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On some properties of *LB*-splines

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Abstract. We give a relation between basic splines in the spaces of Tchebyshev splines and polynomial splines. By means of a new definition of normalized basic splines in the space of Tchebyshev splines we obtain a few further propreties of basic splines analogous to polynomial splines and we apply the results obtained to L-splines.

1. Introduction. The purpose of this paper is to generalize some properties of polynomial B-splines to basic splines in the space of Tchebyshevian spline functions, to obtain a relation between those basic splines and then to apply it to generalized splines.

Further, we need the following notations, definitions and properties of extended complete Tchebyshev systems (ECT-systems, see [5]).

DEFINITION 1. We say that $\{u_i\}_{i=0}^n$, $u_i \in C^n[a, b]$ is an *ECT-system on* $[a, b]$ if, for any points $a \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k \leqslant b$, $k = 0, \ldots, n$,

$$D\begin{pmatrix} t_0, \ldots, t_k \\ u_0, \ldots, u_k \end{pmatrix} = \det\left(D^{d_j} u_i(t_j)\right)_{i,j=0}^k > 0,$$

where $d_j = \max\{l\colon t_j = t_{j-1} = \ldots = t_{j-l}\}$, $j = 0, \ldots, k$, and D is the differentiation operator.

An ECT-system $\{u_i\}_0^n$, $u_0 = 1$ admits the representation

$$\begin{aligned}
& u_0(t) = 1, \\
& u_1(t) = \int_a^t w_1(\tau_1)\, d\tau_1, \\
(1)\qquad & u_2(t) = \int_a^t w_1(\tau_1) \int_a^{\tau_1} w_2(\tau_2)\, d\tau_2\, d\tau_1, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& u_n(t) = \int_a^t w_1(\tau_1) \int_a^{\tau_1} w_2(\tau_2) \ldots \int_a^{\tau_{n-1}} w_n(\tau_n)\, d\tau_n \ldots d\tau_1,
\end{aligned}$$

where $w_i \in C^{n-i}[a, b]$ and $w_i(t) > 0$ for $t \in [a, b]$, $i = 0, \ldots, n$.

The adjoint system $\{v_i\}_0^n$ is defined as follows:

$$
\begin{aligned}
&v_0(t) = 1,\\
&v_1(t) = \int_a^t w_n(\tau_1)\,d\tau_1,\\
(2)\qquad &v_2(t) = \int_a^t w_n(\tau_1) \int_a^{\tau_1} w_{n-1}(\tau_2)\,d\tau_2\,d\tau_1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&v_n(t) = \int_a^t w_n(\tau_1) \int_a^{\tau_1} w_{n-1}(\tau_2) \ldots \int_a^{\tau_{n-1}} w_1(\tau_n)\,d\tau_n \ldots d\tau_1.
\end{aligned}
$$

Define

$$
D_j\varphi = \frac{1}{w_j}\frac{d}{dt}\varphi, \qquad D_j^*\varphi = \frac{d}{dt}\left(\frac{\varphi}{w_j}\right),
$$
$$
L_j\varphi = D_0 D_j \ldots D_1\varphi, \qquad L_j^*\varphi = D_1^* \ldots D_j^* D_0^*\varphi, \qquad j = 0, \ldots, n.
$$

Systems (1) and (2) span the null spaces of the differential operators L_n and L_n^*, respectively.

Let $\Delta = \{a = t_{-n} = \ldots = t_0 < t_1 = \ldots = t_{\alpha_1} < t_{\alpha_1+1} = \ldots = t_{\alpha_1+\alpha_2} < \ldots < t_N = \ldots = t_{N+n} = b\} = \{a = s_0 < s_1 < \ldots < s_M = b\}$; α_j is the multiplicity of the point s_j, $j = 0, \ldots, M$, $\alpha_0 = \alpha_M = n+1$, $\sum_{j=0}^{M} \alpha_j = N+2n+1$.

Definition 2. A function s is called a *Tchebyshev spline function w.r.t. the operator L_n and the partition Δ* if

(a) $L_n s = 0$ in the intervals (s_{j-1}, s_j), $j = 1, \ldots, M$,

(b) $\exists \varepsilon_j > 0$: $s \subset C^{n-\alpha_j}(s_j - \varepsilon_j, s_j + \varepsilon_j)$, $j = 1, \ldots, M-1$.

