ANNALES
POLONICI MATHEMATICI
XLVII (1986)

Christensen measurability of polynomial functions and convex functions of higher orders

by ZBIGNIEW GAJDA (Katowice)

Abstract. In this paper some results of P. Fischer and Z. Słodkowski [3] concerning the Christensen measurability of homomorphisms and Jensen convex functions are generalized to the case of polynomial functions and convex functions of higher orders. In particular, it is shown that any Christensen measurable n-convex function defined on an open convex subset of a real linear topological Polish space is continuous.

1. The concept of Haar zero sets in Abelian topological Polish groups was introduced by J. P. R. Christensen [1]. P. Fischer and Z. Słodkowski [3] considered a notion of measurability in such groups related to the Haar zero sets and they called it *Christensen measurability*. They have shown that any Christensen measurable homomorphism $f\colon X \to Y$ between two Abelian topological Polish groups $(X, +)$ and $(Y, +)$ is continuous. In the same paper it has been proved that Christensen measurability of a Jensen convex function $f\colon X \to \boldsymbol{R}$ defined on a linear topological Polish space X implies its continuity. The aim of the present paper is to extend the above results on polynomial functions and convex functions of higher orders. In the case of arbitrary Abelian Polish groups we obtain also some theorems similar to the S. Kurepa's results concerning certain properties of sets of positive Lebesgue measure in Euclidean spaces.

2. Let $(X, +)$ be an Abelian topological Polish group. We consider the σ-algebra $\mathfrak{M}$ of all universally measurable subsets of X, i.e., the intersection of all completions of the Borel σ-algebra with respect to probability Borel measures. In $\mathfrak{M}$ we distinguish the family $\mathscr{H}_0$ of all Haar zero sets, i.e., sets $A \in \mathfrak{M}$ with the property that there exists a probability measure μ on $\mathfrak{M}$ such that $\mu(A+x) = 0$ for each $x \in X$ (cf. [1]). By $\mathscr{C}_0$ we denote the class of all subsets of sets belonging to $\mathscr{H}_0$ and we call them *Christensen zero sets* (cf. [3], Definition 1). Finally, $\mathscr{C}$ is the class of all sets $A \subset X$ which have the form $A = B \cup C$, where $B \in \mathfrak{M}$ and $C \in \mathscr{C}_0$. Such sets are referred to as Christensen measurable sets (cf. [3], Definition 2).

The letters $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ will always denote the sets of all positive integers, integers, rationals and reals, respectively.

Let us begin with some facts which will be used in the sequel.

LEMMA 1 (see [1], Theorem 1). *If $\mathscr{A}$ is a countable family of sets from $\mathscr{H}_0$, then $\bigcup \mathscr{A} \in \mathscr{H}_0$.*

LEMMA 2 (see [1] and [3]). *The class $\mathscr{C}_0$ is a proper linearly invariant σ-ideal of subsets of X, that means:*

(1) $\mathscr{C}_0$ *is a σ-ideal of subsets of X such that* $X \notin \mathscr{C}_0$;

(2) $A \in \mathscr{C}_0, \quad x \in X \quad$ *implies* $\quad x + A \in \mathscr{C}_0$;

(3) $A \in \mathscr{C}_0, \quad \alpha \in \mathbf{Z} \setminus \{0\} \quad$ *implies* $\quad \alpha \cdot A \in \mathscr{C}_0$.

If X is not only a topological group but if forms a real linear topological space, then

(3′) $A \in \mathscr{C}_0, \quad \alpha \in \mathbf{R} \setminus \{0\} \quad$ *implies* $\quad \alpha \cdot A \in \mathscr{C}_0$.

The phrase "linearly invariant" refers to conditions (2), (3) and (3′).

LEMMA 3 (see [3], Proposition 1). *The family $\mathscr{C}$ is a linearly invariant σ-algebra of subsets of X containing $\mathscr{C}_0$ and the Borel σ-algebra.*

COROLLARY 1 (see [3], Corollary 1). *If X is a countable union of sets from $\mathscr{C}$, then at least one of them does not belong to $\mathscr{C}_0$.*

LEMMA 4. *If U is an open non-empty subset of X, then U is not in $\mathscr{C}_0$.*

Proof. Let P be a countable dense subset of X. Then:

$$X = \bigcup_{x \in P} (x + U).$$

We deduce from Corollary 1 that $x + U \notin \mathscr{C}_0$ for some $x \in P$, and hence $U \notin \mathscr{C}_0$.

COROLLARY 2. *If an open non-empty subset of X is covered by a countable subfamily of $\mathscr{C}$, then at least one element of the family does not belong to $\mathscr{C}_0$.*

The following lemma is due to S. Kurepa (see e.g. [7], Theorem 1; cf. also [8], Lemma 1). Such and similar results are usually formulated for Lebesgue measure in Euclidean space. Here, we present this lemma in a more general setting:

LEMMA 5. *Let $(Y, +)$ be a topological locally compact Abelian group and let h be a Haar measure on Y. If $E \subset Y$ is a universally measurable set with $h(E) > 0$, then for each $n \in \mathbf{N}$:*

$$0 \in \operatorname{int}\{z \in Y\colon \bigcap_{i=-n}^{n} (E + iz) \neq \emptyset\}.$$

For each $n \in \mathbf{N}$ and any $A \subset X$ we put:

$$F_n^*(A) := \{x \in X\colon \bigcap_{i=-n}^{n} (A + ix) \notin \mathscr{H}_0\},$$

$$F_n(A) := \{x \in X\colon \bigcap_{i=-n}^{n} (A + ix) \neq \emptyset\}.$$

Now, using a similar method as in the proof of Theorem 2, [1], we can obtain a generalization of Lemma 5.

THEOREM 1. *For any $n\in N$ and any $A\in\mathfrak{M}$ the set $F_n^*(A)$ is open (possibly empty).*

Proof. Let ϱ be a translation invariant metric on X generating the topology in X, i.e.,

$$\varrho(x+z, y+z) = \varrho(x, y), \qquad x, y, z\in X$$

(cf. [6], p. 210, and [2], p. 86). Such a metric has to be complete (cf. [2], Theorem 5.4).

