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Theorems analogous to those of Weierstrass and Mittag-Leffler for harmonic functions of n variables *

by ALAN S. COVER (Arizona)

1. Introduction. Let E^n be the space of n real variables $x_1, \ldots, x_n$, D a domain, and $x = (x_1, \ldots, x_n)$ a point of E^n. In this paper we consider real or complex valued harmonic functions, $u(x)$, defined on D and in particular functions called characteristic harmonic functions which are defined below. The well known theorems of Weierstrass and Mittag-Leffler concerning entire and meromorphic functions, $f(z)$, can be extended to harmonic functions, $u(x)$, of n real variables. The author wishes to acknowledge the useful suggestions of F. Leja.

The results of this paper can be generalized to solutions, $u(x)$, of elliptic differential equations

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$$

with constant coefficients which can be transformed into the Laplacian Δu by a linear transformation.

2. Characteristic harmonic functions. A real or complex valued harmonic function, $u(x)$, defined in a domain $D \subset E^n$ is called a *characteristic harmonic function* or more briefly, a c.h. function in D, if for every function $f(z)$ of a complex variable z and which is analytic in the set $u(D)$ the composed function

$$F(x) = f[u(x)] \tag{1}$$

is also harmonic in D.

LEMMA 1. *A necessary and sufficient condition that a harmonic function $u(x)$, in a domain D be c.h. is that in D $u(x)$ satisfies the equation*

$$\delta u \equiv \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j}\right)^2 = 0\,. \tag{2}$$

* Most of the results of this paper are contained in the author's doctoral thesis, The Pennsylvania State University, June 1964.

Proof. The condition is necessary, for suppose that $u(x)$ is c.h. in D. Then since the function in (1) is harmonic,

$$\Delta F = f'\Delta u + f''\delta u, \tag{3}$$

and $\Delta u = 0$, condition (2) follows since f is arbitrary.

Condition (2) is sufficient, since if $\Delta u = 0$ and $\delta u = 0$ then it follows from (3) that $\Delta F = 0$.

Let S be any $n-2$ dimensional hyperplane of space E^n. It can always be represented by two linear equations

$$\alpha_1 x_1 + \ldots + \alpha_n x_n + \alpha_0 = 0, \qquad \beta_1 x_1 + \ldots + \beta_n x_n + \beta_0 = 0 \tag{4}$$

where the coefficients are real and satisfy the conditions

$$\sum_{j=1}^{n} \alpha_j^2 = \sum_{j=1}^{n} \beta_j^2 = 1 \quad \text{and} \quad \sum_{j=1}^{n} \alpha_j \beta_j = 0. \tag{5}$$

It is clear that the equations (4) can be replaced by the single equation with complex coefficients, namely

$$u_S(x) \equiv \sum_{j=1}^{n} a_j x_j + a_0 = 0 \quad \text{where} \quad a_k = \alpha_k + i\beta_k. \tag{6}$$

We note that the function $u_S(x)$ is harmonic in E^n, satisfies equation (2) and is zero only on the hyperplane S. We call this function the *linear c.h. function of S*. If $a_0 = 0$, S contains the origin of the coordinates.

LEMMA 2. *If $f(z)$ is an entire function which is zero only at a finite or infinite number of points, $z_1, z_2, \ldots$ and $u_S(x)$ is the linear c.h. function defined by* (6) *where $a_0 = 0$, then*

$$F(x) = f[u_S(x)]$$

is a c.h. function in E^n which is zero only on the set $E = \bigcup_k S_k$, where S_k, $k = 1, 2, \ldots$, is the hyperplane parallel to S which is defined by the equation

$$\sum_{j=1}^{n} a_j x_j = z_k.$$

Proof. From (3)

$$\Delta F = f'\Delta u_S + f''\delta u_S$$

and since $u_S(x)$ is c.h., we have that $\Delta F = 0$, hence F is harmonic in E^n. Furthermore, $\delta F = (f')^2 \delta u_S = 0$, hence F is c.h. and is zero only if $u_S(x) = z_k$, that is, on the hyperplanes S_k, $k = 1, 2, \ldots$, which is what we need to show.

We denote by C^n the space of n complex variables $z_1, \ldots, z_n$, where $z_j = x_j + iy_j$, and denote by z the point in C^n with the coordinates $z_1, \ldots, z_n$. The space E^n is a real hyperplane of n dimensions of C^n.

Let S be a hyperplane given by (6), N a neighborhood of S in E^n, and $h(x)$ a harmonic function in the domain $N \setminus S$. We have ([1]) that $h(x)$ can be (i) developed in a power series of real variables $x_1, \ldots, x_n$ in some neighborhood of any point of $x^0 \in N \setminus S$ and (ii) analytically continued to complex variables to the function $h(z) = h(z_1, \ldots, z_n)$ analytic in a domain $D \setminus S$, where D is a neighborhood of S in C^n and D contains the subset of N of E^n. Let $u_S(z)$ be the analogous continuation of $u_S(x)$.