We denote the set of these functions by $S_\Delta^n[a, b]$.

Lemma 1 (see [5]). *The function*

$$
\varphi_n(t, x) = \begin{cases} \int_x^t w_n(\tau_1) \int_x^{\tau_1} w_{n-1}(\tau_2) \ldots \int_x^{\tau_{n-1}} w_1(\tau_n)\,d\tau_n \ldots d\tau_1 & \text{for } x \leqslant t \leqslant b,\\ 0 & \text{for } t < x, \end{cases}
$$

satisfies the following condition:

$$
D\begin{pmatrix} t_0, \ldots, t_n, t_{n+1} \\ v_0, \ldots, v_n, \varphi_n \end{pmatrix} \geqslant 0
$$

for any system $\{t_j\}_0^{n+1}$ such that $a \leqslant t_0 \leqslant \ldots \leqslant t_{n+1} \leqslant b$.

For $n = 0$

$$
\varphi_n(t, x) = \begin{cases} 1 & \text{for } x \leqslant t \leqslant b,\\ 0 & \text{for } t < x. \end{cases}
$$

Further, for $a \leqslant x \leqslant t_0 \leqslant \ldots \leqslant t_{n+1} \leqslant b$

$$D\begin{pmatrix} t_0, \ldots, t_n, t_{n+1} \\ v_0, \ldots, v_n, \varphi_n \end{pmatrix} = 0$$

and for $t_0 < x < t_{n+1}$

$$D\begin{pmatrix} t_0, \ldots, t_n, t_{n+1} \\ v_0, \ldots, v_n, \varphi_n \end{pmatrix} > 0.$$

The function φ_n *is a Green function associated with the operator* L_n^* *and the function* $w(t) = \int_a^b \varphi_n(t, x)dx$ *is a solution of the equation* $L_n^* w = 1$.

Definition 3. We define the divided difference of a function f at the points $t_0 \leqslant \ldots \leqslant t_{n+1}$, $t_0 < t_{n+1}$ w.r.t. the operator L_n^* by

$$[t_0, \ldots, t_{n+1}; f]_* = \frac{D\begin{pmatrix} t_0, \ldots, t_n, t_{n+1} \\ v_0, \ldots, v_n, f \end{pmatrix}}{D\begin{pmatrix} t_0, \ldots, t_n, t_{n+1} \\ v_0, \ldots, v_n, w \end{pmatrix}}. \tag{3}$$

Remark. For the system $\{t^i\}_0^n$, $w_0 = 1$, $w_i = i$, $i = 1, \ldots, n$, and $[t_0, \ldots, t_{n+1}; f]_* = (n+1)[t_0, \ldots, t_{n+1}; f]$, where the last expresion is the usual divided difference (see [3]). Properties of generalized divided differences may be found in [4] and [7].

Definition 4. The ith $L_n B$-spline (B-spline w.r.t. the system $\{u_i\}_0^n$) w.r.t. the partition Δ is defined by

$$M_{i,n}(x) = [t_i, \ldots, t_{i+n+1}; \varphi_n(t, x)]_*, \qquad i = -n, \ldots, N-1. \tag{4}$$

We can see that

$$\frac{d}{dx}\varphi_n(t, x) = -w_1(x)\varphi_{n-1}(t, x), \tag{5}$$

where

$$\varphi_{n-1}(t, x) = \begin{cases} \int\limits_x^t w_n(\tau_1) \int\limits_x^{\tau_1} w_{n-1}(\tau_2) \ldots \int\limits_x^{\tau_{n-2}} w_2(\tau_{n-1}) d\tau_{n-1} \ldots d\tau_1 & \text{for } x \leqslant t \leqslant b, \\ 0 & \text{for } t < x. \end{cases}$$

Hence $M_{i,n} \in S_\Delta^n[a, b]$.