Suppose that $F_n^*(A)\neq\emptyset$ and fix an $x\in F_n^*(A)$. Setting $B:=\bigcap_{i=-n}^{n}(A+ix)$, we have $B\notin\mathscr{H}_0$. We shall show that $x+F_n^*(B)\subset F_n^*(A)$. Indeed, $B\subset A+ix$, for $i=-n,\ldots,n$, whence

$$\begin{aligned} x+F_n^*(B) &= \{y\in X:\ \bigcap_{i=-n}^{n}(B+i(y-x))\notin\mathscr{H}_0\} \\ &\subset \{y\in X:\ \bigcap_{i=-n}^{n}[(A+ix)+i(y-x)]\notin\mathscr{H}_0\} \\ &= \{y\in X:\ \bigcap_{i=-n}^{n}(A+iy)\notin\mathscr{H}_0\} \\ &= F_n^*(A). \end{aligned}$$

It is enough to show that $0\in\operatorname{int}F_n^*(B)$. If not, we can choose a sequence $(x_k)_{k\in N}$ of elements of X such that

$$x_k\notin F_n^*(B), \qquad k\in N, \tag{4}$$

$$\varrho(x_k, 0)\leqslant 1/2^k, \qquad k\in N. \tag{5}$$

Put $C:=B\setminus\bigcup_{k=1}^{\infty}\big[\bigcap_{i=-n}^{n}(B+ix_k)\big]$. Since (4) implies $\bigcap_{i=-n}^{n}(B+ix_k)\in\mathscr{H}_0$, $k\in N$, and countable union of sets from $\mathscr{H}_0$ remains in $\mathscr{H}_0$, we conclude that $C\notin\mathscr{H}_0$.

Take an arbitrary sequence $a\in\{-n,\ldots,-1,0,1,\ldots,n\}^N$ and let $p, q\in N$, $p<q$. Then we have

$$\begin{aligned} \varrho\Big(\sum_{k=1}^{p}a(k)x_k, \sum_{k=1}^{q}a(k)x_k\Big) &= \varrho\Big(\sum_{k=p+1}^{q}a(k)x_k, 0\Big)\leqslant\sum_{k=p+1}^{q}\varrho(a(k)x_k, 0) \\ &= \sum_{k=p+1}^{q}\varrho(|a(k)|x_k, 0)\leqslant\sum_{k=p+1}^{q}|a(k)|\varrho(x_k, 0) \\ &\leqslant n\sum_{k=p+1}^{q}\varrho(x_k, 0)\leqslant n\sum_{k=p+1}^{q}1/2^k\leqslant n/2^p. \end{aligned}$$

Thus, for any $a \in \{-n, \ldots, -1, 0, 1, \ldots, n\}^N$, the sequence $(\sum_{i=1}^{k} a(i) x_i)_{k \in N}$, being fundamental, is convergent (even uniformly with respect to a).

We define the mapping

$$\alpha\colon \{0, 1, \ldots, 2n\} \to \{-n, \ldots, -1, 0, 1, \ldots, n\}$$

by

$$\alpha(j) := \begin{cases} j & \text{for } j \in \{0, 1, \ldots, n\}, \\ j-(2n+1) & \text{for } j \in \{n+1, \ldots, 2n\}. \end{cases}$$

For each $j \in \{0, 1, \ldots, 2n\}$ we determine the function

$$\beta_j\colon \{0, 1, \ldots, 2n\} \to \{-n, \ldots, -1, 0, 1, \ldots, n\} - \alpha(j);$$

$$\beta_j(i) := \begin{cases} i & \text{for } i \in \{0, \ldots, n-\alpha(j)\}, \\ i-(2n+1) & \text{for } i \in \{n+1-\alpha(j), \ldots, 2n\}. \end{cases}$$

Let $(\{0, 1, \ldots, 2n\}, \oplus) = Z_{2n+1}$ denote the group of integers modulo $2n+1$. By an immediate calculation one checks that the equality

$$\alpha(j \oplus i) = \alpha(j) + \beta_j(i) \tag{6}$$

holds for all $i, j \in \{0, 1, \ldots, 2n\}$. Let $Y := \{0, 1, \ldots, 2n\}^N$. The transformation

$$\varphi\colon Y \to X, \quad \varphi(a) := \sum_{k=1}^{\infty} \alpha(a(k)) x_k, \quad a \in Y,$$

is well defined.

We equip Y with the addition operation $+$ determined as follows:

$$a+b := (a(1) \oplus b(1), a(2) \oplus b(2), \ldots)$$

for $a := (a(1), a(2), \ldots)$, $b := (b(1), b(2), \ldots)$.

The structure $(Y, +) = (Z_{2n+1})^N$ is an Abelian compact group with the product topology (in $\{0, 1, \ldots, 2n\}$ the discrete topology is considered). It is easily seen that the transformation φ is continuous.

Let h be the normed Haar measure on Y. If it were $h(\varphi^{-1}(y+C)) = 0$ for each $y \in X$, the formula

$$\mu(E) := h(\varphi^{-1}(E)), \quad E \in \mathfrak{M},$$

would define a probability measure μ on $\mathfrak{M}$ such that $\mu(y+C) = 0$, $y \in X$, contrary to the condition $C \notin \mathscr{H}_0$.

