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About a conjecture of F. W. Gehring on the boundary correspondence by quasi-conformal mappings

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Abstract. Let $f\colon B \rightleftharpoons R^n$ be a *K-quasi-conformal mapping* of the unit ball B. The *logarithmic capacity* of a measure $\mu > 0$ is given as $C_0(E) = [\inf_\mu I_0(\mu)]^-$ where $I_0(\mu) = \int_{R^n} u_0^\mu(x)\,d\mu(x)$ is the *energy integral*,

$$u_0^\mu(x) = \int_{R^n} \log\frac{1}{|x-y|}\,d\mu(y)$$

is the *logarithmic potential* and the infimum is taken over all measures $\mu > 0$ with total mass 1 and the support $S_\mu \subset E$. Let $D^* = f(B)$, let ∂D^* be the boundary of D and E' the set of points of the unit sphere S corresponding to points of ∂D^* accessible from D^* by rectifiable arcs. We try to give an answer to the question whether $E_0 = S - E'$ is of logarithmic capacity zero or not.

Introduction. D. Storvick proposed that I should prove the following conjecture of F. W. Gehring:

If $f\colon B \rightleftharpoons D^*$ is a K-quasi-conformal mapping ($K-qc$) of the unit ball B onto the domain D^*, where B and D^* are contained in the Euclidean n-space R^n, and if E'^* is the set of points of the boundary ∂D^* of D^* accessible by rectifiable arcs, E' is the corresponding set on $S = \partial B$ and $E_0 = S - E'$, then the logarithmic capacity of E_0 is zero.

This is true for $n = 2$ (A. J. Jenkins [10], J. A. Lohwater [12], [13] and A. Mori [14]).

We begin with some historical considerations. M. O. Reade [15] announced at the 532-nd meeting of the Amer. Math. Soc. (held at Yale University in New Haven, Connecticut, in 1957) that, under the additional hypotheses that $n = 3$, $mD^* < \infty$ and f is differentiable, almost all the radii (in the sense of the measure on the sphere) have rectifiable images. He obtained even more, namely that if $E_1 \subset S$ is a set such that, for each $\xi \notin E_1$, almost all the radii of the ball $B(\xi, \frac{1}{2})$ (centred at ξ and with radius $\frac{1}{2}$) lying in B have rectifiable images, then the Newtonian capacity of E_1 is zero. Clearly $E_0 \subset E_1$. D. Storvick [16] proved that the two-dimensional measure (on S) of E_0 is zero if f is $K-qc$, $f \in C^1$ and $D^* \subset R^3$ is simply connected, with a connected complement CD^* and $mD^* < \infty$. He gives also (in the same paper) an argument due to F. Gehring for the same result, but only under the conditions $n = 3$ and $mD^* < \infty$.

At the Conference on Analysis held in Jyväskylä (Finland) (15–19. VIII. 1973) and at the Colloquium on "Constructive function theory" held in Cluj (Romania) (6–12. IX. 1973), I established that the $(n-1)$-dimensional Hausdorff measure $H^{n-1}(E_0)$ is equal to 0 for $f\colon B \rightleftharpoons D^* qc$ without any restrictive condition; then I found that E_0 is closed and of conformal capacity zero; I deduced hence, by means of a result of H. Wallin [20], that even the a-capacity $(a > 0)$ of E_0 is zero. (A similar result was obtained by V. A. Zorič [22] for the points of S, where f does not admit an angular value.) At the conference on "Transformazioni quasi-conformi e questioni connesse" held in Rome (12–15. III. 1974), I established that, under the additional restrictive condition $\int\limits_B |J|^{1+\epsilon} d\tau < \infty$, E_0 is of logarithmic capacity zero, but, during the discussions, Jaqueline Lelong-Ferrand observed that in this case $E_0 = \emptyset$, as follows directly from a result which she had just obtained [11] (not yet printed). After a more thorough analysis, I observed that this conclusion follows directly also from my own proof.

In the present paper I establish that E_0 is of Φ-capacity zero, where $\Phi = (\log 1/r)^\beta$ $(\beta > n-1)$, and that, under the restrictive condition $\int\limits_B |J| [\log 1/(1-r)]^a d\tau < \infty$ $(a > n-2)$, E_0 is of logarithmic capacity zero. As a consequence, we obtain the same evaluation also for other exceptional sets.

