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On the asymptotic behaviour of the solutions of a system of ordinary differential equations

by C. Kluczny (Katowice)

1. This note remains in close connection with paper [1]. To avoid repetitions we observe all the notations and definitions used in [1]. To the results obtained there we now add some remarks. Moreover, we generalize in a certain way lemma 3 of [1] (p. 229). These additions together with the results of [1] permit us to prove Theorem D formulated beneath (see [1], § 4, p. 227 as well as Theorem C, p. 228).

Theorem D deals with the system

$$X' = AX + B(X, t) + C(X, t)\,, \tag{II}$$

where $X = (x_1, x_2, \ldots, x_n)$, A is an $n\times n$ square real constant matrix, and $B(X, t)$ and $C(X, t)$ are vector-functions with n component each, continuous for $t \geqslant t_1$ and for arbitrary X.

Assumption H. 1° There exists a scalar function $\chi(t)$, continuous for $t \geqslant t_1$, such that

$$\chi(t) = \chi_1(t) + \chi_2(t)\,, \qquad \chi_1(t) \geqslant 0\,, \qquad \chi_2(t) \geqslant 0\,,$$

$$\lim_{t\to\infty} \chi_1(t) = 0\,, \qquad \int_{t_1}^{\infty} \chi_2(t)\,dt < \infty\,.$$

2° There exists a scalar function $w(t) \geqslant 0$ continuous for $t \geqslant t_1$ and a constant s $(-\infty \leqslant s < +\infty)$, such that the characteristic exponent of one at least of the two functions $\int_{t_1}^{t} w(\tau)\,d\tau$, $\int_{t}^{\infty} w(\tau)\,d\tau$ is not greater than s. Moreover, for $t \geqslant t_1$ and for arbitrary X the following inequalities hold:

$$|B(X, t)| \leqslant |X|\chi(t)\,, \qquad |C(X, t)| \leqslant |X|^q w(t)\,, \qquad \text{where} \qquad 0 \leqslant q < 1\,.$$

Just as in [1] let us put $v_0 = s/(1-q)$ and form the sequence

$$v_0 < v_1 < v_2 < \ldots < v_p\,, \tag{k}$$

where $v_1, v_2, \ldots, v_p$ are all the real parts of the characteristic roots of the matrix A which are greater than v_0. Let n_i $(i = 0, 1, \ldots, p)$ be the number of characteristic roots of A whose real parts are not greater than v_i (each i-fold characteristic root is counted i times).

THEOREM D. *Suppose that Assumption* H *holds and that the sequence* (k) *contains at least two elements. Assume that the matrix A has the canonical real Jordan form* (*see* [1], § 2, *p.* 226). *Let j be one of the integers* $1, 2, \ldots, p$. *Put* $X = (X_1, X_2, X_3)$, *where* X_1, X_2, X_3 *are vectors formed of the first* n_{j-1}, *of the next* $n_j - n_{j-1}$ *and of the rest of the coordinates of the vector* X *respectively.*

If $X(t)$ *is any of the solutions of* (II) *with its characteristic exponent equal to* j, *then the following conditions hold*:

$$|X_1(t)| = o(|X_2(t)|), \quad |X_3(t)| = o(|X_2(t)|). \tag{x}$$

Theorem D solves in regard to system (II) a problem similar to that discussed by O. Perron in [2] and [3] concerning a linear system. Z. Szmydtówna has dealt with an analogous problem, [4], concerning a non linear system $\Xi' = U(t, \Xi)$, which, though in some ways more general than (II), yet does not contain system (II). Namely, it is supposed in [4] that $U(t, 0) = 0$, and the other assumptions appearing there imply $\chi_2(t) = 0$. Moreover, our note establishes a close relation between conditions (x) and the characteristic exponents of the solutions.

2. The remarks which we are going to make concern Examples 1 and 2 of [1] (p. 219-220) as well as Theorem B (p. 224).

From formula (1,10) we immediatly infer that, if $b < a$, then all the solutions of the equation $u' = a(t)u + \beta(t)$ [1] whose characteristic exponents are equal to a are of the form:

$$u_c(t) = \pm e^{ta_c(t)} \tag{i}$$

(for t large enough), parameter c assuming all values but zero. Moreover, $\lim_{\to\infty} a_c(t) = a$ for $c \neq 0$.

Consequently,

$$\underline{\lim}_{t\to\infty} \frac{\ln|u_c(t)|}{t} = \overline{\lim}_{t\to\infty} \frac{\ln|u_c(t)|}{t} = a. \tag{ii}$$

From this we find without any difficulty that, if $b < a(1-q)$, then every positive solution of the equation

$$u' = a(t)u + \beta(t)u^q, \quad 0 \leqslant q < 1,$$

(example 2, p. 220), whose characteristic exponent is equal to a is also of the form (i), so that condition (ii) is satisfied.