Consequently, we can choose a $y \in X$ with $h(\varphi^{-1}(y+C)) > 0$. In view of Lemma 5, we have

$$0 \in \operatorname{int} \{a \in Y\colon \bigcap_{i=-n}^{n} [\varphi^{-1}(y+C)+ia] \neq \emptyset\}.$$

Put $e_k := (0, \ldots, 0, 1, 0, \ldots)$, where 1 stands on the kth place, $k \in N$. Since $e_k \xrightarrow[k\to\infty]{} 0$, we get

$$e_N \in \operatorname{int} \{a \in Y: \bigcap_{i=-n}^{n} [\varphi^{-1}(y+C)+ia] \neq \emptyset\}$$

for sufficiently large $N \in N$, whence

$$\bigcap_{i=-n}^{n} [\varphi^{-1}(y+C)+ie_N] \neq \emptyset.$$

Therefore, there exists an $a \in Y$ such that

$$\varphi(a-ie_N) \in y+C, \quad i = -n, \ldots, -1, 0, 1, \ldots, n. \tag{7}$$

Since the equality $-i = (2n+1)-i$ holds true in Z_{2n+1} for $i = 1, \ldots, n$, we can write (7) in an equivalent form:

$$\varphi(a+ie_N) \in y+C, \quad i = 0, 1, \ldots, 2n. \tag{8}$$

From (6), setting $j := a(N)$, and from the definition of φ, we obtain:

$$\varphi(a+ie_N) = \varphi(a)+\beta_j(i)\,x_N, \quad i = 0, 1, \ldots, 2n.$$

Hence and from (8) it follows that

$$\varphi(a)+\beta_j(i)\,x_N \in y+C, \quad i = 0, 1, \ldots, 2n,$$

and consequently:

$$\varphi(a)-y \in \bigcap_{i=0}^{2n} (C-\beta_j(i)\,x_N). \tag{9}$$

Since β_j is a bijection from $\{0, 1, \ldots, 2n\}$ onto $\{-n-\alpha(j), \ldots, n-\alpha(j)\}$, (9) gives

$$\varphi(a)-y \in \bigcap_{i=-n-\alpha(j)}^{n-\alpha(j)} (C-ix_N)$$

and finally

$$\varphi(a)-y-\alpha(j)\,x_N \in \bigcap_{i=-n}^{n} (C-ix_N) = \bigcap_{i=-n}^{n} (C+ix_N).$$

On the other hand, $C \subset B$, whence

$$\bigcap_{i=-n}^{n} (C+ix_N) \subset C \cap \bigcap_{i=-n}^{n} (B+ix_N) = \emptyset,$$

a contradiction. The proof is finished.

Corollary 3. *If* $A \in \mathfrak{M} \setminus \mathscr{H}_0$, *then* $0 \in F_n^*(A) \subset F_n(A)$ *for each* $n \in N$; *in particular* $0 \in \operatorname{int} F_n(A)$, $n \in N$.

COROLLARY 4. *For any* $A\in\mathscr{C}$ *and any* $n\in N$ *the set* $\{x\in X\colon \bigcap_{i=-n}^{n}(A+ix)\notin\mathscr{C}_0\}$ *is open (possibly empty).*

If $A\in\mathscr{C}\setminus\mathscr{C}_0$, *then for each* $n\in N$ *the relation* $0\in\{x\in X\colon \bigcap_{i=-n}^{n}(A+ix)\notin\mathscr{C}_0\}\subset F_n(A)$ *holds; in particular,* $0\in\operatorname{int}F_n(A)$, $n\in N$.

Proof. Suppose that $A\in\mathscr{C}$ has the decomposition $A=E\cup F$, where $E\in\mathfrak{M}$ and $F\in\mathscr{C}_0$. Then we get

$$\bigcap_{i=-n}^{n}(A+ix)=\bigcap_{i=-n}^{n}\big((E\cup F)+ix\big)$$
$$=\bigcap_{i=-n}^{n}(E+ix)\cup\bigcap_{i=-n}^{n}(F+ix)\cup\bigcup_{\substack{L\cup M=\{-n,\dots,n\}\\ L\neq\emptyset,\ M\neq\emptyset,\ L\cap M=\emptyset}}\Big[\bigcap_{i\in L}(E+ix)\cap\bigcap_{j\in M}(F+jx)\Big].$$

Since $F\in\mathscr{C}_0$ and $\mathscr{C}_0$ is a linearly invariant ideal, we have

$$\bigcap_{i=-n}^{n}(F+ix)\cup\bigcup_{\substack{L\cup M=\{-n,\dots,n\}\\ L\neq\emptyset,\ M\neq\emptyset,\ L\cap M=\emptyset}}\Big[\bigcap_{i\in L}(E+ix)\cap\bigcap_{j\in M}(F+jx)\Big]\in\mathscr{C}_0.$$

Hence we deduce that

$$\{x\in X\colon \bigcap_{i=-n}^{n}(A+ix)\notin\mathscr{C}_0\}=\{x\in X\colon \bigcap_{i=-n}^{n}(E+ix)\notin\mathscr{C}_0\}=F_n^*(E)$$

which, in view of Theorem 1, ends the proof of the first part of Corollary 4. The other part is obvious.

From now on, X will denote a real linear topological Polish space.

In the sequel the following sets will be useful (cf. [5], Definition 2):

$$H_n(A):=\{x\in X\colon \text{there exists an } h\in X \text{ such that } x-ih,\ x+ih\in A,$$
$$i=1,\dots,n\}=\Big\{x\in X\colon \bigcap_{i=1}^{n}\Big[\frac{1}{i}(A-x)\cap\frac{1}{i}(x-A)\Big]\neq\emptyset\Big\},\qquad A\subset X,\ n\in N.$$

LEMMA 6. *If* $A\in\mathscr{C}\setminus\mathscr{C}_0$, *then*

$$0\in\operatorname{int}\{x\in X\colon \bigcap_{i=1}^{n}[(A-x/i)\cap(A+x/i)]\neq\emptyset\}\quad \textit{for each } n\in N.$$

Proof.

$$\{x\in X\colon \bigcap_{i=1}^{n}[(A-x/i)\cap(A+x/i)]\neq\emptyset\}$$
$$=n!\,\{x\in X\colon \bigcap_{i=1}^{n}[(A-n!\,x/i)\cap(A+n!\,x/i)]\neq\emptyset\}$$
$$\supset n!\,\{x\in X\colon \bigcap_{i=-n!}^{n!}(A+ix)\neq\emptyset\}=n!\,F_{n!}(A).$$

It remains to use Corollary 4.

