ANNALES
POLONICI MATHEMATICI
XXIX (1974)

Remarks on the concept of mean value

by K. URBANIK (Wrocław)

Abstract. The mean value of probability measures can be introduced in terms of the centering in mean. Replacing the last notion by the centering in probability we get what we call the substitute of the mean value. Our aim is to discuss the relationship between these two concepts. Moreover, we define a more general concept — the convex mean value.

In this paper we are concerned with the possibility of an extension of the concept of mean value for probability distributions. The necessity of such a generalization arises in the prediction theory when one deals with probability distributions without finite moments. In the first two sections we give a short survey of some fundamental facts about the substitute of the mean value introduced in [2]. In the third section we define a more general concept that of — convex mean value — and we discuss the relation between mean value, the substitute of the mean value and convex mean value.

1. Preliminary notions. Let $\mathfrak{P}$ be the set of all probability measures on the real line, i.e. the set of all normalized Borel measures. Let $\mathfrak{L}$ be the subset of $\mathfrak{P}$ consisting of all probability measures P with finite mean value $l(P) = \int_{-\infty}^{\infty} xP(dx)$. Further, μ_P and φ_P will denote the median and the characteristic function of P, respectively.

The concept of mean value for probability measures from $\mathfrak{L}$ can be also introduced in the following way, which, of course, is not the simplest one. Consider a sequence $X_1, X_2, \ldots$ of independent random variables with the same probability distribution P from $\mathfrak{L}$. Throughout this paper we identify random variables which are equal with probability 1. Let $\mathscr{L}_P$ be the linear space generated by the random variables $X_1, X_2, \ldots$ and closed under the convergence in mean. It is clear that the space $\mathscr{L}_P$ is uniquely determined up to an obvious isomorphism by the probability measure P. We say that the probability measure P is *centered in mean* if and only if 0 is the only constant random variable belonging to the space $\mathscr{L}_P$. Given a real number c, we denote by P_c the shifted probability measure, i.e. $P_c(E) = P(E+c)$, where $E+c = \{x+c: x \in E\}$. It is very easy to verify that c is the mean value of P if and only if P_c is centered in mean. This property can be regarded as a definition of the mean value. In an analogous way we can define a substitute of the mean value (see [2]).

2. A substitute of mean value. Consider an arbitrary probability measure P from $\mathfrak{P}$. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with the same probability distribution P. Let $\mathcal{M}_P$ be the linear space generated by the random variables $X_1, X_2, \ldots$ and closed under the convergence in probability. It is clear that the space $\mathcal{M}_P$ is uniquely determined up to an obvious isomorphism by the probability measure P. We say that P is *centered in probability* if and only if 0 is the only constant random variable belonging to $\mathcal{M}_P$.

The following statement was proved in [2]: *for each probability measure P from $\mathfrak{P}$ one of the following three cases holds*:

(i) *there exists exactly one value c for which P_c is centered in probability,*

(ii) *for all c the probability measures P_c are centered in probability,*

(iii) *for all c the probability measures P_c are not centered in probability.*

Let us denote by $\mathfrak{S}$ the set of all probability measures for which case (i) holds. Further, by $s(P)$ we shall denote the value of c for which P_c is centered in probability. The number $s(P)$ is called the *substitute of the mean value.*

LEMMA 2.1. *If $P \in \mathfrak{L}$, then for every real number $c \neq l(P)$ the probability measure P_c is not centered in probability.*

Proof. Since

$$\lim_{T\to\infty} \Big(\mu_{P_c} + \int_{-T}^{T} x P_c(dx+\mu_{P_c})\Big) = l(P) - c$$

and

$$\lim_{T\to\infty} T \int_{-\infty}^{\infty} \frac{x^2}{T^2+x^2} P_c(dx+\mu_{P_c}) = 0,$$

we have, by Theorem 2.1 in [2], the assertion of the lemma.