Now we shall introduce a few concepts:

Let Γ be an arc family and $F(\Gamma)$ a family of admissible functions ϱ satisfying the following conditions:

(i) $\varrho(x) \geqslant 0$ in $\mathbf{R}^n$,

(ii) $\varrho(x)$ is Borel measurable in $\mathbf{R}^n$,

(iii) $\int\limits_\gamma \varrho ds \geqslant 1$ for every $\gamma \in \Gamma$.

Then the modulus of Γ is given as

$$M(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int\limits_{\mathbf{R}^n} \varrho^n d\tau,$$

where $d\tau$ is the volume element (corresponding to the n-dimensional Lebesgue measure).

According to J. Väisälä's geometric definition [19] a $K-qc$ is characterized by

$$\frac{M(\Gamma)}{K} \leqslant M(\Gamma^*) \leqslant K M(\Gamma), \tag{1}$$

where Γ is an arbitrary arc family contained in D and $\Gamma^* = f(\Gamma)$.

A function $u\colon D \to \boldsymbol{R}^n$ is said to be ACL (*absolutely continuous on lines*) in a domain D if, for each interval $I = \{x;\ a^i < x^i < \beta^i\ (i = 1, \ldots, n)\}$, $I \subset\subset D$ (i.e., $\bar{I} \subset D$), u is AC (*absolutely continuous in the ordinary sense*) on a.e. (*almost every*) line segment parallel to the coordinate axes.

The conformal capacity of a bouned set $E \subset \boldsymbol{R}^n$ is

$$\text{(2)} \qquad \operatorname{cap}_n E = \operatorname{cap} E = \inf_u \int_{\boldsymbol{R}^n} |\nabla u|^n d\tau,$$

where the infimum is taken over all functions u which are continuous and ACL in $\boldsymbol{R}^n$, have a compact support S_u contained in a fixed ball and are equal to 1 on E.

We recall that the *support of a function* u is the closure of the set $E\{x;\ u(x) \neq 0\}$ and is denoted by S_u.

Let $\mu \geqslant 0$ be a measure in $\boldsymbol{R}^n$. The support S_μ of μ is a closed set $F \subset \boldsymbol{R}^n$ such that, for each $x \in F$ and every neighbourhood V_x of x, $\mu(V_x) > 0$.

The *Φ-potential of a measure* μ is defined as

$$u_\varphi^\mu(x) = \int_{\boldsymbol{R}^n} \Phi(|x-y|)\,d\mu(y),$$

where the kernel $\Phi(r)$ is supposed to be strictly decreasing, continuous and satisfying $\lim\limits_{r\to 0} \Phi(r) = +\infty$. By means of the *energy integral*

$$I_\varphi(\mu) = \int_{\boldsymbol{R}^n} u_\varphi^\mu(x)\,d\mu(x)$$

one obtains the *Φ-capacity* of a set E, as

$$C_\Phi(E) = [\inf_\mu I_\Phi(\mu)]^{-1},$$

where the infimum is taken over all measures $\mu \geqslant 0$ of total mass 1 and the support $S_\mu \subset E$. If $\Phi(r) = 1/r^a$, or $\Phi(r) = \log 1/r$, then the corresponding capacities are the *a-capacity* $C_a(E)$ and the *logarithmic capacity* $C_0(E)$, respectively. For $a = 1$ we have the *Newtonian capacity*. The diameter $d(E)$ is supposed to be less than r_0, where $\Phi(r_0) = 0$ [in the case of the logarithmic capacity $d(E) \leqslant 1$]. For an arbitrary Borel set E, $C_\Phi(E) = 0$ iff (if and only if) $C_\Phi(E \cap B_r) = 0$ for every ball $B_r = B(x, r)$ centred at x and with the radius $r \in (0, r_0)$.

We shall use the well-known fact (see, for instance, B. Fuglede [6], remark 2 of theorem 2.4, p. 160) that if the kernel $\Phi(r)$ is as defined above and the compact set F has $d(F) \leqslant r_0$, then there exists a unique measure $\tau_\Phi \geqslant 0$ with total mass 1 and $S_{\tau_\Phi} \subset F$ such that the infimum of the energy

$$C_\Phi(F)^{-1} = \inf_\mu I_\Phi(\mu) = \inf_\mu \int_{\boldsymbol{R}^n} \int_{\boldsymbol{R}^n} \Phi(|x-y|)\,d\mu(x)\,d\mu(y)$$

is attained for $\mu = \tau_\Phi$, where μ ranges over the class of all $\mu \geqslant 0$, with total mass 1 and $S_\mu \subset F$. The measure τ_Φ is called the *capacitary distribution with total mass* 1 *and kernel* Φ *of* F.