Let $M_{j,n-1}$ be the basic spline w.r.t. the operator $\tilde{L}_{n-1} = D_0 D_n \ldots D_2$ (all points of multiplicity $n+1$ are exchanged for points of multiplicity n). Put

$$\tilde{M}_{j,n-1}(x) = M_{j,n-1}(x) / \int\limits_{t_j}^{t_{j+n}} w_1(\tau) M_{j,n-1}(\tau) d\tau.$$

DEFINITION 5. The jth normalized $L_n B$-spline w.r.t. the partition Δ is defined by

(6) $N_{j,n}(x)$

$$= \begin{cases} \chi_{t_j}(x) - \int_a^x w_1(\tau)\tilde{M}_{j+1,n-1}(\tau)\,d\tau & \text{for } t_j = \ldots = t_{j+n} < t_{j+n+1}, \\ \int_a^x w_1(\tau)[\tilde{M}_{j,n-1}(\tau) - \tilde{M}_{j+1,n-1}(\tau)]\,d\tau & \text{for } t_j < t_{j+n} \text{ and } t_{j+1} < t_{j+n+1}, \\ \int_a^x w_1(\tau)\tilde{M}_{j,n-1}(\tau)\,d\tau - \chi_{t_{j+n}}(x) & \text{for } t_j < t_{j+1} = \ldots = t_{j+n+1}, \end{cases}$$

where $\chi_t(x) = 1$ for $x > t$ and 0 for $x \leqslant t$, and for $j = -n$ we put $N_{-n,n}(a) = 1$.

Again by (5) we obtain $N_{j,n} \in S_\Delta^n[a, b]$.

Basic splines w.r.t. the Tchebyshev systems were defined by Karlin in [4] and in the special case of trigonometric splines by Schoenberg in [9]. Normalized basic splines were defined in another way by Marsden in [6]. With the help of Definition 5 we shall obtain a few further properties of basic splines analogous to polynomial splines.

2. Properties of basic splines. For polynomial B-splines we have (see [1], [2])

$$[t_0, \ldots, t_{n+1}; f] = \frac{1}{(n+1)!}\int_{t_0}^{t_{n+1}} f^{(n+1)}(t)\hat{M}_{0,n}(t)\,dt,$$

where $\hat{M}_{0,n}$ is the B-spline w.r.t. the system $\{t^i\}_0^n$. Since $\int_{t_0}^{t_{n+1}} \hat{M}_{0,n}(t)\,dt = 1$ (see [1], [2]), we have

$$\lim_{t_{n+1}\to t_0} [t_0, \ldots, t_{n+1}; f] = \frac{1}{(n+1)!} f^{(n+1)}(t_0).$$

By (3) and (4) we obtain

$$\int_{t_0}^{t_{n+1}} M_{0,n}(t)\,dt = 1.$$

As for polynomial splines, we apply the generalized Taylor formula for Tchebyshev systems (see [1], [2], [5]) and obtain

$$[t_0, \ldots, t_{n+1}; f]_* = \int_{t_0}^{t_{n+1}} L_n^* f(t)\, M_{0,n}(t)\,dt.$$

Hence

$$\lim_{t_{n+1}\to t_0} [t_0, \ldots, t_{n+1}; f]_* = L_n^* f(t_0). \tag{7}$$

Applying (7) and the properties of determinants, we obtain

THEOREM 1. *Let $\Delta' = \{a \leqslant t'_{-n} < t'_{-n+1} < \ldots < t'_{N+n} \leqslant b\}$ be a partition of the interval $[a, b]$ with distinct points and let $f \in C^{n+1}[a, b]$. Then there exists*

$$\lim_{\substack{t'_i \to t_i \\ i=0,\ldots,n+1}} [t'_0, \ldots, t'_{n+1}: f]_* = [t_0, \ldots, t_{n+1}: f]_*.$$

Hence it follows that it suffices to prove the properties of splines for partitions with distinct points.

Further we need the following

LEMMA 2. *Every spline $\varphi \in S^n_{\Delta_j}[a, b]$ satisfying the conditions*

$$D^i \varphi(t_k) = 0, \quad i = 0, \ldots, n-1, \ k = j, j+n+1,$$

can be represented in the form

$$\varphi(x) = \alpha M_{j,n}(x),$$

where α is a constant depending only on the function φ and $\Delta_j = \{t_j < \ldots < t_{j+n+1}\}$.

