r_k, r_{k+1}+1], \qquad k = 2, 3, \ldots,$$

have the following property:

$$M_{p_k}^{-1}(x) \leqslant (M_{p_k}^{-1} * \omega)(x) \leqslant M_{p_k}^{-1}(x-1) \leqslant M_{p_{k-1}}^{-1}(x),$$

$$x \in [r_k, r_{k+1}+1], \qquad k = 2, 3, \ldots$$

(6) α is introduced in (3) and X_{p_k}, $k \in N$, in (5)′.

(7) $M_{p_k}^{-1} = 1/M_{p_k}$.

(8) The function φ is quoted in [5], p. 154.

Let us put

$$\gamma(x) = (M_{p_1}^{-1} * \omega)(x), \qquad x \in [r_1, r_2);$$

$$\gamma(x) = (M_{p_{k-1}}^{-1} * \omega)(x)\,\varphi(x - r_k) + (M_{p_k}^{-1} * \omega)(x)\big(1 - \varphi(x - r_k)\big),$$

$$x \in [r_k, r_k + 1), \quad k = 2, 3, \ldots;$$

$$\gamma(x) = (M_{p_k}^{-1} * \omega)(x), \qquad x \in [r_k + 1, r_{k+1}),\ k = 2, 3, \ldots$$

In the same way, by using the function ω_1, $\omega_1 \in C_0^\infty$, $\omega_1 \geqslant 0$, $\operatorname{supp} \omega_1 \subset [-1, 0]$, $\int_{-1}^{0} \omega_1(x)\,dx = 1$, we construct the function γ on the intervals $(-r_2, -r_1]$, $(-r_k - 1, -r_k]$, $(-r_{k+1}, -r_k - 1]$, $k = 2, 3, \ldots$ We define the function γ on the interval $(-r_1, r_1)$ to be smooth, non-negative and to vanish on $(-\alpha, \alpha)$.

Since γ satisfies condition (b) by construction, we shall show that γ satisfies (a).

For a fixed $s \in N$ and $i \leqslant s$, on the intervals $[r_k + 1, r_{k+1})$, $k = 2, 3, \ldots$, we have

$$|M_s(x)\gamma^{(i)}(x)| \leqslant M_s(x)\,|M_{p_k}^{-1} * \omega^{(i)}(x)| \leqslant M_s(x)\,M_{p_k}^{-1}(x-1)\int_0^1 |\omega^{(i)}(x)|\,dx$$

$$\leqslant C_i\,M_s(x)\,M_{p_{k-1}}^{-1}(x), \qquad \text{where } C_i = \int_0^1 |\omega^{(i)}(x)|\,dx,$$

and on the intervals $[r_k, r_k + 1)$, $k = 3, 4, \ldots$, we have

$$M_s(x)\,|\gamma^{(i)}(x)| \leqslant M_s(x)\Big(\sum_{j=0}^{i}\binom{i}{j}|(M_{p_{k-1}}^{-1} * \omega^{(i-j)})(x)|\,|\varphi^{(j)}(x - r_k)| +$$

$$+ \sum_{j=0}^{i}\binom{i}{j}|(M_{p_k}^{-1} * \omega^{(i-j)})(x)|\,\big|\big(1 - \varphi(x - r_k)\big)^{(j)}\big|\Big)$$

$$\leqslant M_s(x)\Big(\sum_{j=0}^{i}\binom{i}{j}C_{i-j}D_j M_{p_{k-2}}^{-1}(x) + \sum_{j=0}^{i}\binom{i}{j}C_{i-j}D_j M_{p_{k-1}}^{-1}(x)\Big),$$

where $D_j = \sup\{|\varphi^{(j)}(x)|;\ x \in [0, 1]\}$.

Similar inequalities hold for $x \in (-r_{k+1}, -r_k - 1]$, $k = 2, 3, \ldots$, and $x \in (-r_k - 1, -r_k]$, $k = 3, 4, \ldots$

These inequalities imply $\gamma \in \mathscr{K}\{M_p\}$.

Remark. For the investigations of non-negative generalized functions from $\mathscr{K}'\{M_p\}$ the following conditions on the sequence M_p are assumed in [3], p. 147:

(a) condition (1);

(b) the M_p, $p \in N$, are infinitely differentiable outside some neighbourhood of zero (the same for all p) and are nowhere infinite;

(c) for any $p \in N$ there are numbers $q(p)$, a_p and C_p such that if $x \geqslant a_p$ and $k = 0, 1, \ldots, p$, then

$$\left|(1/M_{q(p)}(x))^{(k)}\right| \leqslant C_p/M_p(x).$$

These conditions directly imply that there exists a function σ from $\mathscr{K}\{M_p\}$ such that $\sigma(x) = 1/M_{q(p)}(x)$ for $r_p+1 \leqslant x \leqslant r_{p+1}$, where r_p is a sequence of positive numbers such that $r_p > a_p$ and $r_{p+1} > r_p+1$, $p \in N$. The proof of this assertion is the same as the proof of Lemma 1 in [6].