1. Evaluation of E_0.

PROPOSITION 1. *E_0 is closed.*

PROPOSITION 2. $\operatorname{Cap} E_0 = 0$.

(For the proof of these two propositions see our paper in [4].)

PROPOSITION 3. *$u_\Phi^{\tau_\Phi}(x) \leqslant M C_\Phi(F)^{-1}$ (M constant) everywhere in $\mathbf{R}^n$* (T. Ugaheri [18]).

Now we recall another characterization of the $K-qc$.

A homeomorphism $f: D \rightleftarrows D^*$ is said to be $K-qc$ $(1 \leqslant K < \infty)$ *according to Gehring's metric definition* if the linear local dilatation $\delta_L(x)$ is bounded in D and $\delta_L(x) \leqslant K$ a.e. in D, where

$$L(x, r) = \max_{|\Delta x| = r} |f(x + \Delta x) - f(x)|, \quad l(x, r) = \min_{|\Delta x| = r} |f(x + \Delta x) - f(x)|,$$

$$\delta_L(x) = \overline{\lim_{r \to 0}} \frac{L(x, r)}{l(x, r)}.$$

PROPOSITION 4. *A homeomorphism f: $D \rightleftarrows D^*$, $K-qc$ according to Väisälä's geometric definition characterized by* (1), *is a $K-qc$ also according to Gehring's metric definition.*

For the proof see our paper [2], theorems 1 and 3, or our book [3], theorem 2, p. 127.

PROPOSITION 5. *If a homeomorphism f is $K-qc$ according to Gehring's metric definition and differentiable at a point x_0, then the maximal dilatation is*

$$\Lambda_f(x_0) \equiv \overline{\lim_{x \to x_0}} \frac{|f(x') - f(x_0)|}{|x' - x_0|} \leqslant K^{(n-1)/n} \sqrt[n]{|J(x_0)|}.$$

For the proof see our book [3] (corollary of theorem 6, p. 136).

PROPOSITION 6. *If $f: D \rightleftarrows D^*$ is $K-qc$ according to Väisälä's geometric definition, then f is differentiable with the Jacobian $J(x) \neq 0$ a.e. in D.*

The proof follows from Lemma 6.3 and Theorem 6.10 of Väisälä's paper [19] and from the equivalence of all Väisälä's definitions of $K-qc$.

And now we shall introduce, according to V. Zorič [21], the concept of *boundary elements* (as a generalization of the prime ends).

A sequence of domains $\{U_m\}$, $U_m \subset D$ $(m = 1, 2, \ldots)$ is said to be *regular* if

(a) $U_{m+1} \subset U_m$ $(m = 1, 2, \ldots)$,

(b) $\left(\bigcap_{m=1}^{\infty} \overline{U}_m\right) \subset \partial D$,

(c) $\sigma_m = \partial U_m \cap D$ (the relative boundary of U_m in D) is a connected set,

(d) there is at most an accessible boundary point of D which is an accessible boundary point for each domain of the sequence $\{U_m\}$.

Two sequences of domains $\{U_m\}$, $\{U'_m\}$ are called *equivalent* if each term of either of them contains all the terms of the other one beginning with a sufficiently great index.

A *boundary element* of a domain D is the pair $(F, \{U_m\})$ consisting of a regular sequence $\{U_m\}$ and a continuum $F = \bigcap_{m=1}^{\infty} \overline{U}_m$. The boundary elements $(F, \{U_m\})$, $(F, \{U'_m\})$ are considered as *identical* if the two regular sequences $\{U_m\}$ and $\{U'_m\}$ defining them are equivalent. In this way, any of the equivalent sequences determine uniquely a boundary element.

PROPOSITION 7. *For every $K-qc$, f: $B \rightleftharpoons D^*$, it is possible to establish a one-to-one correspondence between the boundary elements $(F^*, \{U_m^*\})$ of D^* and the points of S, so that to each boundary element $(F^*, \{U_m^*\})$ there corresponds on S a point determined by the sequence $\{U_m\} = f^{-1}(\{U_m^*\})$* (Zorič [21]).