Definition. If there exists an integer $m \neq 0$ such that for every point $x \in S$ the limit

$$\lim_{\substack{z \to x \\ (z \in D \setminus S)}} h(z)[u_S(z)]^{-m} = g(x),$$

exists, is finite, and non-zero on S, then we say that $h(x)$ has a *characteristic zero of order m*, if $m > 0$, a *characteristic pole of order $-m$*, if $m < 0$.

On the other hand, we say that two harmonic functions $h_1(x)$ and $h_2(x)$, having poles on a hyperplane S, have the same *principal parts* if for every point $x \in S$ the limit

$$\lim_{\substack{z \to x \\ (z \in D \setminus S)}} [h_2(z) - h_1(z)] = a(x)$$

exists, is finite, and the extension of the difference $h_2(x) - h_1(x)$ by this limit is harmonic on S.

3. Theorem analogues to those of Weierstrass and Mittag-Leffler. Let

$$S_1, S_2, \ldots \tag{8}$$

be a finite or infinite sequence of $n-2$ dimensional hyperplanes of E^n not having any points in common and n_k a positive integer corresponding to S_k for $k = 1, 2, \ldots$ If the sequence (8) is infinite then the distance of S_k to the origin of the coordinates tends to infinity with k.

Theorem 1. *If the hyperplanes* (8) *are parallel then there exists a harmonic function $w(x)$ in E^n having a characteristic zero of order n_k on each S_k, $k = 1, 2, \ldots$ and elsewhere is non-zero.*

Proof. Let $u_S(x) = \sum_{j=1}^{n} a_j x_j$ be a linear c.h. function which is zero on the hyperplane, S, parallel to each S_k and containing the origin of the coordinate system. Then S_k is given by an equation of the form

$u_S(x) = z_k$, where z_k is a complex number which tends to infinity as k approaches infinity if the sequence (8) is not finite.

Let $f(z)$ be an entire function having a zero of order n_k at z_k for $k = 1, 2, \ldots$ and only these zeros. From Lemma 2 the function $w(x) = f[u_S(x)]$ is c.h. in E^n, has a characteristic zero of order n_k on S_k for $k = 1, 2, \ldots$, and is otherwise non-zero.

COROLLARY. *If $w(x)$ and $v(x)$ are two functions which are harmonic in E^n and satisfy the conditions of Theorem* 1, *then*

$$v(x) = e^{h(x)} w(x)\,, \tag{9}$$

where $h(x)$ is an analytic function of the variables $x_1, \ldots, x_n$ in E^n and satisfies the differential equation

$$w(x)[\Delta h + \delta h] + 2 \sum_{j=1}^{n} \frac{\partial h}{\partial x_j} \cdot \frac{\delta w}{\delta x_j} = 0\,. \tag{10}$$

Conversely, if $w(x)$ satisfies the conditions of Theorem 1 *and $h(x)$ is analytic in E^n and satisfies* (10), *then the function* (9) *satisfies the conditions of Theorem* 1.

Proof. For let $w(z)$ and $v(z)$, respectively, be the analytic continuations of $w(x)$ and $v(x)$ in a domain $D \supset E^n$ of the space C^n. In a neighborhood of each S_k the functions

$$w(z)[u_{S_k}(z)]^{-n_k}, \qquad v(z)[u_{S_k}(z)]^{-n_k}$$

are non-zero and analytic hence the quotient $v(z)/w(z)$ is a non-zero analytic function of the variables $z_1, \ldots, z_n$ in some neighborhood of each point of E^n. Consequently, $h(z) = \log v(z)/w(z)$ is an analytic function in a domain D^* contained in D and containing E^n and relation (9) follows. Since $\Delta v = \Delta(e^h w) = 0$, relation (10) holds.

Conversely, if $h(x)$ is analytic in E^n and satisfies (10) then the function defined in (9) is harmonic in E^n since

$$\Delta v = e^h\left\{\Delta w + w(\Delta h + \delta h) + 2 \sum_{j=1}^{n} \frac{\delta h}{\delta x_j} \cdot \frac{\delta w}{\delta x_j}\right\} = 0 \tag{11}$$

and has the same zeros as $w(x)$.

Remark 1. The function $w(x)$ is c.h. since $\delta w = f'^2 \delta u_S(x) = 0$. Since

$$\delta v = e^{2h}\left\{w + w\left(w\delta h + 2 \sum_{j=1}^{n} \frac{\delta h}{\delta x_j} \cdot \frac{\delta w}{\delta x_j}\right)\right\}, \tag{12}$$

it follows from (11) that if $h(x)$ satisfies (10) and is harmonic then $v(x)$ is also c.h. because if $\Delta h = 0$ then $\delta v = 0$.