Proof. The partition Δ_j has only distinct points. Since $w_j > 0$, the function φ satisfies the following conditions:

$$L_i \varphi(t_k) = 0, \quad k = j, j+n+1, \ i = -1, 0, \ldots, n-2,$$

where $L_0 \varphi = D_0 \varphi$ and $L_{-1} \varphi = \varphi$. Put

$$\tilde{\varphi}_{n-i}(t, x) = \begin{cases} \int_t^x w_n(\tau_1) \int_{\tau_1}^x w_{n-1}(\tau_2) \ldots \int_{\tau_{n-i-1}}^x w_{i+1}(\tau_{n-i}) d\tau_{n-i} \ldots d\tau_1 & \text{for } t \leqslant x, \\ 0 & \text{for } x < t, \end{cases}$$

$i = -1, 0, \ldots, n-2$.

We can write the function φ in the following form (see [5]):

$$\varphi(x) = \sum_{l=j}^{j+n} b_l \tilde{\varphi}_n(t_l, x).$$

Therefore

$$L_i \varphi(t_{i+n+1}) = \sum_{l=j}^{j+n} b_l \tilde{\varphi}_{n-i-1}(t_l, t_{j+n+1}), \quad i = -1, 0, \ldots, n-2.$$

This system has the matrix $A_n = [\tilde{\varphi}_i(t_l, t_{j+n+1}), \ i = n, \ldots, 1, \ l = j, \ldots, j+n]$.

Let B_n be the matrix obtained from A_n by concelling the last column of A_n. We shall prove that $\det B_n > 0$. For $n = 1$, $B_1 = \tilde{\varphi}_1(t_j, t_{j+1}) = \int\limits_{t_j}^{t_{j+1}} w_1(t)\,dt > 0$, because $w_1 > 0$. Assume that $\det B_{n-1} > 0$ for every system of functions $\{\tilde{w}_i\}_0^{n-1}$, $\tilde{w}_i > 0$. Subtracting the ith column from its predecessor and factoring out the integrals from the function w_n, we obtain

$$B_n = \int\limits_{t_j}^{t_{j+1}} w_n(x_1)\ldots \int\limits_{t_{j+n-2}}^{t_{j+n-1}} w_n(x_{n-1}) \int\limits_{t_{j+n-1}}^{t_{j+n+1}} w_n(x_n)\tilde{B}_{n-1}(x_1, \ldots, x_n)\,dx_1\ldots dx_n,$$

where $\tilde{B}_{n-1}(x_1, \ldots, x_n) = \det[\tilde{\varphi}_{n-i}(x_l, t_{j+n+1}),\ i = 1, \ldots, n,\ l = 1, \ldots, n]$.

Repeating this reasoning for the determinant $\tilde{B}_{n-1}$ and expanding the determinant w.r.t. the last row, we obtain

$$\det B_n = \int\limits_{t_j}^{t_{j+1}} w_n(x_1)\ldots \int\limits_{t_{j+n-2}}^{t_{j+n-1}} w_n(x_{n-1}) \int\limits_{t_{i+n-1}}^{t_{j+n+1}} w_n(x_n) \int\limits_{x_1}^{x_2} w_{n-1}(y_1)\ldots$$
$$\ldots \int\limits_{x_{n-1}}^{x_n} w_{n-1}(y_{n-1})B_{n-1}(y_1, \ldots, y_{n-1})\,dx_1\ldots dx_n\,dy_1\ldots dy_{n-1},$$

where $B_{n-1}(y_1, \ldots, y_{n-1}) = \det[\tilde{\varphi}_{n-i}(y_l, t_{j+n+1}),\ i = 2, \ldots, n,\ l = 1, \ldots, n-1]$. Since $x_i < x_{i+1}$, $i = 1, \ldots, n$, we have $B_{n-1} > 0$ by the inductive hypothesis, whence $\det B_n > 0$ and the theorem is proved.

COROLLARY. *There exists a constant r_j such that*

$$N_{j,n}(x) = r_j M_{j,n}(x). \tag{8}$$

Remark. For the system $\{x^i\}_0^n$, $r_j = (t_{j+n+1} - t_j)/(n+1)$.

THEOREM 2. *There exists a constant C depending only on the system $\{u_i\}$ such that*

$$C^{-1}\hat{M}_{j,n}(x) \leqslant M_{j,n}(x) \leqslant C\hat{M}_{j,n}(x),$$

where $\hat{M}_{j,n}$ is the j-th B-spline for the system $\{x^i\}_0^n$.