THEOREM 1. *Let f: $B \rightleftharpoons D^*$ be a $K-qc$, $mD^* < \infty$ and $E_0 \subset S$ the set of points corresponding to boundary elements of D^* inaccessible by rectifiable arcs; then*

$$C_\Phi(E_0) = 0,$$

if

$$\int_0^R \Phi(\varrho)^{-1/(n-1)} \frac{d\varrho}{\varrho} < \infty \qquad (\varrho = |x - \xi|,\ R < r_0), \tag{3}$$

where $\Phi(r_0) = 0$.

Let $\xi \in E_0$ and assume, for simplicity, that $\xi = (1, 0, \ldots, 0)$. It is easy to see that, for any segment $l_e \subset B$ with an endpoint at ξ and corresponding to a versor e,

$$\infty = \int_{\gamma^*} ds^* = \int_{l_e} \frac{ds^*}{ds} ds \leqslant \int_{l_e} \Lambda_f(x)\, ds, \tag{4}$$

where $\gamma^* = f(l_e)$. Since f is $K-qc$, Propositions 5, 6 together with (4) imply, for almost every versor e,

$$\infty = \int_{l_e} \Lambda_f(x)\, ds \leqslant K^{(n-1)/n} \int_0^{\varrho_e} \sqrt[n]{|J(x)|}\, d\varrho \leqslant K \int_0^{\varrho_e} \sqrt[n]{|J(x)|}\, d\varrho, \tag{5}$$

where $\varrho = |x - \xi|$ and ϱ_e is the length of l_e.

We shall prove that $C_\Phi(E_0) = 0$, with Φ satisfying (3) by *reductio ad absurdum*. Thus, suppose that $C_\Phi(E_0) > 0$, let r_0 be such that $\Phi(r_0) = 0$, let $\Delta \subset B$ be a convex domain with $d(\Delta) < R$ and $d(S \cap \Delta) > \frac{1}{2}R$. Of course, S may be covered by a finite number of sets $S \cap \Delta$. Let $E_0^\Delta = E_0 \cap \Delta$. Clearly, there is at least a Δ such that $C_\Phi(E_0^\Delta) > 0$.

Next, let $B_\xi = B\,[\frac{1}{2}\xi(1-r_\Delta), \frac{1}{2}(1-r_\Delta)]$ be the ball tangent to S at ξ and of radius $\frac{1}{2}(1-r_\Delta)$, where r_Δ is chosen so that $B_\xi \subset \Delta$. Let ϱ_e be the length of the segment of direction e joining ξ and the other point of $S_\xi = \partial B_\xi$, where the unit vector e depends on $\vartheta_1, \ldots, \vartheta_{n-1}$. If $(\varrho, \vartheta_1, \ldots, \vartheta_{n-1})$ are the polar coordinates of the point $x - \xi$, then integrating (5) with respect to $\vartheta_1, \ldots, \vartheta_{n-1}$ $(0 \leqslant \vartheta_1 \leqslant \frac{1}{2}\pi,\ 0 \leqslant \vartheta_2 \leqslant \pi, \ldots, 0 \leqslant \vartheta_{n-1} \leqslant 2\pi)$ and taking into account Fubini's theorem, we obtain for every $\xi \in E_0$

$$\infty = \int_{S_1} d\sigma \int_0^{\varrho_e} \sqrt[n]{|J(x)|}\, d\varrho = \int_{B_\xi} \frac{\sqrt[n]{|J(x)|}}{\varrho^{n-1}}\, d\tau, \tag{6}$$

where S_1 is the corresponding hemisphere; $d\sigma$ is the spherical element and $d\tau = \varrho^{n-1} d\varrho\, d\sigma$. We put

$$I = \int_\Delta |J| u_\Phi^{\tau\Phi}\, d\tau.$$

Since $\int_B |J|\, d\tau = mD^* < \infty$, from Proposition 3 we deduce

$$I = \int_\Delta |J|\, u_\Phi^{\tau\Phi}\, d\tau \leqslant M C_\Phi(E_0)^{-1} \int_\Delta |J|\, d\tau < \infty. \tag{7}$$