THEOREM 2. *For every sequence of disjoint hyperplanes* (8), *parallel or not, such that if the sequence is not finite their distance from the origin tends to infinity and for any sequence of positive integers n_k there exists a function, $M(x)$, which is harmonic in E^n except on the hyperplanes* (8) *and on these hyperplanes $M(x)$ has a characteristic pole of order n_k on S_k, $k = 1, 2, \ldots$*

Proof. Let

$$u_{S_k}(x) \equiv \sum_{j=1}^{n} a_j^{(k)} x_j + z_k = 0, \qquad k = 1, 2, \ldots \tag{13}$$

which is the equation of the hyperplane S_k where

$$a_j^{(k)} = a_j^{(k)} + i\beta_j^{(k)}, \qquad \sum_{j=1}^{n} a_j^{(k)2} = \sum_{j=1}^{n} \beta_j^{(k)2} = 1, \qquad \sum_{j=1}^{n} a_j^{(k)} \beta_j^{(k)} = 0 \tag{14}$$

and z_k is a complex number which tends to infinity with $k \to \infty$. We can assume that $z_k \neq 0$ for $k \geqslant \nu$. In the ball

$$B_k = \left\{x \,\middle|\, \sum_{j=1}^{n} x_j^2 < 1/2 |z_k|^2\right\}, \qquad k \geqslant \nu, \tag{15}$$

we see from (14) that $u_k(x) = \sum_{1}^{n} a_j^{(k)} x_j$ satisfies the inequality

$$|u_k|^2 \leqslant \Big(\sum_{j=1}^{n} a_j^{(k)} x_j\Big)^2 + \Big(\sum_{j=1}^{n} \beta_j^{(k)} x_j\Big)^2 \leqslant \Big(\sum_{j=1}^{n} a_j^{(k)2} + \sum_{j=1}^{n} \beta_j^{(k)2}\Big)\Big(\sum_{j=1}^{n} x_j^2\Big) < |z_k|^2 .$$

Since for $k \geqslant \nu$ and $x \in B_k$ we have

$$\frac{1}{u_{S_k}(x)} = \frac{1}{u_k(x) + z_k} = \frac{1}{z_k} \sum_{p=0}^{\infty} \Big(-\frac{u_k(x)}{z_k}\Big)^p .$$

Hence, for $k \geqslant \nu$ and any number A_k,

$$\frac{A_k}{[u_{S_k}(x)]^{n_k}} = \sum_{p=0}^{\infty} C_p^{(k)} [u_k(x)]^p, \qquad x \in B_k ,$$

where $c_p^{(k)}$ are constants and the series converges uniformly in every closed set contained in B_k.

Let B_k^1 be the closed ball $\{x \mid \sum x_j^2 \leqslant 1/4 |z_k|^2\}$. There exists a positive integer N_k such that

$$\left|\frac{A_k}{[u_{S_k}(x)]^{n_k}} - \sum_{p=0}^{N_k} C_p^{(k)} u_k(x)^p\right| < \Big(\frac{1}{2}\Big)^k \qquad \text{for} \qquad x \in B_k^1 . \tag{17}$$

Set

$$G_k(x) = \sum_{p=0}^{N_k} C_p^{(k)} u_k(x)^p \text{ if } k \geqslant \nu \quad \text{and} \quad G_k(x) \equiv 0 \text{ if } k < \nu .$$

Therefore, the series

$$M(x) = \sum_{k=0}^{\infty} \left\{ \frac{A_k}{[u_{S_k}(x)]^{n_k}} - G_k(x) \right\} \tag{18}$$

converges uniformly in every closed bounded set which is disjoint from $\bigcup S_k$. For every k the kth term of (18) is a harmonic function in $E^n \setminus S_k$, since it is an analytic function of the c.h. function $u_{S_k}(x)$. Hence, $M(x)$ is a harmonic function in $E^n \setminus \bigcup S_k$. Moreover, if the A_k's are different from zero then $M(x)$ has a characteristic pole of order n_k on each S_k which proves the theorem.

Corollary. *If two harmonic functions $M(x)$ and $N(x)$ satisfy the conditions of Theorem 2 and have the same principal parts on each S_k, then*

$$N(x) = M(x) + h(x)$$

where $h(x)$ is a harmonic function in all of the space E^n.

Indeed, the difference $N(x) - M(x)$ is harmonic on S_k follows from the definition of principal parts, hence, the function $h(x)$ is harmonic in all of the space E^n.

Reference

[1] S. Bochner and W. Martin, *Several Complex Variables*, Princeton 1948.

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