Proof. Assume that Δ has only distinct points.

$$M_{j,n}(x) = D\begin{pmatrix} t_0, \ldots, t_n, t_{n+1} \\ v_0, \ldots, v_n, \varphi_n(t, x)\end{pmatrix} \Big/ D\begin{pmatrix} t_0, \ldots, t_n, t_{n+1} \\ v_0, \ldots, v_n, w\end{pmatrix} = L_n(x)/M_n(x)$$

and analogously $\hat{M}_{j,n}(x) = \hat{L}_n(x)/\hat{M}_n(x)$.

$L_n(x) = \det[a_{ij}]_{i,j=0}^{n+1}$, where $a_{0j} = 1$, $j = 0, \ldots, n+1$,

$$a_{ij} = \int\limits_a^{t_j} w_n(\tau_1)\int\limits_a^{\tau_1} w_{n-1}(\tau_2)\ldots\int\limits_a^{\tau_{n-i}} w_{n-i+1}(\tau_i)\,d\tau_i\ldots d\tau_1,$$
$$i = 1, \ldots, n,\ j = 0, \ldots, n+1,$$

$$a_{n+1,j} = \varphi_n(t_j, x) = \int\limits_a^{t_j} w_n(\tau_1)\int\limits_a^{\tau_1} w_{n-1}(\tau_2)\ldots\int\limits_a^{\tau_{n-1}} w_1(\tau_n)\varrho_x(\tau_n)\,d\tau_n\ldots d\tau_1,$$

$j = 0, \ldots, n+1$, where $\varrho_x(t) = 1$ for $t \geqslant x$ and 0 for $t < x$.

Subtracting the jth column from its successor, then expanding the determinant w.r.t. the first row and applying the properties of determinants, we obtain

$$L_n(x) = \int_{t_0}^{t_1} w_n(x_1)\ldots\int_{t_n}^{t_{n+1}} w_n(x_{n+1})\det[b_{ij}]\,dx_1\ldots dx_{n+1},$$

where $b_{0j} = 1$, $j = 1,\ldots,n+1$,

$$b_{ij} = \int_a^{x_j} w_{n-1}(\tau_1)\int_a^{\tau_1} w_{n-2}(\tau_2)\ldots\int_a^{\tau_{i-1}} w_{n-i}(\tau_i)\,d\tau_i\ldots d\tau_1,$$

for $i = 1,\ldots,n-1$, $j = 1,\ldots,n+1$,

$$b_{n+1,j} = \int_a^{x_j} w_{n-1}(\tau_1)\int_a^{\tau_1} w_{n-2}(\tau_2)\ldots\int_a^{\tau_{n-2}} w_1(\tau_{n-1})\varrho_x(\tau_{n-1})\,d\tau_{n-1}\ldots d\tau_1,$$

$j = 1,\ldots,n+1$.

Repeating this reasoning, we obtain

$$(9)\quad L_n(x) = \int_{t_0}^{t_1} w_n(x_{1,1})\ldots\int_{t_n}^{t_{n+1}} w_n(x_{1,n+1})\int_{x_{1,1}}^{x_{1,2}} w_{n-1}(x_{2,1})\ldots$$
$$\ldots\int_{x_{1,n}}^{x_{1,n+1}} w_{n-1}(x_{2,n})\int_{x_{2,1}}^{x_{2,2}} w_{n-2}(x_{3,1})\ldots$$
$$\ldots\int_{x_{n-1,1}}^{x_{n-1,2}} w_1(x_{n,1})\int_{x_{n-1,2}}^{x_{n-1,3}} w_1(x_{n,2})[\varrho_x(x_{n,2})-$$
$$-\varrho_x(x_{n,1})]\,dx_{1,1}\ldots dx_{1,n+1}\,dx_{2,1}\ldots dx_{n,1}\,dx_{n,2}.$$

Since $x_{j,k} \leqslant x_{j,k+1}$, the difference $\varrho_x(x_{n,2})-\varrho_x(x_{n,1})$ admits only two values, 0 and 1. Remember that, for the system $\{x^i\}_0^n$, $w_0 = 1$, $w_i = i$, $i = 1,\ldots,n$. Since the functions w_i are continuous and positive in the interval $[a, b]$, there exist positive constants c_i and d_i such that

$$(10)\qquad 0 < ic_i \leqslant w_i \leqslant id_i \quad \text{for } i = 1,\ldots,n.$$