On the other hand,

$$\begin{aligned}\int_{B_\xi} \frac{\sqrt[n]{|J|}}{\varrho^{n-1}}\, d\tau &= \int_{B_\xi} [|J|\,\Phi(\varrho)]^{1/n} \frac{d\tau}{\Phi(\varrho)^{1/n} \varrho^{n-1}} \leqslant \Big(\int_{B_\xi} |J|\,\Phi d\tau\Big)^{1/n} \Big(\int_{B_\xi} \frac{d\tau}{\Phi^{1/(n-1)} \varrho^n}\Big)^{\frac{n-1}{n}} \\ &\leqslant \Big(\int_{B_\xi} |J|\Phi d\tau\Big)^{1/n} \Big(\int_0^R \Phi^{-1/(n-1)} \frac{d\varrho}{\varrho}\Big)^{(n-1)/n} (n\,\omega_n)^{(n-1)/n} \\ &= N\Big(\int_{B_\xi} |J|\,\Phi\, d\tau\Big)^{1/n},\end{aligned}$$

where $N = (n\omega_n)^{(n-1)/n} \int_0^R \Phi^{-1/(n-1)} \frac{d\varrho}{\varrho} < \infty$ according to (3). Hence, and by (6), we obtain

$$\infty = \int_{B_\xi} \frac{\sqrt[n]{|J(x)|}}{\varrho^{n-1}}\, d\tau \leqslant N\Big(\int_{B_\xi} |J|\Phi d\tau\Big)^{1/n} \leqslant N \int_{B_\xi} |J|\Phi\, d\tau \leqslant N \int_\Delta |J|\,\Phi\, d\tau.$$

But $\varrho = \varrho_{x\xi} = |x - \xi|$, so that, integrating with respect to the capacitary distribution $\tau_\Phi(\xi)$ over E_0 and taking into account Fubini's theorem, we get

$$\infty = \int_{E_0^\Delta} d\tau_\Phi(\xi) \int_\Delta |J(x)| \Phi(|\xi - x|) d\tau = \int_\Delta |J(x)| \Big[\int_{E_0^\Delta} \Phi(|\xi - x|) d\tau_\Phi(\xi) \Big] d\tau$$

$$= \int_\Delta |J(x)| u_\Phi^{\tau_\Phi}(x) d\tau = I.$$

But this contradicts (7), and thus the hypothesis $C_\Phi(E_0) > 0$ was false, allowing us to conclude that only the points of S belonging to a set of Φ-capacity zero can correspond to boundary elements of D^* inaccessible by rectifiable arcs.

COROLLARY. *In the conditions of the preceding theorem,* $C_\Phi(E_0) = 0$ *with* $\Phi(r) = \left(\log \frac{1}{r}\right)^\beta$ $(\beta > n-1)$.

Remarks. 1. The preceding theorem may be obtained as a consequence of Proposition 2 and of the following theorem of H. Wallin (communicated to me in a letter) and deduced from some results of V. P. Havin and V. G. Mazja [9]:

If E is of conformal capacity zero, then $C_\Phi^{\mathsf{1}}(E) = 0$ *if* (3) *holds.*

2. $E_0 \neq \emptyset$. Indeed, let D^* be the union of a sequence of cylinders all having the height 1 and as bases $(n-1)$-dimensional balls with area equal to $1/m^2$. Suppose all the axes are on a ray, covering it. Then D^* is a convex domain with $mD^* < \infty$; hence D^* is quasi-conformally equivalent to a ball, i.e., $D^* = f(B)$ (see F. Gehring and J. Väisälä [8]) and the point at infinity is a boundary point not accessible by rectifiable arcs. The corresponding boundary element contains only ∞, so that its image on S is $x_0 = f^{-1}(\infty) \in E_0$, and then $E_0 \neq \emptyset$, as desired. By a slight modification of the preceding example, it is possible to obtain a starlike domain D^* with a countable set of such sequences of cylinders. But a starlike domain D^* is quasi-conformally equivalent to a ball (see F. Gehring and J. Väisälä [8]) and the corresponding exceptional set E_0 is countable. In this case $mD^* = \infty$, but it is possible to modify D^* in order to have $mD^* < \infty$. It is enough to take the bases of the cylinders of each sequence in such a manner that the corresponding series converges to a value equal to a certain term of a convergent series.

THEOREM 2. *In the hypotheses of the preceding theorem, if*

$$\int_B |J| \left(\log \frac{1}{1-r}\right)^a d\tau < \infty \quad (a > n-2) \ (r = |x|), \tag{8}$$

then E_0 is of logarithmic capacity zero.