THEOREM 2.1. *If P has a finite second moment, then $P \in \mathfrak{L} \cap \mathfrak{S}$ and $l(P) = s(P)$.*

Proof. It was proved in [2] (Theorem 2.2) that a probability measure with a finite second moment centered in mean is also centered in probability. Hence it follows that $P_{l(P)}$ is centered in probability. By Lemma 2.1 for every $c \neq l(P)$ the probability measure P_c is not centered in probability. Consequently, P has the substitute of the mean value and $s(P) = l(P)$, which completes the proof.

THEOREM 2.2. $\mathfrak{L}\setminus\mathfrak{S} \neq \emptyset$.

Proof. Put

$$a^{-1} = \int_e^\infty \frac{du}{u^2\log^2 u}$$

and

$$Q(E) = \tfrac{1}{2}\delta_{-a}(E) + \frac{a}{2}\int_{E\cap[e,\infty)} \frac{du}{u^2\log^2 u},$$

where δ_b denotes the probability measure concentrated at the point b. It is evident that $Q \in \mathfrak{L}$ and $l(Q) = 0$. On the other hand, it was proved in [2] (p. 68) that Q is not centered in probability. Moreover, by Lemma 2.1, for every $c \neq 0$ the probability measure Q_c is not centered in probability. Hence it follows that Q does not belong to $\mathfrak{S}$. The theorem is thus proved.

THEOREM 2.3. $\mathfrak{S}\setminus\mathfrak{L} \neq \emptyset$.

Proof. Put

$$b^{-1} = \int_e^\infty \frac{du}{u^2\log u}$$

and

$$R(E) = \frac{b}{2}\int_{E\cap[e,\infty)} \frac{du}{u^2\log u} + \frac{b}{2}\int_{(-E)\cap[e,\infty)} \frac{du}{u^2\log u},$$

where $-E = \{-x\colon x \in E\}$. The measure R being symmetric, is centered in probability. Given $c \neq 0$, we have the relation

$$\mu_{R_c} + \int_{-T}^{T} xR_c(dx+\mu_{R_c}) = -c$$

and

$$T\int_{-\infty}^{\infty} \frac{x^2}{T^2+x^2} R_c(dx+\mu_{R_c}) = bT\int_e^\infty \frac{dx}{(T^2+x^2)\log x}.$$

Since for $T \geqslant e^e$

$$\int_e^{\log T} \frac{dx}{(T^2+x^2)\log x} \leqslant \frac{\log T}{T^2}$$

and

$$\int_{\log T}^{\infty} \frac{dx}{(T^2+x^2)\log x} \leqslant \frac{\pi}{2T\log\log T},$$

we infer that

$$\lim_{T\to\infty}\frac{T\int\limits_{-\infty}^{\infty}\frac{x^2}{T^2+x^2}R_c(dx+\mu_{R_c})}{\mu_{R_c}+\int\limits_{-T}^{T}xR_c(dx+\mu_{R_c})}=0.$$

Thus, by Theorem 2.1 in [2], for all $c\neq 0$ the measures R_c are not centered in probability. In other words, $R\in\mathfrak{S}$ and $s(R)=0$. Finally, we have the evident relation $R\notin\mathfrak{L}$ which completes the proof.

3. Convex mean value. Let $P\in\mathfrak{P}$ and let $X_1, X_2, \ldots$ be a sequence of independent random variables with the same probability distribution P. Let $\mathscr{H}_P$ be the convex hull generated by $X_1, X_2, \ldots$ and closed under the convergence in probability. It is evident that $\mathscr{H}_P$ is uniquely determined up to an obvious isomorphism by the probability measure P. Let $\mathscr{C}_P$ be the subset of $\mathscr{H}_P$ consisting of all constant random variables. Evidently, the set $\mathscr{C}_P$ is closed and convex and, consequently, can be regarded as a subinterval of the real line. We shall prove that every closed subinterval of the real line can be obtained in this way. We start with some lemmas.