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(1) According to [1], $a = \lim_{t\to\infty} \frac{1}{t}\int_{t_1}^{t} a(\tau)d\tau$, and b is such a number that the characteristic exponent of one at least of the two functions $\int_{t_1}^{t}|\beta(\tau)|d\tau$ and $\int_{t}^{\infty}|\beta(\tau)d\tau$ does not exceed b.

The last remark permits us to generalize part (b) of Theorem B (p. 224, see also the proof of that theorem) as follows: if the characteristic exponent of the solution $X(t)$ of the system $X' = F(X, t)$ is greater than $s/(1-q)$, then the following inequality is satisfied:

$$\text{(iii)} \qquad \varliminf_{t\to\infty} \frac{\ln|X(t)|}{t} \geqslant \mu \text{ (2)} .$$

Indeed, in the proof of part (b) of Theorem B we have made use of the inequality $X^2(t) > \varphi_1(t)$, where $\varphi_1(t)$ is a solution of equation (2,7) of [1]. In view of the above remarks this implies

$$2\varliminf_{t\to\infty} \frac{\ln|X(t)|}{t} \geqslant \lim_{t\to\infty} \frac{\ln\varphi_1(t)}{t} = 2\mu .$$

3. Now, in order to generalize Lemma 3 of [1] we present the following

LEMMA 4. *Let the right-hand member of the system*

$$\text{(I)} \qquad X' = F(X, t) , \qquad X = (Y, Z)$$

be continuous for $t \geqslant t_1$ and for arbitrary X. Suppose that the assumptions of Lemma 3 are satisfied. Then for every σ such that

$$\text{(1)} \qquad \frac{s}{1-q} < \sigma < \lambda$$

and for arbitrary $p > 0$ there exists such a number t_{2p} that for $t \geqslant t_{2p}$ and for every solution of system (I) *such that its characteristic exponent is not greater than λ we have*

$$\text{(a)} \qquad |Z(t)| < \frac{2}{p}(|Y(t)| + e^{\sigma t}) .$$

Moreover, for arbitrary $p > 0$ and for every $X(t)$ whose characteristic exponent is greater than λ (consequently, greater than or equal to μ) there exists a number $t_{3p} \geqslant t_{2p}$ such that for $t \geqslant t_{3p}$ the following inequality holds:

$$\text{(b)} \qquad |Z(t)| > \frac{1}{p} |Y(t)| .$$

We have also

$$\text{(c)} \qquad \varliminf_{t\to\infty} \frac{\ln|X(t)|}{t} \geqslant \varliminf_{t\to\infty} \frac{\ln|Z(t)|}{t} \geqslant \mu .$$

The proof of this lemma is similar to that of lemma 3. We present again the calculating part of it, which is more difficult now.

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(2) In theorem B the function $F(X, t)$ was supposed to satisfy the condition: $X \cdot F(X, t) \leqslant \omega(t) X^2 + w(t)|X|^{1+q}$, and by μ we denoted $\lim_{t\to\infty} \frac{1}{t} \int_{t_1}^{t} \omega(\tau) d\tau$.

Let us put for every $p > 0$

$$N_p = (p^2 + 4p)(p+1)\,, \tag{2}$$

$$h_p(t) = \exp \int_t^\infty \frac{N_p}{p^2} [\chi_2(\tau) + w(\tau) e^{-(1-q)\sigma\tau}] d\tau\,. \tag{3}$$

We have

$$N_p > 0\,, \quad h_p(t) > 1 \quad \text{and} \quad \lim_{t\to\infty} h_p(t) = 1\,. \tag{4}$$

Let the number t_{2p} be chosen in such a manner that

$$h_p(t) \leqslant 2\,, \quad p^2(\mu - \lambda) \geqslant N_p \chi_1(t) \tag{5}$$

for $t \geqslant t_{2p}$, which is possible since $\lim_{t\to\infty} \chi_1(t) = 0$.