Arguing as in the preceding theorem, we get (6). Next, taking $\Phi(r) = \log 1/r$, if $C_0(E_0) > 0$, then on account of (8), we obtain

$$(9)\quad I = \int_B |J| u_0^{\tau_0} \left(\log\frac{1}{1-r}\right)^a d\tau \leqslant MC_0(E_0)^{-1} \int_B |J| \left(\log\frac{1}{1-r}\right)^a d\tau < \infty.$$

On the other hand,

$$\int_{B_\xi} \frac{\sqrt[n]{|J|}}{\varrho^{n-1}} d\tau = \int_{B_\xi} \left[|J| \log\frac{1}{\varrho}\left(\log\frac{1}{\varrho}\right)^a\right]^{1/n} \frac{d\tau}{\left(\log\frac{1}{\varrho}\right)^{(a+1)/n} \varrho^{n-1}}$$

$$\leqslant \left[\int_{B_\xi} |J| \log\frac{1}{\varrho}\left(\log\frac{1}{\varrho}\right)^a d\tau\right]^{1/n} \left[\int_{B_\xi} \frac{d\tau}{\left(\log\frac{1}{\varrho}\right)^{(a+1)/(n-1)} \varrho^n}\right]^{(n-1)/n}$$

$$\leqslant \left[\int_B |J| \log\frac{1}{\varrho}\left(\log\frac{1}{\varrho}\right)^a d\tau\right]^{1/n} (n\omega_n)^{(n-1)/n} \left[\int_0^R \frac{d\varrho}{\left(\log\frac{1}{\varrho}\right)^{(a+1)/(n-1)} \varrho}\right]^{(n-1)/n}$$

$$= N\left[\int_{B_\xi} |J| \log\frac{1}{\varrho}\left(\log\frac{1}{\varrho}\right)^a d\tau\right]^{1/n},$$

where

$$N = (n\omega_n)^{(n-1)/n} \left[\int_0^R \frac{d\varrho}{\left(\log\frac{1}{\varrho}\right)^{(a+1)/(n-1)} \varrho}\right]^{(n-1)/n} < \infty.$$

Hence, and by (6), we deduce

$$\infty = \int_{B_\xi} \frac{\sqrt[n]{|J(x)|}}{\varrho^{n-1}} d\tau \leqslant N\left[\int_{B_\xi} |J| \log\frac{1}{\varrho}\left(\log\frac{1}{\varrho}\right)^a d\tau\right]^{1/n}$$

$$\leqslant N \int_{B_\xi} |J| \log\frac{1}{\varrho}\left(\log\frac{1}{\varrho}\right)^a d\tau \leqslant N \int_{B_\xi} |J| \left(\log\frac{1}{1-r}\right)^a \log\frac{1}{\varrho}\, d\tau$$

$$\leqslant M \int_B |J| \left(\log\frac{1}{1-r}\right)^a \log\frac{1}{\varrho}\, d\tau,$$

so that, integrating over E_0 with respect to the capacitary distribution with kernel $\log 1/\varrho$, and taking into account Fubini's theorem and (9), we derive

$$\infty = \int_{E_0} d\tau_0(\xi) \int_B |J(x)| \left(\log\frac{1}{1-|x|}\right)^{\alpha} \log\frac{1}{|\xi - x|}\, d\tau_x$$

$$= \int_B |J(x)| \left(\log\frac{1}{1-|x|}\right)^{\alpha} \left[\int_{E_0} \log\frac{1}{|\xi - x|}\, d\tau_0(\xi)\right] d\tau_x$$

$$= \int_B |J(x)| \left(\log\frac{1}{1-|x|}\right)^{\alpha} u_0^{\tau_0}(x)\, d\tau_x = I < \infty,$$

allowing us to conclude (arguing as above) that E_0 is of logarithmic capacity zero.

By a similar argument we obtain

THEOREM 3. *In the hypotheses of Theorem* 1, *if*

$$\int_B |J| \left(\log\frac{1}{1-r}\right)^{\alpha} d\tau < \infty \qquad (r = |x|),$$

then E_0 *is of* Φ*-capacity zero, where* $\Phi(r) = (\log 1/r)^{\beta}$ *and* $\alpha + \beta > n-1$.

Let E_1 be the exceptional set mentioned in the introduction and considered by M. Reade [15], characterized by the condition that for each $\xi \notin E_1$ almost all the radii of the ball $B(\xi, \frac{1}{2})$ lying in B have rectifiable images. Then, arguing as in the preceding theorems, we get the following

COROLLARY. *Theorems* 1, 2, 3 *hold for* E_1.

