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Remarks on absolutely regular and regular $\mathcal{K}'\{M_p\}$-distributions

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Abstract. We give the definition of $\mathcal{K}'\{M_p\}$-regular distributions and some characterizations of absolutely regular and regular $\mathcal{K}'\{M_p\}$-distributions.

1. Absolutely regular tempered distributions were studied by Szmydt [6], [7] who noted an interesting property of a locally integrable function which defines a tempered distributions. This was the motivation for several papers concerning the relations between locally integrable functions and various subspaces of Schwartz distributions. Dierolf [2] studied regular and absolutely regular tempered distributions; he gave a detailed topological analysis of such spaces. Kliś and Pilipović [3] gave some remarks on these spaces.

The space $\mathcal{S}'$ is a $\mathcal{K}'\{M_p\}$-type space where $\mathcal{K}'\{M_p\}$ is the Gel'fand–Shilov space [1]. Pilipović [4] characterized absolutely regular $\mathcal{K}'\{M_p\}$-distributions.

The following properties of the sequence of functions M_p are assumed in [4]:

(1) $1 \leqslant M_p(x) \leqslant M_{p+1}(x)$, $x \in \mathbf{R}$, $p \in \mathbf{N}$;

(2) For any $x \in \mathbf{R}$ there are only two possible cases: $M_p(x) = \infty$ for all p or $M_p(x) < \infty$ for all p;

(3) M_p, $p \in \mathbf{N}$, is continuous with respect to x where the function is finite. The set of points at which $M_p(x) = \infty$ is contained in $(-\alpha, \alpha)$ for some $\alpha > 0$;

(N) For every $p \in \mathbf{N}$ there is $p' \in \mathbf{N}$ such that $M_p/M_{p'} \in L^1$;

(4) For every $p \in \mathbf{N}$ there is $Y_p > 0$ such that M_p is non-decreasing on (Y_p, ∞) and non-increasing on $(-\infty, -Y_p)$;

(5) For every $p \in \mathbf{N}$ there are $p' \in \mathbf{N}$ and $X_p > 0$ such that $M_p(x+1) \leqslant M_{p'}(x)$, $x > X_p$, $M_p(x-1) \leqslant M_{p'}(x)$, $x < -X_p$. (Assume $X_p = Y_p$.)

Note that (1), (2), the first part of (3) and a stronger version of (N) are well-known conditions from [1], p. 86, and p. 111.

Recall that $\mathscr{K}\{M_p\}$ is the space of all $\phi \in C^\infty(\mathbf{R})$ for which the norms

$$\|\phi\|_p = \sup\{M_p(x)|\phi^{(q)}(x)|;\ q \leqslant p,\ x \in \mathbf{R}\}, \quad p \in \mathbf{N},$$

are finite.

An $f \in L^1_{\text{loc}}$ is called an *absolutely regular generalized function* from $\mathscr{K}'\{M_p\}$ if $f\phi \in L^1$ for any $\phi \in \mathscr{K}\{M_p\}$. We denote by $\mathscr{K}'_{ar}\{M_p\}$ the space of such elements from $\mathscr{K}'\{M_p\}$. Conditions (1), (2), (3), (N) imply that const. $\in \mathscr{K}'_{ar}\{M_p\}$ ([4]).

2. For the definition of a regular $\mathscr{K}'\{M_p\}$-distribution we need that $\mathscr{K}'\{M_p\}$ is a subspace of $\mathscr{D}'$. Because of that, in the sequel we shall assume that M_p satisfies conditions (1), (N), (4), (5) and the following two:

(3′) M_p, $p \in \mathbf{N}$, are continuous functions,

and ([1])

(P) For any $p \in \mathbf{N}$, there is $p' \in \mathbf{N}$ such that

$$\lim_{|x| \to \infty} M_p(x)/M_{p'}(x) = 0.$$

Clearly, condition (3′) is stronger than (2) and (3).