LEMMA 3.1. *Suppose that there exists a subsequence $n_1<n_2<\ldots$ of integers for which*

$$\lim_{k\to\infty} n_k\int\limits_{-\infty}^{\infty}\frac{x^2}{n_k^2+x^2}P(dx+\mu_P)=0$$

and

$$\lim_{k\to\infty}\int\limits_{-n_k}^{n_k}xP(dx+\mu_P)=c-\mu_P.$$

Then $c\in\mathscr{C}_P$.

Proof. Setting $Y_k=\frac{1}{n_k}\sum_{j=1}^{n_k}X_j$, where $X_1, X_2, \ldots$ generate the set $\mathscr{H}_P$ we have $Y_k\in\mathscr{H}_P$ $(k=1,2,\ldots)$. Moreover, by Feller Theorem (Theorem 2 in [1]), the sequence $Y_1, Y_2, \ldots$ tends to c in probability. Thus $c\in\mathscr{C}_P$.

COROLLARY. *If*

$$\lim_{T\to\infty} T\int\limits_{-\infty}^{\infty}\frac{x^2}{T^2+x^2}P(dx+\mu_P)=0,$$

then the set $\mathscr{C}_P$ contains all limit points of $\{\mu_P+\int\limits_{-T}^{T}xP(dx+\mu_P)\}$ *when* $T\to\infty$.

LEMMA 3.2. *Setting*

$$a_P = \varliminf_{T\to\infty}\Big(\mu_P + \int_{-T}^{T} xP(dx+\mu_P)\Big)$$

and

$$b_P = \varlimsup_{T\to\infty}\Big(\mu_P + \int_{-T}^{T} xP(dx+\mu_P)\Big)$$

we have the inclusion $\mathscr{C}_P \subset [a_P, b_P]$.

Proof. For probability measures P concentrated at a single point, say q, we have $\mathscr{C}_P = \{q\}$ and $a_P = b_P = q$, which implies the assertion of the Lemma.

Now suppose that P is not concentrated at a single point. Given $c \in \mathscr{C}_P$, we can find a sequence $\{\sum_{j=1}^{k_n} a_{jn} X_j\}$ of convex combinations of generators in $\mathscr{H}_P$ tending to c in probability. Since the generators $X_1, X_2, \ldots$ are not constant, we infer that

$$\lim_{n\to\infty} \max_{1\leqslant j\leqslant k_n} a_{jn} = 0.$$

In other words, the random variables $\{a_{jn}X_j\}$ $(j = 1, 2, \ldots, k_n;$ $n = 1, 2, \ldots)$ form a uniformly infinitesimal triangular array. Consequently, by Theorem 1 in [1],

$$\lim_{n\to\infty}\sum_{j=1}^{k_n} a_{jn}\Big(\mu_P + \int_{|x|\leqslant a_{jn}^{-1}} xP(dx+\mu_P)\Big) = c$$

whence the inequalities $a_P \leqslant c \leqslant b_P$ follow. The Lemma is thus proved.

THEOREM 3.1. *Each closed subinterval of the real line is equal to the set* $\mathscr{C}_P$ *for a certain probability measure* P.

Proof. It is very easy to verify that each interval (empty or non-empty, bounded or unbounded) coinsides with the set of all limit points of the sequence $\{q\sum_{j=1}^{n} u_j/j\}$ for suitably chosen $u_1, u_2, \ldots$ with $u_i = \pm 1$. Here q denotes the number $\frac{1}{2}\big(\sum_{k=1}^{\infty} 1/ke^{k!}\big)^{-1}$. Put $x_0 = 0$, $x_k = u_k e^{k!}$ $(k = 1, 2, \ldots)$, $p_0 = \frac{1}{2}$, $p_k = q/ke^{k!}$ $(k = 1, 2, \ldots)$ and