2. Evaluation of E_γ. Let us denote by E_γ the set of points $\xi \in S$ with the property that there is no endcut of B from ξ (i.e., an arc $\gamma_\xi \subset B$ through ξ) along which f would have a finite limit.

THEOREM 4. *Theorems* 1, 2, 3 *hold also for* E_γ.

Clearly, $E_\gamma \subset E_0$ since every point ξ such that there is no endcut of B from ξ along which f would have a finite limit is (at the same time) inaccessible from B by rectifiable arcs. Hence, the desired result follows from the preceding three theorems.

Remark. The inclusion $E_\gamma \subset E_0$ is strict, since it is possible to have a point $\xi \in S$ and an endcut γ_ξ of ξ from B such that $\lim f(x) < \infty$ exist along γ_ξ but $f(\gamma_\xi)$ is an unrectifiable arc. Thus, for instance, for $n = 2$ we may consider as domain $D^* = f(B)$ the intersection of the unit ball and the domain contained between two logarithmic spirals $\log r = a\vartheta$ and $\log r = a'\vartheta$ $(a' \neq a)$. Then D^* is conformally equivalent to B and the origin is not accessible by rectifiable arcs. However, if $\xi \in S$ is the point corresponding to the origin, then $\lim f(x)$ exists along any endcut of B from ξ and is finite.

3. Evaluation of E_c. By means of the preceding theorems, it is possible to improve a result of V. A. Zorič [22].

Let us recall first that a sequence $\{x_m\}$ of points of B is said *to converge in a cone to a point* $\xi \in S$ if x_m converge to ξ and there exists a constant a, $1 \leqslant a < \infty$, such that

$$|x_m - \xi| \leqslant a d(x_m, S)$$

for all m. ξ is a point at which the mapping f has an *angular boundary value* $\xi^* \in R^n$ if ξ^* is the only boundary value of all the sequences $\{x_m^*\}$ corresponding to the sequences $\{x_m\}$ converging in a cone to ξ; i.e. $\lim_{x \to \xi} f(x) < \infty$ exists if $x \to \xi$ in an arbitrary way in a cone. Let us denote by E the set of points of S at which f does not have an angular boundary value.

THEOREM OF ZORIČ. *If f: $B \to \boldsymbol{R}^n$ is a qc, then*

(a) *the logarithmic capacity $C_0 E_c = 0$ for $n = 2$,*

(b) *the a-capacity $C_a E_c = 0$ for $n \geqslant 2$ and arbitrary $a > 0$* (Zorič [22]).

V. A. Zorič raises the problem whether even for $n > 2$ we have $C_0 E_c^c = 0$, as in the case $n = 2$, solved by A. Beurling [1]. A partial answer is given by

THEOREM 5. *Theorems* 1, 2, 3 *hold also for E_c.*

This is a direct consequence of the preceding theorem, since $E_c = E_\gamma$, as follows from the

THEOREM OF GEHRING. *If f: $B \rightleftarrows D^*$ is qc and f converges to ξ^* as x converges to $\xi \in S$ along some endcut γ_ξ of B from ξ, then f converges to ξ^* as x converges to ξ in a cone* (F. Gehring [7], corollary of theorem 6).

5. Other properties of E_0, E_γ, E_c. Corresponding to an arbitrary set $E \subset \boldsymbol{R}^n$, there is exactly one real number a, $0 \leqslant a \leqslant n$, such that

(a) $C_{a+\varepsilon} E = 0$ for every $\varepsilon > 0$ and

(b) $C_{a-\varepsilon} E > 0$ for every $\varepsilon > 0$.

[When $a = 0$, only part (a) applies, and when $a = n$, only part (b) applies.] The number $a = \dim_c E$ is called the *capacitary dimension* of E.

THEOREM 6. $\dim_c E_c = \dim_c E_\gamma = \dim_c E_c = 0$.

This is a direct consequence of

PROPOSITION 8. $C_a(E_0) = 0$ *for every* $a > 0$.

For the proof, see our paper in [4].

Now let $F \subset \boldsymbol{R}^n$ be a closed set; then the Hausdorff dimension of F, $\dim_H F$, is the infimum of all numbers $a > 0$ such that the a-dimensional Hausdorff measure of F is zero.

PROPOSITION 9. *If $F \subset \boldsymbol{R}^n$ is a closed set, then*

$$\dim_H F = \dim_c F.$$

(For the proof, see for instance O. Frostman [5], p. 90.)