Put

$$g_p(u, t) = \frac{1}{p^2} h_p^2(t)(u + e^{2\sigma t})\,, \tag{6}$$

and denote by ω_p and $\mathfrak{M}_p$ the sets of points, $P = (Y, Z, t)$, defined as follows:

$$\omega_p\colon \quad Z^2 < g_p(Y^2, t)\,,\ t \geqslant t_{2p}\,, \tag{7}$$

$$\mathfrak{M}_p\colon \quad Z^2 = g_p(Y^2, t)\,,\ t \geqslant t_{2p}\,. \tag{8}$$

First we prove that the assumptions of Lemma 2 of [1] are satisfied if we substitute any $g_p(u, t)$ for $g(u, t)$. In order to demonstrate that inequality (x) appearing in that lemma holds on the surface $\mathfrak{M}_p$, we put

$$\Gamma(P) = 2\left[(0, Z)\cdot F(P) - (Y, 0)\cdot F(P)\frac{\partial g_p(Y^2, t)}{\partial u}\right] - \frac{\partial g_p(Y^2, t)}{\partial t}\,.$$

It follows from (6) and from the assumptions 6° and 7° of Lemma 3 that

$$\frac{p^2}{2}\Gamma(P) > \mu p^2 Z^2 - p^2|Z||X|\chi(t) - p^2|Z||X|w(t) - \\ - h_p^2(t)[\lambda Y^2 + |Y||X|\chi(t) + |Y||X|^q w(t)] - h_p(t) h_p'(t)(Y^2 + e^{2\sigma t}) - h_p^2(t)\sigma e^{2\sigma t}\,. \tag{9}$$

Since on the surface $\mathfrak{M}_p$ we have

$$p|Z| > |Y|\,, \quad p|Z| > e^{\sigma t}\,, \quad |X| \leqslant |Y| + |Z| \leqslant (1+p)|Z|\,, \\ h_p^2(t) Y^2 = p^2 Z^2 - h_p^2(t) e^{2\sigma t}\,, \tag{10}$$

from the last equality we get by (1)

$$\mu p^2 Z^2 - \lambda h_p^2(t) Y^2 - \sigma h_p^2(t) e^{2\sigma t} = \mu p^2 Z^2 + \lambda h_p^2(t) e^{2\sigma t} - \lambda p^2 Z^2 - \sigma h_p^2(t) e^{2\sigma t} \\ = (\mu - \lambda) p^2 Z^2 + (\lambda - \sigma) h_p^2(t) e^{2\sigma t} > p^2(\mu - \lambda) Z^2 \geqslant N_p \chi_2(t) Z^2; \tag{11}$$

then using the inequalities of (10) we infer from (5) and (2) that

$$(12)\qquad -p^2|Z||X|\chi(t)-h_p^2(t)|Y||X|\chi(t) \geqslant -\big(p^2+ph_p^2(t)\big)|Z||X|\chi(t)$$
$$\geqslant -N_pZ^2\chi(t) = -N_pZ^2\big(\chi_1(t)+\chi_2(t)\big)\,,$$

$$(13)\qquad -p^2|Z||X|^q w(t)-h_p^2(t)|Y||X|^q w(t)$$
$$\geqslant -(p^2+4p)(1+p)^q|Z|^{1+q}w(t) \geqslant -N_pZ^2e^{-(1-q)\sigma t}w(t)\ (^3)\,.$$

Since for $t\geqslant t_{2p}$ we have $p^2(\mu-\lambda)\geqslant N_p\chi_1(t)$, adding inequalities (11)-(13) and using once more the last relation of (10) we infer that on the surface $\mathfrak{M}_p$ the following inequality holds:

$$\frac{p^2}{2}\Gamma(P) > \frac{Z^2}{h_p(t)}\Big[-N_ph_p(t)\big(\chi_2(t)+w(t)e^{-(1-q)\sigma t}\big)-p^2h_p(t)\Big]\,.$$

Hence, in view of (3), it follows that on $\mathfrak{M}_p$ we have $\Gamma(P)>0$.

By a similar argument to that used in the proof of Lemma 3 ([1], p. 231-233) we infer that, given any $p>0$, the following condition,

$$(14)\qquad Z^2(t) < \frac{h_p^2(t)}{p^2}\big(Y^2(t)+e^{2\sigma t}\big)\,,$$

is satisfied for $t\geqslant t_{2p}$ and for every solution of (I) whose characteristic exponent does not exceed λ. At the same time, given any $p>0$ and an arbitrary solution of (I) with its characteristic exponent greater than λ (consequently greater than or equal to μ), there exists a number $t_{3p}\geqslant t_{2p}$ such that for $t\geqslant t_{3p}$ the following inequality holds:

$$(15)\qquad Z^2(t) > \frac{h_p^2(t)}{p^2}\big(Y^2(t)+e^{2\sigma t}\big)\,.$$

Since $1<h_p(t)\leqslant 2$ for $t\geqslant t_{2p}$, by (14) and (15) we obtain formulas (a) and (b) of lemma 4. Formula (c) of the same lemma results from the generalization, given above, of Theorem B (relation (iii), see also (4,21)-(4,32) of [1]).