$$P(E) = \sum_{x_k\in E} p_k.$$

Of course, $\mu_P = 0$. Since, by a simple calculation,

$$T\sum_{k=1}^{\infty}\frac{e^{k!}}{k(T^2+e^{2k!})} \leqslant \frac{a}{n(T)}, \tag{3.1}$$

where a is a constant and the integer $n(T)$ is defined by the condition $n(T)! \leqslant \log T < (n(T)+1)!$, we have the relation

$$\lim_{T\to\infty} T \int_{-\infty}^{\infty} \frac{x^2}{T^2+x^2} P(dx) = 0.$$

Moreover,

$$\int_{-T}^{T} xP(dx) = q \sum_{j=1}^{n(T)} \frac{u_j}{j},$$

which, by corollary to Lemma 3.1 and by Lemma 3.2, shows that the set $\mathscr{C}_P$ coincides with the interval in question. The theorem is thus proved.

Our aim is to define a numerical constant associated with probability measures which could be regarded as a simultaneous generalization of both notions — the mean value and the substitute of the mean value. Let $\mathfrak{C}$ be the set of all probability measures P for which $\mathscr{C}_P$ is a one-point set. The only element of $\mathscr{C}_P$ will be denoted by $c(P)$ and will be called the *convex mean value* of P.

THEOREM 3.2. *$\mathfrak{L} \subset \mathfrak{C}$ and $l(P) = c(P)$ for $P \in \mathfrak{L}$.*

Proof. Given $P \in \mathfrak{L}$, we have, by the ergodic theorem, the convergence $\frac{1}{n}\sum_{j=1}^{n} X_j \to l(P)$ in probability. Here $X_1, X_2, \ldots$ denote the generators of $\mathscr{H}_P$. Consequently, $l(P) \in \mathscr{C}_P$. On the other hand $a_P = b_P = l(P)$, which, by Lemma 3.2, gives the formula $\mathscr{C}_P = \{l(P)\}$. The theorem is thus proved.

THEOREM 3.3. *$\mathfrak{S} \subset \mathfrak{C}$ and $s(P) = c(P)$ for $P \in \mathfrak{S}$.*

Proof. Let $P \in \mathfrak{S}$. Given $b \neq s(P)$, we have the relation $1 \in \mathscr{M}_P$. Consequently, by Remark 2.1 in [2], there exist a sequence $n_1 < n_2 < \ldots$ of integers and a sequence $a_1, a_2, \ldots$ of real numbers such that

$$a_k \sum_{j=1}^{n_k} (X_j - b) \to 1 \tag{3.2}$$

in probability. Here $X_1, X_2, \ldots$ denote the generators of $\mathscr{M}_P$. Of course, without loss of generality we may assume that the sequence $\{a_k n_k\}$ has a finite or infinite limit, say g. If g is infinite, then (3.2) yields the relation

$$\frac{1}{n_k}\sum_{j=1}^{n_k}\big(X_j - s(P)\big) = \frac{1}{a_k n_k}\sum_{j=1}^{n_k}(X_j - b) + b - s(P) \to b - s(P)$$

in probability. Further, the equation $g = 0$ and (3.2) imply the relation

$$a_k \sum_{j=1}^{n_k}\big(X_j - s(P)\big) = a_k \sum_{j=1}^{n_k}(X_j - b) + a_k n_k\big(b - s(P)\big) \to 1$$

in probability. Consequently, in both cases the space $\mathcal{M}_{P_{s(P)}}$ would contain a non-zero constant random variable which contradicts the definition of the quantity $s(P)$. Thus $0 < |g| < \infty$ and, consequently, by

$$\frac{1}{n_k}\sum_{j=1}^{n_k}(X_j-b)\to g^{-1} \tag{3.2}$$

in probability. Hence we get the relation

$$\frac{1}{n_k}\sum_{j=1}^{n_k}\big(X_j-s(P)\big)\to g^{-1}+b-s(P).$$

Since 0 is the only constant random variable in $\mathcal{M}_{P_{s(P)}}$, we have the formula $g^{-1}+b-s(P) = 0$. Consequently,