THEOREM 10. $\dim_H E_0 = \dim_H E_\gamma = \dim_H E_c = 0.$

Clearly, if for $0 < \alpha < 1$ the α-dimensional Hausdorff measure of a set is zero, then such a set cannot contain any continuum, i.e., is totally disconnected, so that from the preceding theorem we obtain

COROLLARY. *E_0, E_γ, E_c are totally disconnected.*

6. An extension of a theorem of M. Tsuji.

TSUJI'S THEOREM. *If $u(x)$ is harmonic in $B \subset \boldsymbol{R}^3$ and*

$$\int_B |\operatorname{grad} u|^2 \frac{d\tau}{\sqrt{1-r}} < \infty, \qquad r = |x|,$$

then there exists a set $E \subset S$ which is of Newtonian capacity zero and such that if $\xi \in S$ does not belong to E, then

$$\lim_{x \to \xi} u(x) = u(\xi) < \infty \qquad \textit{exists uniformly}$$

when x tends to ξ inside a Stolz domain whose vertex is at ξ and, for any rectilinear segment l_ξ which connects ξ with a point of E,

$$\int_{l_\xi} |\operatorname{grad} u|\, ds < \infty.$$

(For the proof, see M. Tsuji [17], theorem 3.)

We recall that a *Stolz domain* is a domain which is bounded by a cone whose vertex is at ξ and whose generator forms an angle ϑ_0 $(< \pi/2)$ with the radius $\overline{O\xi}$.

Now, let $L^p(E)$ be the class of all Lebesgue-measurable functions f in $\boldsymbol{R}^n$ such that $\int_E |f|^p d\tau < \infty$. By the same argument as in Theorem 1, we obtain a result which is more general (in some respects) than Tsuji's theorem:

THEOREM 8. *Let $u \in L^{n+\varepsilon}[B(R)]$ $(0 < R < \infty,\ \varepsilon > 0)$. There exists a set $E \subset S(R)$ of logarithmic capacity zero such that if $\xi \in S(R)$ does not belong to E, then for almost all linear segments l_ξ joining ξ and an arbitrary point of $B(R)$*

$$\int_{l_\xi} |u(x)|\, ds < \infty.$$

Indeed, suppose that $C_0(E) > 0$ and put

$$I = \int_{\Delta} |u|^{n+\varepsilon} u_0^{\tau_0} d\tau,$$

where Δ is as in Theorem 1. Then, by hypothesis, and taking into account also Proposition 3, we have

$$I = \int_{\Delta} |u|^{n+\varepsilon} u_0^{\tau_0} d\tau < M_1 < \infty. \tag{10}$$

On the other hand, for $\xi \in E$,

$$\infty = \int_{l_\xi} |u(x)|\, ds = \int_0 \varrho_\varphi |u(\varrho, \vartheta_1, \ldots, \vartheta_{n-1})|\, d\varrho,$$

where ϱ_φ denotes the length of the segment of direction φ joining ξ with the corresponding point of the sphere $S\left[\frac{\xi(1-r)}{2}, \frac{1-r}{2}\right]$, where $r = r_\Delta$ is chosen so that the ball $B\left[\frac{\xi(1-r)}{2}, \frac{1-r}{2}\right] \subset \Delta$. Hence, arguing as in Theorem 1, we get

$$\infty = \int_S d\sigma \int_0^{\varrho_\varphi} |u|\, d\varrho = \int_{B\left[\frac{\xi(1-r)}{2}, \frac{1-r}{2}\right]} |u| \frac{d\tau}{\varrho^{n-1}} \leqslant N \int_{\Delta} |u|^{n(1+\varepsilon_1)} \log \frac{1}{\varrho_{\xi x}}\, d\tau,$$

and integrating over $E^\Delta = E \cap \Delta$ with respect to $\tau_0(\xi)$, we get

$$\infty = \int_{E^\Delta} d\tau_0(\xi) \int |u|^{n(1+\varepsilon_1)} \log \frac{1}{\varrho_{\xi x}}\, d\tau_x = \int |u|^{n+\varepsilon} \left[\int_{E^\Delta} \log \frac{1}{\varrho_{\xi x}}\, d\tau_0(\xi) \right] d\tau_x = I,$$

where $\varepsilon = n\varepsilon_1$ and $d\tau_x$ is written instead of $d\tau$ in order to establish precisely the variable of integration. But this contradicts (10), establishing that $C_0(E) = 0$, as desired.

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