Lemma 2 directly implies:

LEMMA 3 (9). *If $f \in L^1_{\text{loc}}$, $f \notin (\mathscr{K}'\{M_p\})_r$, then there exists a non-negative function $\gamma \in \mathscr{K}\{M_p\}$ such that $f\gamma \notin L^1$.*

Proof. See [6], the proof of Lemma 2.

Following [6], we define Λ to be the set of locally integrable functions f such that for some $p \in N$, $fM_p^{-1} \in L^1$.

If for some x, $M_p(x) = \infty$, then we take $1/M_p(x) = 0$.

Also, we denote by $\mathscr{K}\{M_p\}_\infty$ the vector space of functions f defined almost everywhere on $\boldsymbol{R}$ for which the countably many norms

$$q_k(f) := \text{ess sup}\,\{M_k(x)|f(x)|;\ x \in \boldsymbol{R}\}, \qquad k \in N,$$

are all finite; we equip this space with the topology defined by this sequence of seminorms. Clearly, $\mathscr{K}\{M_p\}_\infty \subset L^1_{\text{loc}}$.

If $\sigma \in \mathscr{K}\{M_p\}$ and $f \in \Lambda$ then

$$f(x)\sigma(x) \leqslant \big(f(x)/M_p(x)\big)\big(\sigma(x)M_p(x)\big)$$

almost everywhere on $\boldsymbol{R}$. Using this fact we obtain the following characterization of the space $(\mathscr{K}'\{M_p\})_r$:

THEOREM 2. (10). *Let $f \in L^1_{\text{loc}}$. The following conditions are equivalent:*

(i) $f \in (\mathscr{K}'\{M_p\})_r$;

(ii) $f \in \Lambda$;

(iii) $\mathscr{K}\{M_p\} \ni \theta \mapsto f\theta \in L^1$;

(iv) $\mathscr{K}\{M_p\}_\infty \ni \theta \mapsto f\theta \in L^1$;

(v) *mapping* (iv) *is continuous.*

Proof. See [6], the proof of Theorem 1.

We shall give one more useful characterization of the space $(\mathscr{K}'\{M_p\})_r$. But one more assumption has to be made. First we introduce the following notion:

A function $f \in L^1_{\text{loc}}$ is called an *M_p-function* iff there exists $k \in N$ such that $fM_k^{-1} \in L^\infty$, where L^∞ is the space of measurable essentially bounded functions on $\boldsymbol{R}$.

(9) See [6], Lemma 2.

(10) See [6], Theorem 1.

THEOREM 3. *If a sequence M_p satisfies conditions (1)–(5), $(\mathrm{N})_1$ and the following one*:

$(\mathrm{N})_2$ [11] *for every $p \in N$ there exists $p' \in N$ such that*

$$M_p(x)\, M_{p'}^{-1}(x) \to 0 \quad \text{as } |x| \to \infty,$$

then for $f \in L^1_{\mathrm{loc}}$ the following conditions are equivalent:

(i) $f \in (\mathscr{K}'\{M_p\})_r$;

(ii) *The function $x \mapsto \int_0^x |f(t)|\, dt$, $x \in \boldsymbol{R}$, is an M_p-function.*

Proof. (i) ⇒ (ii). Let $f \in (\mathscr{K}'\{M_p\})_r$ and let $A > 0$ be such that

$$\int_{\boldsymbol{R}} |f(t)/M_p(t)|\, dt \leqslant A.$$

For $|x| > \alpha$ and for suitable C_1 and C_2 we have

$$\Big|\int_0^x |f(t)|\, dt\Big| \leqslant \Big|\int_0^\alpha |f(t)|\, dt\Big| + \Big|\int_\alpha^x |f(t)|\, dt\Big| \leqslant C_1\Big(1 + M_p(x)\Big|\int_\alpha^x |f(t)|\, M_p^{-1}(t)\, dt\Big|\Big)$$
$$\leqslant C_1\big(1 + M_p(x)\, A\big) \leqslant C_2\, M_p(x).$$

This implies the assertion.

(ii) ⇒ (i). Let us suppose that $F(x) = \int_0^x |f(t)|\, dt$ is an M_p-function, i.e., $FM_r^{-1} \in L^\infty$ for some $r \in N$. Let $r' \in N$ be the number corresponding to r in $(\mathrm{N})_1$, and r'' the number corresponding to r' in condition (5).

We define a function $x \mapsto N_{r''}(x)$ by

$$N_{r''}(x) = (M_{r''}^{-1} * \omega)(x), \quad x > X_{r''} + 1, \quad \text{and}$$
$$N_{r''}(x) = (M_{r''}^{-1} * \omega)(x), \quad x < -X_{r''} - 1 \ \text{[12]}.$$

Let $a > X_{r''} + 1$. From $M_{r''}^{-1}(x) = \int_0^1 M_{r''}^{-1}(x)\, \omega(t)\, dt \leqslant \int_0^1 M_{r''}^{-1}(x - t)\, \omega(t)\, dt$ we obtain $M_{r''}^{-1}(x) \leqslant N_{r''}(x)$ if $x \geqslant a$. Similarly one can prove that $M_{r''}^{-1}(x) \leqslant N_{r''}(x)$ if $x \leqslant -a$.