$$\frac{1}{n_k}\sum_{j=1}^{n_k}X_j\to s(P)$$

in probability which shows that $s(P)\in\mathscr{C}_P$. On the other hand, $\mathcal{M}_{P_{s(P)}}$ contains exactly one constant random variable and, consequently, $\mathscr{C}_{P_{s(P)}}$ is at most a one-point set. Hence it follows that $\mathscr{C}_P$ is also at most a one--point set. This yields the equality $\mathscr{C}_P = \{s(P)\}$. The theorem is thus proved.

THEOREM 3.4. $\mathfrak{C}\setminus(\mathfrak{L}\cup\mathfrak{S}) \neq \emptyset$.

Proof. We define an auxiliary sequence $v_1, v_2, \ldots$ recursively. Put $v_1 = 1$. Further, if $v_1, v_2, \ldots, v_n$ are already defined, then we put $v_{n+1} = 1$ whenever $\sum_{j=1}^{n}\frac{v_j}{j} \leqslant (n+1)^{-\frac{1}{2}}$ and $v_{n+1} = -1$ in the remaining case.

Set $y_0 = 0$, $y_k = v_k e^{k!}$ $(k = 1, 2, \ldots)$, $p_0 = \frac{1}{2}$, $p_k = q/ke^{k!}$ $(k = 1, 2, \ldots)$, where $q = \frac{1}{2}\big(\sum_{k=1}^{\infty} 1/ke^{k!}\big)^{-1}$ and

$$P(E) = \sum_{y_k\in E} p_k.$$

Evidently, $\mu_P = 0$ and $P\notin\mathfrak{L}$. Moreover, by (3.1), we get the inequality

$$T\int_{-\infty}^{\infty}\frac{x^2}{T^2+x^2}P(dx)\leqslant\frac{a}{n(T)}, \tag{3.3}$$

where a is a constant and the integer $n(T)$ is defined by the condition $n(T)!\leqslant\log T < \big(n(T)+1\big)!$

It is very easy to prove, for instance by induction with respect to n, the inequalities

$$n^{-\frac{1}{2}}-n^{-1}\leqslant\sum_{j=1}^{n}\frac{v_j}{j}\leqslant n^{-\frac{1}{2}}+n^{-1}.$$

Since

$$\int_{-T}^{T} xP(dx) = q\sum_{j=1}^{n(T)} \frac{v_j}{j},$$

we have the inequality

$$\left|\int_{-T}^{T} xP(dx) - \frac{q}{\sqrt{n(T)}}\right| \leqslant \frac{q}{n(T)}. \tag{3.4}$$

Hence, in particular, it follows that

$$\lim_{T\to\infty} \int_{-T}^{T} xP(dx) = 0,$$

which by (3.3) and Lemmas 3.1 and 3.2 yields the equality $\mathscr{C}_P = \{0\}$. Thus $P \in \mathfrak{C}$ and $c(P) = 0$. Moreover, for every b, $\mathscr{C}_{P_b} = \{b\}$. Consequently, for every $b \neq 0$ the measure P_b is not centered in probability. Further, by (3.3) and (3.4),

$$\lim_{T\to\infty} \frac{T\int_{-\infty}^{\infty} \frac{x^2}{T^2+x^2} P(dx)}{\int_{-T}^{T} xP(dx)} = 0$$

which, by Theorem 2.1 in [2] proves that P itself is not centered in probability. Thus $P \notin \mathfrak{S}$, which completes the proof.

References

[1] B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, Moscow–Leningrad 1949 (in Russian).

[2] A. Kamiński and K. Urbanik, *Centered probability distributions*, Ann. Soc. Math. Polon., Ser. I: Comm. Math. 14 (1970), p. 65–73.

Reçu par la Rédaction le 18. 5. 1973