We introduce the following helpful

DEFINITION 1. Let $A \subset X$ and $U \subset X$, $U \neq \emptyset$. A point $x \in A$ is said to be *Q_U-internal point of A* iff

$$\bigwedge_{h \in U} \bigvee_{\varepsilon > 0} \bigwedge_{\lambda \in (-\varepsilon, \varepsilon) \cap Q} x + \lambda h \in A.$$

LEMMA 7. *Let $A \subset X$, $A \in \mathscr{C}$ and suppose that $U \subset X$ is not a Christensen zero set. If $x \in A$ is a Q_U-internal point of the set A, then*

$$x \in \operatorname{int} H_n(A) \quad \textit{for each } n \in N.$$

Proof. Let $x \in A$ be a Q_U-internal point of the set A and put

$$B := \bigcap_{i=1}^{n} \left[\frac{1}{i}(A - x) \cap \frac{1}{i}(x - A) \right].$$

We shall show that $U \subset \bigcup_{\alpha \in (0, \infty) \cap Q} \alpha \cdot B$.

Indeed, if $h \in U$, then there exists a $\lambda \in (0, \infty) \cap Q$ such that $x + i\lambda h \in A$ and $x - i\lambda h \in A$, for $i = 1, \ldots, n$. Such a λ exists in view of the fact that x is Q_U-internal. Thus, we have

$$h \in \frac{1}{\lambda} B \subset \bigcup_{\alpha \in (0, \infty) \cap Q} \alpha \cdot B.$$

If it were $\alpha \cdot B \in \mathscr{C}_0$ for each $\alpha \in (0, \infty) \cap Q$ we would have

$$\bigcup_{\alpha \in (0, \infty) \cap Q} \alpha \cdot B \in \mathscr{C}_0,$$

and consequently $U \in \mathscr{C}_0$; a contradiction. Thus, there exists an $\alpha \in (0, \infty) \cap Q$ such that $\alpha \cdot B \notin \mathscr{C}_0$, whence $B \in \mathscr{C} \setminus \mathscr{C}_0$. Setting

$$E := \left\{ y \in X \colon \bigcap_{i=1}^{n} [(B - y/i) \cap (B + y/i)] \neq \emptyset \right\}$$

by virtue of Lemma 6, we obtain $0 \in \operatorname{int} E$.

To prove that $x \in \operatorname{int} H_n(A)$ it suffices to show that $x + E \subset H_n(A)$. For, observe that

$$B \subset \frac{1}{i}(A - x), \quad B \subset \frac{1}{i}(x - A), \quad i = 1, \ldots, n,$$

and hence

$$x + E = \left\{ y \in X \colon \bigcap_{i=1}^{n} \left[\left(B - \frac{y - x}{i} \right) \cap \left(B + \frac{y - x}{i} \right) \right] \neq \emptyset \right\}$$

$$\subset \left\{ y \in X\colon \bigcap_{i=1}^{n} \left[\left(\frac{A-x}{i} - \frac{y-x}{i} \right) \cap \left(\frac{x-A}{i} + \frac{y-x}{i} \right) \right] \neq \emptyset \right\}$$

$$= \left\{ y \in X\colon \bigcap_{i=1}^{n} \left[\left(\frac{A-y}{i} \right) \cap \left(\frac{y-A}{i} \right) \right] \neq \emptyset \right\} = H_n(A).$$

This completes the proof.

3. In what follows X denotes a real linear topological Polish space, $D \subset X$ is a non-empty open convex set and $S \subset X$ is a cone with the property that $S \cup (-S) \cup \{0\} = X$.

Let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a relation $\leqslant$ which is compatible with the linear structure of Y, i.e.,

$$\bigwedge_{x,y,z \in Y} \bigwedge_{\alpha \in [0,\infty)} [x \leqslant y \text{ implies } x+z \leqslant y+z \text{ and } x \leqslant y \text{ implies } \alpha x \leqslant \alpha y].$$

Moreover, we assume that

$$(10) \qquad \bigwedge_{x,y \in Y} 0 \leqslant x \leqslant y \quad \text{implies} \quad \|x\| \leqslant \|y\|.$$

For example, the space $Y = C(T)$ of all real valued continuous functions defined on a compact space T with the norm

$$\|y\| := \sup\{|y(t)|\colon t \in T\}$$

and with the order

$$y_1 \leqslant y_2 \quad \text{iff} \quad y_1(t) \leqslant y_2(t), \quad t \in T,$$

has all the above properties.

Definition 2 (cf. [5], Definition 1). A function $f\colon D \to Y$ is called *n-convex* ($n \in N$) with respect to the cone S iff

$$\Delta_h^{n+1} f(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x+jh) \geqslant 0$$

holds for all $x \in D$ and $h \in S$ such that $x+ih \in D$ for $i = 0, 1, \ldots, n+1$.

Following T. Popoviciu [10], we note

Lemma 8. *If a function $f\colon D \to Y$ is n-convex with respect to the cone S, then for all rational numbers $\lambda_0, \lambda_1, \ldots, \lambda_{n+1}$ such that $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \lambda_{n+1} = 1$, the following inequality is satisfied*:

$$\sum_{j=0}^{n+1} (-1)^{n+1-j} V(\lambda_0, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{n+1}) f(x+\lambda_j h) \geqslant 0,$$

$$x \in D,\ h \in S,\ x+h \in D,$$

where

$$V(\alpha_1,\ldots,\alpha_{n+1}) := \begin{vmatrix} 1, & \alpha_1, & \alpha_1^2,\ldots, & \alpha_1^n \\ 1, & \alpha_2, & \alpha_2^2,\ldots, & \alpha_2^n \\ \cdots & \cdots & \cdots & \cdots \\ 1, & \alpha_{n+1}, & \alpha_{n+1}^2,\ldots, & \alpha_{n+1}^n \end{vmatrix} = \prod_{\substack{k,l=1\\ k>l}}^{n+1} (\alpha_k - \alpha_l)$$

for $\alpha_1,\ldots,\alpha_{n+1} \in \mathbf{R}$.