Under these two conditions ([1], p. 96), $\mathscr{D}$ is a dense subspace of $\mathscr{K}\{M_p\}$ and the convergence of a sequence in $\mathscr{D}$ implies its convergence in $\mathscr{K}\{M_p\}$. So, $\mathscr{K}'\{M_p\} \subset \mathscr{D}'$.

An $f \in L^1_{\text{loc}}$ is called a *regular $\mathscr{K}'\{M_p\}$-distribution* if $f \in \mathscr{K}'\{M_p\}$, i.e. if for any ϕ and any sequence η_k from $\mathscr{D}$ which converges to ϕ in $\mathscr{K}\{M_p\}$ (as $k \to \infty$),

$$\lim_{k \to \infty} \langle f, \eta_k \rangle = \lim_{k \to \infty} \int_{\mathbf{R}} f(t)\eta_k(t)\,dt = \langle f, \phi \rangle < \infty.$$

Denote by $\mathscr{K}'_r\{M_p\}$ the space of such distributions. Obviously, $\mathscr{K}'_{ar}\{M_p\} \subset \mathscr{K}'_r\{M_p\}$. An example of a regular $\mathscr{K}'\{M_p\}$-distribution which is not absolutely regular is given in [4], Theorem 1.

Note that if $f \in \mathscr{K}'_r\{M_p\}$ and $f^{(j)} \in L^1_{\text{loc}}$, $j \in \mathbf{N}$, then $f^{(j)} \in \mathscr{K}'_r\{M_p\}$.

3. We need the following lemma.

LEMMA 1. *Let $\phi \in \mathscr{K}\{M_p\}$. Then there is $\psi \in \mathscr{K}\{M_p\}$ such that $\psi \geqslant |\phi|$.*

Proof. Let us define $J_k = (k-3/2, k+3/2)$, $I_k = (k-1, k+1)$, $k \in \mathbf{Z}$ $(= -\mathbf{N} \cup \{0\} \cup \mathbf{N})$, $a_k = \sup\{|\phi(x)|;\ x \in I_k\}$, $k \in \mathbf{Z}$. Let $\omega \in C^\infty$ be such that $\omega \geqslant 0$, $\operatorname{supp}\omega \subset (-1/2, 1/2)$ and $\int \omega = 1$. Let $\tilde{\psi}_k(x) = a_k$, $x \in J_k$ and $\tilde{\psi}_k(x) = 0$, $x \notin J_k$, $k \in \mathbf{Z}$. Put $\psi_k = \tilde{\psi}_k * \omega$, $k \in \mathbf{Z}$.

We have: $\psi_k \geqslant 0$, $\operatorname{supp}\psi_k \subset (k-2, k+2)$, $\psi_k(x) = a_k$, $x \in I_k$, $k \in \mathbf{Z}$. Put $\psi = \sum_{k \in \mathbf{Z}} \psi_k$. Note that on I_k

$$\psi = \psi_{k-2} + \psi_{k-1} + \psi_k + \psi_{k+1} + \psi_{k+2}, \quad k \in \mathbf{Z}.$$

Let

$$d_j = \sup\{|\omega^{(j)}(x)|;\ x \in \mathbf{R}\}, \quad j = 0, 1,$$

$$b_{p,k} = \sup\{M_p(x);\ x \in I_k\}, \quad k \in \mathbf{Z}.$$

Using (5) twice we see that for given $p \in \mathbf{N}$ there are p_2 and $\bar{X}_p$ such that

$$\sup\{M_p(x);\ x \in I_k\} \leqslant \inf\{M_{p_2}(x);\ x \in I_k\}, \quad |k| > \bar{X}_p. \tag{6}$$