Since $N'_{r''}(x) = (M_{r''}^{-1} * \omega')(x)$, $x \geqslant a$ and $N'_{r''}(x) = (M_{r''}^{-1} * \omega_1')(x)$, $x \leqslant -a$, we obtain for some $C > 0$

$$|N'_{r''}(x)| \leqslant CM_{r''}^{-1}(x - 1), \quad x \geqslant a$$
$$\text{and} \quad |N'_{r''}(x)| \leqslant CM_{r''}^{-1}(x + 1), \quad x \leqslant -a.$$

We will prove that $fM_r^{-1} \in L^1$. Since

$$|f(x)|/M_{r''}(x) \leqslant |f(x)|\, N_{r''}(x) \quad \text{for } |x| \geqslant a$$

[11] Condition $(\mathrm{N})_1$ and $(\mathrm{N})_2$ together constitute condition (N) from [1], p. 111.

[12] ω and ω_1 are introduced in the proof of Lemma 2 and $X_{r''}$ is from (5).

and $f \in L^1_{loc}$, it is enough to prove that

$$\lim_{A\to\infty} \int_a^A |f(x)| N_{r''}(x)\,dx \quad \text{and} \quad \lim_{A\to-\infty} \int_A^{-a} |f(x)| N_{r''}(x)\,dx$$

exist. This will imply that

$$\int_a^\infty |f(x)| N_{r''}(x)\,dx < \infty, \qquad \int_{-\infty}^{-a} |f(x)| N_{r''}(x) < \infty$$

and that $f M_{r''}^{-1} \in L^1$.

Condition $(N)_2$ implies that $F(A) N_{r''}(A) \to 0$ as $A \to \infty$. Thus

$$\lim_{A\to\infty} \int_a^A |f(x)| N_{r''}(x)\,dx = -F(a) N_{r''}(a) - \lim_{A\to\infty} \int_a^A F(x) N'_{r''}(x)\,dx.$$

The last limit exists because

$$\Big|\int_a^\infty F(x) N'_{r''}(x)\,dx\Big| \leqslant \int_a^\infty F(x)\,|N'_{r''}(x)|\,dx \leqslant C \int_a^\infty F(x) M_{r''}^{-1}(x-1)\,dx$$

$$\leqslant C \operatorname*{ess\,sup}_{x\in\mathbf{R}} \{|F(x)|\, M_r(x)\} \int_a^\infty M_r(x) M_{r'}^{-1}(x)\,dx < \infty.$$

Similarly one can prove that $\lim_{A\to-\infty} \int_A^{-a} |f(x)| N_{r''}(x)\,dx$ exists. This completes the proof.

Remark. If f is an M_p-function, then $\int_0^x |f(t)|\,dt$, $x \in \mathbf{R}$, is also an M_p-function. We shall prove it.

Let $f M_r^{-1} \in L^\infty$ and r' be the natural number corresponding to r in $(N)_1$. If $Y = \mathbf{R} \backslash (-\alpha, \alpha)$ we have

$$\int_Y |f(t)/M_{r'}(t)|\,dt \leqslant \operatorname*{ess\,sup}_{x\in\mathbf{R}} \{|f(x)|\, M_r^{-1}(x)\} \int_{\mathbf{R}} M_r(t) M_{r'}^{-1}(t)\,dt.$$

Thus $f \in (\mathcal{K}'\{M_p\})_r$ and Theorem 3 implies the assertion.

Examples 1. The space $\mathcal{S}'$ is generated by the sequence $x \mapsto M_p(x) = (1+|x|)^p$, $p \in N$, which satisfies conditions (1)–(5) and (N) ([13]).

2. The space $\mathcal{K}'_r$, $r \geqslant 1$, investigated in [4] and [7] is generated by the sequence $x \mapsto M_p(x) = \exp(p|x|^r)$, $p \in N$. This sequence satisfies conditions (1)–(5) and (N).

3. The space $W_{M,a}$, $a \in \mathbf{R}$, where $x \mapsto M(x)$, $x \in \mathbf{R}$, is a convex function defined in [2], Chapter 1, investigated in [2], is generated by the sequence $x \mapsto M_p(x) = M\big(a(1-1/p)x\big)$, $p \in N$. This sequence satisfies (1)–(5) and (N).

([13]) See ([11]).

4. The space $\operatorname{proj}_{q\to\infty} W^q_{M,1/q}$, where $x \mapsto M(x)$, $x \in \boldsymbol{R}$, is a convex function as in 3, and $W^q_{M,A}$, $q \in N$, $A > 0$, is the space defined in [8], p. 330, is generated by the sequence $x \mapsto M_p(x) = M(px)$ which also satisfies (1)–(5) and (N).

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