Let us remark that $V(\alpha_1,\ldots,\alpha_{n+1}) > 0$ whenever $\alpha_1 < \ldots < \alpha_{n+1}$, and $V(\alpha_1,\ldots,\alpha_{n+1}) = 0$ if there exist $i, j \in \{1,\ldots,n+1\}$, $i \neq j$, such that $\alpha_i = \alpha_j$.

LEMMA 9. *Suppose $f\colon D \to Y$ to be n-convex with respect to the cone S and put*

$$A := \{x \in D\colon \|f(x)\| < r\} \quad \textit{for some } r > 0.$$

Take an $x \in A$ and let $U := S \cap (D-x) \cap (x-D)$.

Then x is an Q_U-internal point of the set A.

Proof. Fix an $x \in A$. At first, we shall show that $U \neq \emptyset$. Indeed, since the mapping

$$\mathbf{R} \ni \alpha \to \alpha h \in X$$

is continuous and $(D-x) \cap (x-D)$ is a neighbourhood of zero, for an arbitrarily chosen $h \in S$, we can find an $\alpha_0 > 0$ such that $\alpha_0 h \in (D-x) \cap \cap (x-D)$. Since, evidently, $\alpha_0 h \in S$, we get $\alpha_0 h \in U$. Observe that $U = \{h \in S\colon x+h,\ x-h \in D\}$.

Take an $h \in U$. Let us choose rational numbers $\lambda, \lambda_1,\ldots,\lambda_{n-1}$ such that

$$0 = \lambda_n < \lambda < \lambda_{n-1} < \ldots < \lambda_1 < \lambda_0 = 1.$$

Put $y := x-h$, $\mu = 1-\lambda$, $\mu_j = 1-\lambda_j$, $j = 0, 1,\ldots,n$. Then $\mu, \mu_1,\ldots,\mu_{n-1} \in \mathbf{Q}$, $0 = \mu_0 < \mu_1 < \ldots < \mu_{n-1} < \mu < \mu_n = 1$ and $x-\lambda h = y+\mu h$, $x-\lambda_j h = y+\mu_j h$, $j = 0, 1,\ldots,n$. Using these denotations, we get

$$\sum_{j=0}^{n-1} (-1)^{n+1-j} V(\mu_0,\ldots,\mu_{j-1},\mu_{j+1},\ldots,\mu_{n-1},\mu,\mu_n) f(y+\mu_j h) - \\ - V(\mu_0,\ldots,\mu_{n-1},\mu_n) f(y+\mu h) + V(\mu_0,\ldots,\mu_{n-1},\mu) f(x) \geqslant 0,$$

whence

$$f(x-\lambda h) = f(y+\mu h) \leqslant \frac{V(\mu_0,\ldots,\mu_{n-1},\mu)}{V(\mu_0,\ldots,\mu_{n-1},\mu_n)} f(x) + \\ + \sum_{j=0}^{n-1} (-1)^{n+1-j} \frac{V(\mu_0,\ldots,\mu_{j-1},\mu_{j+1},\ldots,\mu_{n-1},\mu,\mu_n)}{V(\mu_0,\ldots,\mu_{n-1},\mu_n)} f(y+\mu_j h).$$

Now, the numbers $\lambda_1,\ldots,\lambda_{n-1}$ are regarded as fixed and λ is allowed to vary in the set $(0, \lambda_{n-1}) \cap \boldsymbol{Q}$. Since $\mu \to 1 = \mu_n$ whereas $\lambda \to 0^+$, we obtain

$$V(\mu_0, \ldots, \mu_{n-1}, \mu) \xrightarrow[\lambda \to 0^+]{} V(\mu_0, \ldots, \mu_{n-1}, \mu_n)$$

and

$$V(\mu_0, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_{n-1}, \mu, \mu_n)$$
$$\xrightarrow[\lambda \to 0^+]{} V(\mu_0, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_{n-1}, \mu_n, \mu_n) = 0 \quad \text{for } j = 0, 1, \ldots, n-1.$$

Therefore, there exist functions

$$a_1\colon (0, \lambda_{n-1}) \cap \boldsymbol{Q} \to \boldsymbol{R}, \quad b_1\colon (0, \lambda_{n-1}) \cap \boldsymbol{Q} \to Y$$

such that

(11) $$f(x-\lambda h) \leqslant a_1(\lambda) f(x) + b_1(\lambda), \quad \lambda \in (0, \lambda_{n-1}) \cap \boldsymbol{Q}$$

and

$$a_1(\lambda) \xrightarrow[\lambda \to 0^+]{} 1, \quad \|b_1(\lambda)\| \xrightarrow[\lambda \to 0^+]{} 0.$$

Choose, again, rational numbers $\lambda, \lambda_1, \ldots, \lambda_{n-2}$ satisfying the condition: $0 = \lambda_{n-1} < \lambda < \lambda_{n-2} < \ldots < \lambda_1 < \lambda_0 = 1$.

By the continuity of the transformation

$$\boldsymbol{R} \ni \lambda \to x - \lambda h \in X$$

one can find such a $\lambda_n \in (-\infty, 0) \cap \boldsymbol{Q}$ that $x - \lambda_n h \in D$.

Put $p := x - \lambda_n h$, $h' := p - (x-h) = (1-\lambda_n) h \in S$,

$$y := x - h, \quad \mu := \frac{1-\lambda}{1-\lambda_n}, \quad \mu_j := \frac{1-\lambda_j}{1-\lambda_n}, \quad j = 0, 1, \ldots, n.$$

Then $\mu_1, \ldots, \mu_{n-1}, \mu \in \boldsymbol{Q}$ and

$$0 = \mu_0 < \mu_1 < \ldots < \mu_{n-2} < \mu < \mu_{n-1} < \mu_n = 1.$$

With the aid of these notions we get

$$x - \lambda h = y + \mu h', \quad x - \lambda_j h = y + \mu_j h', \quad j = 0, 1, \ldots, n.$$