Let p_3 correspond to p_2 in (N) and p_5 and $\bar{X}_{p_3}$ correspond to p_3 in (6). For $|k| > \bar{X}_{p_3}$ we have

$$\begin{aligned}
&\inf\{M_{p_2}(x);\ x \in I_k\} \cdot \sup\{|\phi(x)|;\ x \in I_k\} \\
&\quad \leqslant \inf\{M_{p_2}(x)/M_{p_3}(x);\ x \in I_k\} \cdot \sup\{M_{p_3}(x);\ x \in I_k\} \cdot \sup\{|\phi(x)|;\ x \in I_k\} \\
&\quad \leqslant \inf\{M_{p_2}(x)/M_{p_3}(x);\ x \in I_k\} \cdot \inf\{M_{p_5}(x);\ x \in I_k\} \cdot \sup\{|\phi(x)|;\ x \in I_k\} \\
&\quad \leqslant \sup\{M_{p_5}(x)|\phi(x)|;\ x \in I_k\} \cdot \inf\{M_{p_2}(x)/M_{p_3}(x);\ x \in I_k\}.
\end{aligned}$$

Since any $x \in \mathbf{R}$ belongs to two intervals from the family I_k, from (N) we get

$$\begin{aligned}
&2 \sum_{|k| > \bar{X}_{p_3}} \inf\{M_{p_2}(x)/M_{p_3}(x);\ x \in I_k\} \\
&\qquad \leqslant \sum_{|k| > \bar{X}_{p_3}} \int_{I_k} \big(M_{p_2}(x)/M_{p_3}(x)\big)\, dx \leqslant 2 \int_{\mathbf{R}} \big(M_{p_2}(x)/M_{p_3}(x)\big)\, dx < \infty.
\end{aligned}$$

By using these inequalities we have

$$\begin{aligned}
\sum_{|k| > \bar{X}_{p_3}} b_{p,k} a_k &\leqslant \sum_{|k| > \bar{X}_{p_3}} \inf\{M_{p_2}(x);\ x \in I_k\} \cdot \sup\{|\phi(x)|;\ x \in I_k\} \\
&\leqslant \sum_{|k| > \bar{X}_{p_3}} \sup\{M_{p_5}(x)|\phi(x)|;\ x \in I_k\} \cdot \inf\{M_{p_2}(x)/M_{p_3}(x);\ x \in I_k\} \\
&\leqslant \sup\{M_{p_5}(x)\,|\phi(x)|;\ x \in \mathbf{R}\} \int_{\mathbf{R}} \big(M_{p_2}(x)/M_{p_3}(x)\big)\, dx < \infty.
\end{aligned}$$

So, for any $p \in \mathbf{N}$ we have $\sum_{k \in \mathbf{Z}} b_{p,k} a_k < \infty$. In the same way we prove that

$$\sum_{k \in \mathbf{Z}} b_{p,k-i} a_k < \infty \qquad \text{for any } p \in \mathbf{N} \text{ and } i = -2, -1, 1, 2.$$

Fix p and j. We have

$$\begin{aligned}
\sup\{|M_p(x)\psi^{(j)}(x)|;\ x \in \mathbf{R}\} &\leqslant \sum_{k \in \mathbf{Z}} \sup\{M_p(x)|\psi^{(j)}(x)|;\ x \in I_k\} \\
&\leqslant \sum_{k \in \mathbf{Z}} b_{p,k} \sup\{|\psi^{(j)}(x)|;\ x \in I_k\} \\
&\leqslant \sum_{k \in \mathbf{Z}} b_{p,k} \Big(\sup\Big\{\sum_{i=0}^{4} |\psi_{k-2+i}^{(j)}|;\ x \in I_k\Big\}\Big) \\
&\leqslant d_j \sum_{k \in \mathbf{Z}} b_{p,k} (a_{k-2} + \quad + a_{k+2}) < \infty.
\end{aligned}$$

So, $\psi \in \mathcal{K}\{M_p\}$. Obviously $\psi \geqslant |\phi|$.