Writing $\hat{L}_n(x)$ in the form (9), we obtain

$$(11)\quad \hat{L}_n(x) = 2^3 3^4\ldots n^{n+1}\int_{t_0}^{t_1}\ldots\int_{t_n}^{t_{n+1}}\int_{x_{1,1}}^{x_{1,2}}\ldots\int_{x_{n-1,2}}^{x_{n-1,3}} [\varrho_x(x_{n,2})-\varrho_x(x_{n,1})]\,dx_{1,1}\ldots dx_{n,2}.$$

Since $M_n(x) = \int_{t_0}^{t_{n+1}} L_n(x)$ and $\hat{M}_n(x) = \int_{t_0}^{t_{n+1}} \hat{L}_n(x)\,dx$, we have by (9)–(11)

$$(12)\quad \left(\frac{c_1}{d_1}\right)^2\left(\frac{c_2}{d_2}\right)^3\cdot\ldots\cdot\left(\frac{c_n}{d_n}\right)^{n+1}\hat{M}_{j,n}(x) \leqslant M_{j,n}(x) \leqslant \left(\frac{d_1}{c_1}\right)^2\left(\frac{d_2}{c_2}\right)^3\cdot\ldots\cdot\left(\frac{d_n}{c_n}\right)^{n+1}\hat{M}_{j,n}(x).$$

Since the above constants depend only on the functions w_j, by Theorem 1 we obtain the theorem under consideration.

THEOREM 3. *Basic splines have the following properties*:

(M.1) $\operatorname{supp} M_{j,n} = [t_j, t_{j+n+1}], \quad M_{j,n}(x) > 0 \quad$ *for* $t_j < x < t_{j+n+1}$,

(N.1) $\operatorname{supp} N_{j,n} = [t_j, t_{j+n+1}], \quad N_{j,n}(x) > 0 \quad$ *for* $t_j < x < t_{j+n+1}$,

(M.2)
$$\int_{t_j}^{t_{j+n+1}} M_{j,n}(x)\,dx = 1,$$

(N.2)
$$\sum_{j=-n}^{N-1} N_{j,n}(x) = 1.$$

Proof. The first three properties follow from the definition of basic splines, Lemma 1, (8) and the properties of polynomial splines. Let $x \in [t_k, t_{k+1}]$. Then

$$\begin{aligned}\sum_{j=-n}^{N-1} N_{j,n}(x) &= \sum_{j=k-n}^{k} N_{j,n}(x)\\ &= \int_a^x w_1(\tau) \sum_{j=k-n}^{k} [\hat{M}_{j,n-1}(\tau) - \tilde{M}_{j+1,n-1}(\tau)]\,d\tau\\ &= \int_a^x w_1(\tau)[\tilde{M}_{k-n,n-1}(\tau) - \tilde{M}_{k+1,n-1}(\tau)]\,d\tau\\ &= \int_a^x w_1(\tau)\tilde{M}_{k-n,n-1}(\tau)\,d\tau = 1,\end{aligned}$$

for $t_j < t_{j+n}$ and $t_{j+1} < t_{j+n+1}$. To the remaining cases of (N.2) we apply Theorem 1.

Now, we estimate the constants r_j, $r_j = N_{j,n}(x)/M_{j,n}(x)$, $x \in (t_j, t_{j+n+1})$. Hence for $x \in (t_j, t_{j+1})$ by (12) we obtain

$$r_j = \frac{\int_{t_j}^{x} w_1(\tau) M_{j,n-1}(\tau)\,d\tau}{M_{j,n}(x) \int_{t_j}^{t_{j+n+1}} w_1(\tau) M_{j,n-1}(\tau)\,d\tau} \leqslant (n+1)\beta_n \frac{\hat{N}_{j,n}(x)}{\hat{M}_{j,n}(x)} = (t_{j+n+1} - t_j)\beta_n$$

and analogously $r_j \geqslant \alpha_n(t_{j+n+1} - t_j)$, where α_n and β_n depend only on the system $\{u_i\}$ and $\hat{N}_{j,n}$ is the jth normalized B-spline for the system $\{x^i\}_0^n$. Thus we have proved