4. Proof of Theorem D. First let us assume that the numbers ε_i appearing under the main diagonal of the matrix A are not greater than the number ε_0 defined by (4,24) ([1], p. 233). Putting $\sigma=\frac{1}{2}(v_0+v_1)$ we have $v_0<\sigma<v_1-\varepsilon_0$. Let j be any of the integers $1,2,\ldots,p$, and $X(t)=\big(X_1(t), X_2(t), X_3(t)\big)$ any of the solution of (II) with its characteristic exponent equal to v_j. In the case where $j\leqslant p-1$, assumptions 6° and 7° of Lemma 3, which at the same time are those of Lemma 4, will be satisfied if we put

$$Y=(X_1,X_2)\,,\qquad Z=X_3\,,\qquad \lambda=v_j+\varepsilon_0\,,\qquad \mu=v_{j+1}-\varepsilon_0$$

(3) The inequality $p|Z|>e^{\sigma t}$ implies $p^{1-q}|Z|^{1-q}e^{-(1-q)\sigma t}>1$, and consequently $(1+p)^{1-q}|Z|^{1-q}e^{-(1-q)\sigma t}>1$.

(cp. (4,32) of [1]). Thus, since assumption (1) of Lemma 4 as well as the rest of the assumptions of Lemma 3 are also satisfied, putting $p = 1, 2, \ldots$ we infer from (a) that for every $n = 1, 2, \ldots$ there exists such a number t_{2n} that for $t \geqslant t_{2n}$ the following inequality holds:

$$|X_3(t)| < \frac{2}{n}\left(|X_1(t)| + |X_2(t)| + e^{\sigma t}\right). \tag{16}$$

Of course, if $j = p$, then $X_3(t) = 0$. Thus, the above inequality is satisfied also for $j = p$.

On the other hand, if we put

$$Y = X_1, \quad Z = (X_2, X_3), \quad \lambda = v_{j-1} + \varepsilon_0, \quad \mu = v_j - \varepsilon_0,$$

then assumptions 6° and 7° will be satisfied for $j = 1, 2, \ldots, p$ if $v_0 \geqslant \varrho_1$ as well as for $j = 2, 3, \ldots, p$ if $v_0 < \varrho_1$. Thus putting $p = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and taking into account that if $v_0 < \varrho_1$ and $j = q$ then $X_1(t) = 0$, we infer from (b) that for every $n = 1, 2, \ldots$ there exists such a number t_{3n} that for $t \geqslant t_{3n}$ the following inequality is satisfied:

$$|X_2(t)| + |X_3(t)| > n|X_1(t)|. \tag{17}$$

Moreover, it follows from assertion (c) of Lemma 4 that

$$\varliminf_{t\to\infty} \frac{\ln\left(|X_2(t)| + |X_3(t)|\right)}{t} \geqslant v_j - \varepsilon_0 \geqslant v_1 - \varepsilon_0 > \sigma.$$

This means that $e^{\sigma t} = o\left(|X_2(t)| + |X_3(t)|\right)$, and consequently, $e^{\sigma t} = o\left(|X_1(t)| + |X_2(t)| + |X_3(t)|\right)$. Thus, by condition (16) equivalent to

$$(n+2)|X_3(t)| < 2|X_1(t)| + 2|X_2(t)| + 2|X_3(t)| + 2e^{\sigma t},$$

we obtain $|X_3(t)| = o\left(|X_1(t)| + |X_2(t)|\right)$. Hence and from (17) we find that $|X_1(t)| = o\left(|X_2(t)|\right)$, and consequently, $|X_3(t)| = o\left(|X_2(t)|\right)$.

In the general case, where the numbers ε_i appearing under the main diagonal of A do not satisfy the condition $\varepsilon_i \leqslant \varepsilon_0$, the proof of the theorem results from the following remark.

There exists a non-singular matrix T such that the above condition is already satisfied as regards the matrix $T^{-1}AT$, and moreover, the transformation $X \to TX$ maps each of the hyperplanes X_1, X_2, X_3 on itself and as is well known (see [1] p. 219) we have $\frac{1}{\|T^{-1}\|} \leqslant \frac{|TX|}{|X|} < \|T\|$ for all X.

The theorem is thus proved.

References

[1] C. Kluczny, *On the characteristic exponents of the solutions of a system of ordinary differential equations*, Ann. Pol. Math. 8 (1960), p. 215-239.

[2] O. Perron, *Über lineare Differentialgleichungen, bei denen die unabhängige Variable reel ist*, J. für Math. 142 (1913), p. 254-270.

[3] — *Über lineare Differentialgleichungen, bei denen die unabhängige Variable reel ist*, J. für Math. 143 (1913), p. 25-49.

[4] Z. Szmydtówna, *Sur l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Pol. Math. 24 (1951), p. 17-34.

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