In the next proposition we give a characterization of an absolutely regular $\mathcal{K}'\{M_p\}$-distribution. Note that we gave in [4], Theorems 2, 3, under the assumptions from [4], several characterizations of absolutely regular

$\mathscr{K}'\{M_p\}$-generalized functions. Assumptions from this paper allow us to use the term absolutely regular $\mathscr{K}'\{M_p\}$-distributions for such generalized functions.

PROPOSITION 2. *Let $T\in L^1_{\mathrm{loc}}$. The following conditions are equivalent:*

(i) *T is an absolutely regular $\mathscr{K}'\{M_p\}$-distribution.*

(ii) $|T|\in\mathscr{K}'\{M_p\}$.

Proof. (i)$\Rightarrow$(ii) is obvious so let us prove (ii)$\Rightarrow$(i).

Take $\phi\in\mathscr{K}\{M_p\}$ which is non-negative. There exists a sequence ϕ_n from $\mathscr{D}$ such that $\phi_n\geqslant 0$, $n\in\mathbf{N}$, and $\phi_n\to\phi$ in $\mathscr{K}\{M_p\}$. We have

$$\int_{-\infty}^{\infty}|T(t)|\phi_n(t)\,dt\to\langle|T(t)|,\ \phi(t)\rangle<\infty,\qquad n\to\infty.$$

For almost all $t\in\mathbf{R}$, $|T(t)|\phi_n(t)\to|T(t)|\phi(t)$, $n\to\infty$. By Fatou's Lemma we have

$$\int_{\mathbf{R}}|T(t)|\phi(t)\,dt\leqslant\lim_{n\to\infty}\int_{\mathbf{R}}|T(t)|\phi_n(t)\,dt<\infty.$$

This implies that $T\phi\in L^1$.

If ϕ is an arbitrary element from $\mathscr{K}\{M_p\}$ then by Lemma 1 there exists a $\psi\in\mathscr{K}\{M_p\}$ so that $|\phi|\leqslant\psi$. So, by the previous part of the proof we have $T\phi\in L^1$.

4. Let $T\in L^1_{\mathrm{loc}}$. We have

$$|T(t)|=\left|\frac{d}{dt}\Big(\exp i\Big(\int_0^t|T(u)|\,du\Big)\Big)\right|,$$

where $\exp\big(i(\int_0^t|T(u)|\,du)\big)\in\mathscr{K}'_r\{M_p\}$. So every non-negative function from L^1_{loc} is the absolute value of some regular $\mathscr{K}'\{M_p\}$-distribution.

In order to obtain a useful sufficient condition for a locally integrable function to be a regular element from $\mathscr{K}'\{M_p\}$ we have to assume two more conditions.

Since we shall use the notion of convolution we need:

(S) $\qquad \phi\in\mathscr{K}\{M_p\}\ \Rightarrow\ \check{\phi}\in\mathscr{K}\{M_p\}\qquad$ where $\check{\phi}(x)=\phi(-x)$, $x\in\mathbf{R}$.

Note that (S) holds if M_p, $p\in\mathbf{N}$, are even functions or, more generally, if for every $p\in\mathbf{N}$ there are $p'\in\mathbf{N}$ and $C_{p,p'}$ so that

$$M_p(x)\leqslant C_{p,p'}M_{p'}(-x),\qquad x\in\mathbf{R}.$$

The following condition is essential for the next proposition.

(I) There are functions θ and M defined on $[0,\infty)$ such that

a) $\theta\geqslant 1$; M is continuous and monotonically increases from 0 to ∞;

b) For every $p\in\mathbf{N}$ there is $p'\in\mathbf{N}$ so that

$$pM_p(x)\leqslant M(p'\theta(|x|)),\qquad x\in\mathbf{R};$$

c) For every $D>0$ there is p'' such that

$$M(D\theta(|x|))\leqslant p''M_{p''}(x),\qquad x\in\mathbf{R}.$$

With the assumptions on M_p given above we have:

PROPOSITION 3. *Let $T \in L^1_{\mathrm{loc}}$. Then $T \in \mathscr{K}'_r\{M_p\}$ if for any $\phi \in \mathscr{D}$, $T * \phi$ is an M_p-slowly increasing function.*

Proof. We shall use the idea of the proof of Théorème XXII, p. 195 in [5].