Moreover, $y + \mu_{n-1} h' = x$ and

$$\sum_{j=0}^{n-2} (-1)^{n+1-j} V(\mu_0, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_{n-2}, \mu, \mu_{n-1}, \mu_n) f(y + \mu_j h') +$$
$$+ V(\mu_0, \ldots, \mu_{n-2}, \mu_{n-1}, \mu_n) f(y + \mu h') -$$
$$- V(\mu_0, \ldots, \mu_{n-2}, \mu, \mu_n) f(y + \mu_{n-1} h') +$$
$$+ V(\mu_0, \ldots, \mu_{n-2}, \mu, \mu_{n-1}) f(y + \mu_n h') \geqslant 0.$$

Hence

$$f(x-\lambda h)=f(y+\mu h')\geqslant\frac{V(\mu_0,\ldots,\mu_{n-2},\mu,\mu_n)}{V(\mu_0,\ldots,\mu_{n-2},\mu_{n-1},\mu_n)}f(x)-$$

$$-\frac{V(\mu_0,\ldots,\mu_{n-2},\mu,\mu_{n-1})}{V(\mu_0,\ldots,\mu_{n-2},\mu_{n-1},\mu_n)}f(y+\mu_n h')-$$

$$-\sum_{j=0}^{n-2}(-1)^{n+1-j}\frac{V(\mu_0,\ldots,\mu_{j-1},\mu_{j+1},\ldots,\mu_{n-2},\mu,\mu_{n-1},\mu_n)}{V(\mu_0,\ldots,\mu_{n-2},\mu_{n-1},\mu_n)}f(y+\mu_j h').$$

Fix numbers $\lambda_1,\ldots,\lambda_n$ and let λ vary in the set $(0,\lambda_{n-2})\cap\boldsymbol{Q}$. If $\lambda\to 0^+$, then $\mu\to\mu_{n-1}$, whence

$$V(\mu_0,\ldots,\mu_{n-2},\mu,\mu_n)\xrightarrow[\lambda\to 0^+]{}V(\mu_0,\ldots,\mu_{n-2},\mu_{n-1},\mu_n),$$
$$V(\mu_0,\ldots,\mu_{n-2},\mu,\mu_{n-1})\xrightarrow[\lambda\to 0^+]{}V(\mu_0,\ldots,\mu_{n-2},\mu_{n-1},\mu_{n-1})=0$$

as well as

$$V(\mu_0,\ldots,\mu_{j-1},\mu_{j+1},\ldots,\mu_{n-2},\mu,\mu_{n-1},\mu_n)$$
$$\xrightarrow[\lambda\to 0^+]{}V(\mu_0,\ldots,\mu_{j-1},\mu_{j+1},\ldots,\mu_{n-2},\mu_{n-1},\mu_{n-1},\mu_n)=0,$$

for $j=0,1,\ldots,n-2$. Thus, there exist functions

$$c_1\colon(0,\lambda_{n-2})\cap\boldsymbol{Q}\to\boldsymbol{R},\qquad d_1\colon(0,\lambda_{n-2})\cap\boldsymbol{Q}\to Y$$

such that

$$c_1(\lambda)f(x)+d_1(\lambda)\leqslant f(x-\lambda h),\qquad \lambda\in(0,\lambda_{n-2})\cap\boldsymbol{Q}, \tag{12}$$
$$c_1(\lambda)\xrightarrow[\lambda\to 0^+]{}1,\qquad \|d_1(\lambda)\|\xrightarrow[\lambda\to 0^+]{}0.$$

On account of (11) and (12) one can find a $\delta_1>0$ such that

$$c_1(\lambda)f(x)+d_1(\lambda)\leqslant f(x-\lambda h)\leqslant a_1(\lambda)f(x)+b_1(\lambda)\quad\text{ for }\lambda\in(0,\delta_1)\cap\boldsymbol{Q}.$$

This yields

$$0\leqslant f(x-\lambda h)-(c_1(\lambda)f(x)-d_1(\lambda))\leqslant(a_1(\lambda)-c_1(\lambda))f(x)+b_1(\lambda)-d_1(\lambda)$$
$$\text{for }\lambda\in(0,\delta_1)\cap\boldsymbol{Q}.$$

Applying property (10) of the norm in Y, we obtain

$$\|f(x-\lambda h)\|-|c_1(\lambda)|\cdot\|f(x)\|-\|d_1(\lambda)\|\leqslant\|f(x-\lambda h)\|-\|c_1(\lambda)f(x)+d_1(\lambda)\|$$
$$\leqslant\|f(x-\lambda h)-c_1(\lambda)f(x)-d_1(\lambda)\|\leqslant\big\|(a_1(\lambda)-c_1(\lambda))f(x)+b_1(\lambda)-d_1(\lambda)\big\|$$
$$\leqslant|a_1(\lambda)-c_1(\lambda)|\,\|f(x)\|+\|b_1(\lambda)\|+\|d_1(\lambda)\|,\qquad\lambda\in(0,\delta_1)\cap\boldsymbol{Q}.$$

Hence

$$\|f(x-\lambda h)\| \leqslant |c_1(\lambda)|\cdot\|f(x)\| + |a_1(\lambda)-c_1(\lambda)|\cdot\|f(x)\| + \|b_1(\lambda)\| + 2\|d_1(\lambda)\| \xrightarrow[\lambda\to 0^+]{} \|f(x)\| < r.$$

Consequently, for some $\varepsilon_1 > 0$, the inequality $\|f(x-\lambda h)\| < r$ holds whenever $\lambda\in(0, \varepsilon_1)\cap \boldsymbol{Q}$.