THEOREM 4. *There exist constants α_n and β_n depending only on the system $\{u_i\}_0^n$ such that*

$$\alpha_n(t_{j+n+1} - t_j) \leqslant r_j \leqslant \beta_n(t_{j+n+1} - t_j).$$

From (6) we obtain

LEMMA 3. *Put* $\beta_j = \left(\int\limits_{t_j}^{t_{j+n+1}} w_1(\tau) M_{j,n-1}(\tau)\, d\tau \right)^{-1}$. *Then*

$$D_1 N_{j,n} = \begin{cases} \beta_j M_{j,n-1} & \text{for } t_{j+1} = t_{j+n+1}, \\ \beta_j M_{j,n-1} - \beta_{j+1} M_{j+1,n-1} & \text{for } t_j < t_{j+n} \text{ and } t_{j+1} < t_{j+n+1}, \\ -\beta_{j+1} M_{j+1,n-1} & \text{for } t_j = t_{j+n}. \end{cases}$$

Hence follows

THEOREM 5. *Let* $f \in S^n_\Delta[a, b]$ *and* $f = \sum_{j=-n}^{N-1} a_j N_{j,n}$. *Then*

$$D_1 f = -a_{-n\beta-n+1} M_{-n+1,n-1} + \\ + \sum_{\substack{j=-n+2 \\ t_j < t_{j+n}}}^{N-2} (a_j - a_{j-1}) \beta_j M_{j,n-1} + a_{N-1} \beta_{N-1} M_{N-1,n-1}.$$

Hence, as for polynomial splines, we obtain

COROLLARY. *The system* $\{N_{j,n}\}_{-n}^{N-1}$ *is a basis in the space* $S^n_\Delta[a, b]$.

Put $N_{j,n,p} = r_j^{-1/p} N_{j,n}$, $1 \leqslant p \leqslant \infty$. As for polynomial splines, we can prove (see [2], [8], [10])

THEOREM 6. *There exists a constant* $D_n > 0$ *depending only on the system* $\{u_i\}$ *such that*

$$D_n \|a\|_p \leqslant \Big\| \sum_{j=-n}^{N-1} a_j N_{j,n,p} \Big\|_p \leqslant \|a\|_p,$$

where $a = (a_{-n}, \ldots, a_{N-1}) \in l_p$, $1 \leqslant p \leqslant \infty$.

From Theorems 5 and 6 we obtain

COROLLARY. *There exist constants* C_1 *and* C_2 *depending only on* p *and* q $(1 \leqslant p,\ q \leqslant \infty)$ *and the system* $\{u_i\}$ *such that for any* $s \in S^n_\Delta[a, b]$

$$\|Ds\|_p \leqslant C_1 \|s\|_p [\min(t_{j+n} - t_j)]^{-1}$$

and

$$\|s\|_p \leqslant C_2 \|s\|_q \max_j [(t_{j+n+1} - t_j)^{1/q - 1/p}].$$

3. *L*-splines. Let

(13) $$L = D^{n+1} + \sum_{i=0}^{n} a_i(x) D^i$$

be a linear differential operator defined on the interval $[a, b]$ with the null space N_L.

DEFINITION 6. A function s is called an *L-spline w.r.t. the partition Δ* if

(a) $Ls = 0$ on the intervals (s_{j-1}, s_j), $j = 1, \ldots, M$,

(b) $\exists \varepsilon_j > 0$: $s \in C^{n-\alpha_j}(s_j - \varepsilon_j, s_j + \varepsilon_j)$, $j = 1, \ldots, M-1$,

where α_j is the multiplicity of the point s_j.

We can reduce the investigation of L-splines to the investigation of Tchebyshev splines by means of the following theorem:

THEOREM 7 (see [5]). *For every operator L of the form* (13) *there exists $\delta > 0$ such that, for every subinterval $I \subset [a, b]$ with the length $|I| \leqslant \delta$, the space N_L has a basis $\{u_i^I\}_0^n$ which is an ECT-system in the subinterval I.*

Assume that $\max(t_{i+n+1} - t_i) < \delta/4$ and $a_0(x) = 0$. Then the definition and properties of basic splines reduce to those of Tchebyshev splines. If there exists a function $u(x) > 0$ for $x \in [a, b]$ in the space N_L, then we can change the basis $\{u_i\}_0^n$ of the space N_L for the system which includes 1. In the general case we divide the interval $[a, b]$ into a finite number of subintervals I_j with the length $\delta/4 < |I_j| < \delta/2$ and we repeat the above construction of $L_n B$-splines, in each of the subintervals $J_i = I_i \cup I_{i+1}$.

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