We shall prove that T is of the form $T = \sum_{\alpha=0}^{m} f_\alpha^{(\alpha)}$ for some M_p-slowly increasing functions f_α, $\alpha = 0, \ldots, m$. This implies that T can be linearly and continuously extended to $\mathscr{K}\{M_p\}$ by

$$\langle T, \phi \rangle = \lim_{n \to \infty} \langle T, \phi_n \rangle, \quad \phi \in \mathscr{K}\{M_p\},$$

where ϕ_n is any sequence from $\mathscr{D}$ which converges to ϕ in $\mathscr{K}\{M_p\}$, and since $T \in L^1_{\mathrm{loc}}$ we have $T \in \mathscr{K}'_r\{M_p\}$.

For $\phi \in \mathscr{K}\{M_p\}$

$$(T(t) * \check{\phi}(t))(x) = \langle T(t), \check{\phi}(x-t) \rangle = \langle T(u+x), \phi(x) \rangle, \quad x \in \mathbf{R},$$

and thus, for any $\phi \in \mathscr{D}$ there is $p = p_\phi \in \mathrm{N}$ such that

$$|\langle T(x+u), \phi(u) \rangle| \leqslant p M_p(x), \quad x \in \mathbf{R},$$

i.e.

$$M^{-1}(|\langle T(x+u), \phi(u) \rangle|)/M^{-1}(pM_p(x)) \leqslant 1, \quad x \in \mathbf{R}.$$

where M is from (I). (I) implies that for some $p' \in \mathrm{N}$

$$1/\theta(x) \leqslant p'/M^{-1}(pM_p(x)), \quad x \in \mathbf{R}.$$

So, for $\phi \in \mathscr{D}_K$ the functions

$$x \to h(x, \phi) = M^{-1}(|\langle T(x+u), \phi(u) \rangle|)/\theta(x), \quad x \in \mathbf{R},$$

are bounded. For fixed $x \in \mathbf{R}$ and a compact set K we find that

$$\phi \to h(x, \phi), \quad \phi \in \mathscr{D}_K,$$

is a continuous function because M^{-1} is continuous.

Let us view $\phi \to h(\cdot, \phi)$ as a mapping from $\mathscr{D}_K$ into the space of bounded continuous functions on $\mathbf{R}$ with the usual supremum norm, denoted by $\mathscr{C}_b$. Since $\mathscr{D}_K$ is of the second category there is a neighbourhood of zero $\mathscr{U}_K$ in $\mathscr{D}_K$ such that

$$\{x \to h(x, \phi);\ \phi \in \mathscr{U}_K\}$$

is a bounded family in $\mathscr{C}_b$. Thus, for a given constant $D > 0$ we deduce that for every compact set K there exists a neighbourhood of zero $\mathscr{U}_K$ so that for every $\phi \in \mathscr{U}_K$

$$\sup_{x \in \mathbf{R}} \{M^{-1}(|\langle T(x+u), \phi(u) \rangle|)/\theta(x)\} < D.$$

By (I) there exists $p'' \in \mathbf{N}$ such that for every compact set K

$$|\langle T(x+u), \phi(u)\rangle| / M_{p''}(x) \leqslant p'', \quad \phi \in \mathscr{U}_K.$$