Analogously (up to some slight modifications), one can show that there exist functions

$$\begin{matrix} a_2, c_2\colon (0, \delta_2)\cap \boldsymbol{Q}\to \boldsymbol{R} \\ b_2, d_2\colon (0, \delta_2)\cap \boldsymbol{Q}\to Y \end{matrix} \quad (\delta_2 > 0),$$

such that

$$c_2(\lambda)f(x)+d_2(\lambda) \leqslant f(x+\lambda h) \leqslant a_2(\lambda)f(x)+b_2(\lambda) \quad \text{for } \lambda\in(0, \delta_2)\cap \boldsymbol{Q},$$

$$a_2(\lambda)\xrightarrow[\lambda\to 0^+]{} 1, \quad c_2(\lambda)\xrightarrow[\lambda\to 0^+]{} 1, \quad \|b_2(\lambda)\|\xrightarrow[\lambda\to 0^+]{} 0, \quad \|d_2(\lambda)\|\xrightarrow[\lambda\to 0^+]{} 0.$$

Hence, in the same manner as above, we conclude that, for some $\varepsilon_2 > 0$, the inequality $\|f(x+\lambda h)\| < r$ is satisfied whenever $\lambda\in(0, \varepsilon_2)\cap \boldsymbol{Q}$. From what we have just shown, it follows that

$$x+\lambda h\in A \quad \text{for } \lambda\in(-\varepsilon, \varepsilon)\cap \boldsymbol{Q}, \text{ where } \varepsilon := \min(\varepsilon_1, \varepsilon_2).$$

Since the element h has been arbitrarily chosen from U, we deduce that x is a $\boldsymbol{Q}_U$-internal point of A which was to be proved.

In order to prove the main result of this section we refer to the following fact:

LEMMA 10 (see [5], Theorem 1). *If $f\colon D\to Y$ is an n-convex function bounded on a set $A\subset D$ such that* $\operatorname{int} H_{n+1}(A)\neq \emptyset$, *then f is continuous.*

The methods of the proof are similar to those presented in [4] with slight alterations taking into account more general assumptions on spaces X and Y.

Christensen measurability of a function $f\colon D\to Y$ is understood as measurability with respect to the σ-algebra $\mathscr{C}$.

Now, we have all tools to prove the announced theorem.

THEOREM 2. *If a Christensen measurable function $f\colon D\to Y$ is n-convex with respect to the cone S, then it is continuous.*

Proof. Since

$$D = \bigcup_{k=1}^{\infty} \{x\in D\colon \|f(x)\| < k\},$$

there exists an $N\in \boldsymbol{N}$ such that the Christensen measurable set $A := \{x\in D\colon \|f(x)\| < N\}$ is non-empty. Take an arbitrary $x\in A$ and put $U := S\cap(D-x)\cap(x-D)$. We shall show that $X = \bigcup_{\alpha\in \boldsymbol{Q}} \alpha\cdot U$.

For, take an arbitrary $y \in X$. Since the transformation

$$\boldsymbol{R} \ni \lambda \to \lambda y \in X$$

is continuous and $(D-x) \cap (x-D)$ is a neighbourhood of zero, one can find rational numbers $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\lambda_1 y \in (D-x) \cap (x-D)$ and $-\lambda_2 y \in (D-x) \cap (x-D)$. We distinguish three possible cases:

(a) $y \in S$; then $\lambda_1 y \in S$ and consequently $\lambda_1 y \in U$.

(b) $-y \in S$; then $-\lambda_2 y = \lambda_2(-y) \in S$ which implies that $-\lambda_2 y \in U$.

(c) $y = 0$; then $y \in 0 \cdot U$ since $U \neq \emptyset$ as it was shown at the beginning of the proof of Lemma 9.

Since the above cases exhaust all possibilities, there exists a rational number α such that $y \in \alpha \cdot U$, which was to be shown.

Now, since X is not a Christensen zero set, it is clear that $\alpha \cdot U \notin \mathscr{C}_0$ for some $\alpha \in \boldsymbol{Q}$ and therefore $U \notin \mathscr{C}_0$.

On account of Lemma 9, x is a $\boldsymbol{Q}_U$-internal point of the set A and by Lemma 7, $x \in \operatorname{int} H_{n+1}(A)$.

To finish the proof it remains to apply Lemma 10.

4. Now, suppose Y to be an arbitrary linear topological space and let remaining assumptions be unchanged.

DEFINITION 3. A function f: $D \to Y$ is said to be a *polynomial function of n-th order* iff

$$\Delta_h^{n+1} f(x) = 0$$

for all $x \in D$ and $h \in X$ such that $x + ih \in D$, $i = 1, \ldots, n+1$.

The following result corresponds to Lemma 8:

LEMMA 11. *Let f: $D \to Y$ be a polynomial function of n-th order, $x \in D$ and $h \in X$ such that $x+h \in D$. If rational numbers $\lambda_0, \ldots, \lambda_{n+1}$ are chosen so that $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \lambda_{n+1} = 1$, then*

$$\sum_{j=0}^{n+1} (-1)^{n+1-j} V(\lambda_0, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{n+1}) f(x + \lambda_j h) = 0.$$

THEOREM 3. *Every Christensen measurable polynomial function f: $D \to Y$ is continuous.*

Proof. Let f: $D \to Y$ be a Christensen measurable polynomial function of n-th order and let $W \subset Y$ be any open set. We shall show that $f^{-1}(W)$ is open. For, fix an $x_0 \in f^{-1}(W)$ and put $y_0 := f(x_0)$. The function φ: $Y^{n+1} \to Y$ determined by

$$\varphi(y_1, \ldots, y_{n+1}) := (-1)^n \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} y_j, \quad (y_1, \ldots, y_{n+1}) \in Y^{n+1}$$

is continuous and it is easily seen that

$$\varphi(y_0, \ldots, y_0) = y_0.$$

Thus, there exists a neighbourhood $V \subset Y$ of y_0 with the property

$$\varphi(V, \ldots, V) = (-1)^n \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} V \subset W.$$

Now, take arbitrarily an $x \in f^{-1}(V)$ and put $U := (D-x) \cap (x-D)$. Using Lemma 11, to any $h \in U$ we can assign points $x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_n \in D$ and functions a, c, b_j, d_j: $(0, \delta) \cap \mathbf{Q} \to \mathbf{R}$, $j = 1, \ldots, n$ such that

$$\left.\begin{aligned} f(x+\lambda h) &= a(\lambda) f(x) + \sum_{j=1}^{n} b_j(\lambda) f(x_j) \\ f(x-\lambda h) &= c(\lambda) f(x) + \sum_{j=1}^{n} d_j(\lambda) f(\tilde{x}_j) \end{aligned}\right. \quad \text{for } \lambda \in (0, \delta) \cap \mathbf{Q},$$

and

$$\begin{aligned} a(\lambda) &\xrightarrow[\lambda \to 0^+]{} 1, \quad & b_j(\lambda) &\xrightarrow[\lambda \to 0^+]{} 0, \quad & j &= 1, \ldots, n, \\ c(\lambda) &\xrightarrow[\lambda \to 0^+]{} 1, \quad & d_j(\lambda) &\xrightarrow[\lambda \to 0^+]{} 0, \quad & j &= 1, \ldots, n. \end{aligned}$$

Hence, by the continuity of the scalar multiplication and addition in Y, we obtain

$$f(x+\lambda h) \in V \quad \text{for } \lambda \in (-\varepsilon, \varepsilon) \cap \mathbf{Q}$$

and $\varepsilon > 0$ small enough.

This shows that x is a $\mathbf{Q}_U$-internal point of the Christensen measurable set $f^{-1}(V)$, and so, in view of Lemma 7, $x \in \operatorname{int} H_{n+1}(f^{-1}(V))$. Consequently, the inclusion $f^{-1}(V) \subset \operatorname{int} H_{n+1}(f^{-1}(V))$ holds true.

Now, if x is an arbitrarily chosen point of $H_{n+1}(f^{-1}(V))$, then there exists an $h \in X$ such that $f(x+ih) \in V$, $i = 1, \ldots, n+1$. Hence

$$f(x) = (-1)^n \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x+jh) \in W.$$

Thus, the inclusion $H_{n+1}(f^{-1}(V)) \subset f^{-1}(W)$ is proved.

Finally, we have the following relations:

$$x_0 \in f^{-1}(V) \subset \operatorname{int} H_{n+1}(f^{-1}(V)) \subset f^{-1}(W)$$

getting $x_0 \in \operatorname{int} f^{-1}(W)$ and our proof is finished.

Remark 1. Theorem 3 does not follow from Theorem 2 since in Section 4 we do not suppose Y to be an ordered and normed space. Consequently, polynomial functions cannot be considered, in general, as the particular case of n-convex functions.

Remark 2. S. Kurepa was interested in investigation of measurable functions $f\colon \boldsymbol{R}\to\boldsymbol{R}$ satisfying the following equation:

$$\sum_{k=0}^{n+1}\gamma_k f(x+\alpha_k y)=0,\qquad x,y\in\boldsymbol{R}, \tag{13}$$

where γ_k, α_k, $k=0,1,\ldots,n+1$, are some real numbers such that $\gamma_0\cdot\gamma_1<0$ and $\alpha_0<\alpha_1<\ldots<\alpha_{n+1}$, $\alpha_0\neq 0$.

It was pointed out in [7], Theorem 4, that all measurable solutions of (13) are polynomials of at most nth degree.

From the results and methods presented in [9] it follows implicitly that if a function $f\colon X\to Y$ satisfies

$$\sum_{k=0}^{n+1}\gamma_k f(x+\alpha_k y)=0,\qquad x,y\in X,$$

where $\sum_{i\in J}\gamma_i\neq 0$, for every subset $J\subsetneq\{0,1,\ldots,n+1\}$ and the α_k's do not vanish simultaneously, then

$$\Delta_h^{\frac{n(n+1)}{2}+1}f(x)=0\qquad\text{for all } x,h\in X.$$

If such a function f is Christensen measurable, then, in virtue of our previous theorem, f is a continuous polynomial function of at most $\frac{n(n+1)}{2}$-th order.

Remark 3. One can ask whether Theorem 2 holds true if the assumption $S\cup(-S)\cup\{0\}=X$ is replaced by a weaker condition, for instance that S is not a Christensen zero set. The following example excludes the above hypothesis. Let us consider the function $f\colon\boldsymbol{R}\to\boldsymbol{R}$ defined by

$$f(x):=\begin{cases}|x| & \text{for } x\in\boldsymbol{R}\setminus\{0\},\\ 1 & \text{for } x=0\end{cases}$$

and a cone $S:=(1,\infty)$. It is easy to check that f is convex (of the first order) with respect to S, i.e.,

$$f(x+2h)-2f(x+h)+f(x)\geqslant 0\qquad\text{for } x\in\boldsymbol{R},\ h\in(1,\infty).$$

The function f is Borel measurable, S is a non-empty open set and, consequently, S is not a Christensen zero set but nevertheless f is discontinuous.

Acknowledgement. I wish to thank Professor Roman Ger who suggested me the questions considered in this paper.

References

[1] J. P. R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. 13 (1972), 255–260.

[2] –, *Topology and Borel structure*, North-Holland, Amsterdam; American Elsevier, New York 1974.

[3] P. Fischer, Z. Słodkowski, *Christensen zero sets and measurable convex functions*, Proc. Amer. Math. Soc. 79 (1980), 449–453.

[4] R. Ger, *Convex functions of higher orders in Euclidean spaces*, Ann. Polon. Math. 25 (1972), 293–302.

[5] –, *n-convex functions in linear spaces*, Aequationes Math. 10 (1974), 172–176.

[6] J. L. Kelley, *General topology*, Springer Verlag, New York, Heidelberg, Berlin 1975.

[7] S. Kurepa, *A property of a set of positive measure and its application*, J. Math. Soc. Japan 13 (1961), 13–19.

[8] –, *A cosine functional equation in Hilbert space*, Canad. J. Math. 12 (1960), 45–50.

[9] M. A. Mc Kiernan, *Boundedness on a set of positive measure and the mean value property characterize polynomials on a space V^**, Aequationes Math. 4 (1970), 31–36.

[10] T. Popoviciu, *Sur quelques propriétés des fonctions d'une ou de deux variables réeles*, Mathematica 8 (1934), 1–85.

INSTYTUT MATEMATYKI
UNIWERSYTETU ŚLĄSKIEGO, KATOWICE
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY, KATOWICE

Reçu par la Rédaction le 1982.02.11