Since for every compact set K, $\mathscr{U}_K$ absorbs $\mathscr{D}_K$ we have for every $\phi \in \mathscr{D}$

$$\langle T(x+u), \phi(u)\rangle / M_{p''}(x) \in L^\infty.$$

This implies (see Chapitre VI, Théorème XXII in [5]) that for a given open bounded set $\Omega \subset \mathbf{R}$, $0 \in \Omega$, there exist a compact neighbourhood of zero K and $m \in \mathbf{N}$ such that for every $\phi \in \mathscr{D}_K^m$

$$\{x \to (T(t+h) * \phi(t))(x)/M_p(h),\ h \in \mathbf{R}\}$$

is a bounded family of continuous functions on Ω. On setting $x = 0$ we see that

$$h \to (T * \phi)(h)/M_p(h), \quad h \in \mathbf{R},$$

is a bounded function for any $\phi \in \mathscr{D}_K^m$. Now, by (VI, 6; 22) in [5] we obtain

$$T = (\gamma E * T)^{(2N)} - \psi * T$$

where $^{(2N)}$ denotes the distributional derivative of order $2N$, E satisfies $E^{(2N)} = \delta$, $\gamma \in \mathscr{D}_K$, $\gamma \equiv 1$ in a neighbourhood of 0 and $\psi \in \mathscr{D}_K$. If N is sufficiently large, $\gamma E \in \mathscr{D}_K^m$ and we get the required representation of T.

Condition (I) seems complicated but it is satisfied for all concrete $\mathscr{K}'\{M_p\}$-type spaces which have been studied in the literature. For example:

(i) Let $M_p(\cdot) = \exp(p|\cdot|^\alpha)$, $p \in \mathbf{N}$ and let $\alpha > 0$ be fixed. We take $M(x) = e^x - 1$, $\theta(x) = x^\alpha + 1$, $x \geqslant 0$.

(ii) Let m be a real-valued continuous increasing function defined on $[0, \infty)$ such that $m \geqslant 1$ and that for every $p \in \mathbf{N}$ there is $p' \in \mathbf{N}$ such that $pm(x) \leqslant m(p'x)$, $x > 0$. Assume that for $M_p(x) = m(p|x|)$, $x \in \mathbf{R}$, the conditions from Section 2 hold. Then (I) holds for M_p with

$$M(x) = m(x) - m(0), \quad \theta(x) = x + 1, \quad x \geqslant 0.$$

Note that for every $p > 0$ there is p' and $X_p > 0$ such that

$$m(p|x|) \leqslant m(p'|x|) - m(0) \quad \text{for } |x| > X_p.$$

This follows from condition (P).

(iii) Let $M_p(\cdot) = (1 + |\cdot|^2)^{p/2}$, $p \in \mathbf{N}$. We take

$$M(x) = e^x - 1, \quad \theta(x) = \ln(x+1) + 1, \quad x \geqslant 0.$$

(iv) Let $M_p(\cdot) = \exp(|\cdot|^p)$, $p \in \mathbf{N}$. We take

$$M(x) = \exp(\exp x) - e, \quad \theta(x) = \ln(x+1) + 1, \quad x \geqslant 0.$$

Let us remark that the converse assertion in Proposition 3 holds if we assume instead of (I) the following condition:

(R) For any $p \in \mathbf{N}$ there are $C_p > 0$ and $p' \in \mathbf{N}$ such that $M_p(x+u) \leqslant C_p M_{p'}(x) M_{p'}(u)$, $x, u \in \mathbf{R}$.

Namely, for the converse assertion in Proposition 3 one has to use the inequality

$$|\langle T(t), \phi(x-t)\rangle| \leqslant S_p \|\phi(x-t)\|_p$$

which holds for some $p \in \mathbf{N}$ and some $S_p > 0$, and then use (R).

Note that (R) holds in all the examples given above. Let us prove that for example (ii).

Since m is increasing and $m \geqslant 1$ we have

$$m(p|x+u|) \leqslant m(p|x|+p|u|) \leqslant m(2p|x|) \cdot m(2p|u|), \qquad x, u \in \mathbf{R}.$$

This implies the assertion.

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