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# On a weak coregular division of a differential space

by R. MAJCHRZAK (Łódź)

**Abstract.** Let $(M, \mathscr{C})$ be a differential space, $R$ an equivalence relation on $M$. We denote by $p_R$ the natural mapping $p_R\colon (M, \mathscr{C}) \to (M, \mathscr{C})/R$. The definition of coregularity and weak coregularity in the category of differential spaces was formulated by Waliszewski [6]. In the present paper we give a necessary and sufficient condition for the weak coregularity of the mapping $p_R$. We prove a theorem on a weak coregular division of a generalized Lie group by a Lie subgroup.

**Introduction.** In the theory of differential manifolds the notion of a coregular mapping is well known. Equivalent conditions for coregularity are given by Serre [4]. These conditions have their analogues in the category of differential spaces. However, they are not any more equivalent in that category. In paper [7] we can find a necessary and sufficient condition for the coregularity of the natural mapping $p_R\colon (M, \mathscr{C}) \to (M, \mathscr{C})/R$, where $R$ is an equivalence relation on $M$.

The main theorem of the present paper gives a necessary and sufficient condition for the weak coregularity of the mapping $p_R$. As a consequence, a theorem on a weak coregular division of a generalized Lie group by a Lie subgroup is obtained.

**1. Preliminaries.** We use the same terminology and notation as in [6] and [7]. Let $M$ be any set and $\mathscr{C}$ an arbitrary set of real functions defined on $M$. Any pair $(M, \mathscr{C})$ such that $\mathscr{C}_M = \mathscr{C} = \mathrm{sc}\mathscr{C}$ is called a *differential space*.

Let $(M, \mathscr{C})$ be a differential space. If $(M, \cdot)$ is a group, then for any $p, p_1, p_2$ the elements $p_1 \cdot p_2, p^{-1}$ also belong to $M$. If, moreover, the mappings

$$(M, \mathscr{C}) \times (M, \mathscr{C}) \to (M, \mathscr{C}), \quad (p_1, p_2) \mapsto p_1 \cdot p_2,$$
$$(M, \mathscr{C}) \to (M, \mathscr{C}), \quad p \mapsto p^{-1},$$

are smooth, then the ordered system $(M, \mathscr{C}, \cdot)$ is called a *generalized Lie group*.

Let $(M, \mathscr{C})$ and $(N, \mathscr{D})$ be differential spaces. A smooth mapping

$$f\colon (M, \mathscr{C}) \to (N, \mathscr{D}) \tag{1}$$

will be called *coregular at a point* $p \in M$ if and only if there exist a neighbourhood $U$ of $p$ open in the topology $\tau_{\mathscr{C}}$, a set $V$ open in $\tau_{\mathscr{D}}$ such that $f[U] \subset V$, a differential space $(N_0, \mathscr{D}_0)$ and a diffeomorphism

$$\varphi\colon (U, \mathscr{C}_U) \to (V, \mathscr{D}_V) \times (N_0, \mathscr{D}_0)$$

such that $\mathrm{pr}_1 \circ \varphi = f|U$.

A mapping which is coregular at every point $p \in M$ is said to be *coregular* or is called a *submersion.*

A smooth mapping (1) will be called weak coregular at a point $p \in M$ if and only if there exist neighbourhoods $U$ and $V$ or points $p$ and $f(p)$ open in the topologies $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$, respectively, and a mapping

$$\sigma\colon (V, \mathscr{D}_V) \to (U, \mathscr{C}_U)$$

such that $\sigma[V] \subset U, f \circ \sigma = \mathrm{id}_V$ and $\sigma(f(p)) = p$. A mapping which is weak coregular at every point of $M$ is said to be *weak coregular.*

Every coregular mapping is weak coregular. In general, the converse statement is not true (see [6]). The composition of two coregular (weak coregular) mappings is also coregular (weak coregular).

Let $(M, \mathscr{C})$, $(M', \mathscr{C}')$, $(N, \mathscr{D})$, $(N', \mathscr{D}')$ be differential spaces, and $f\colon M \to M'$, $g\colon N \to N'$. By $f \times g$ (cf. [5], p. 108) we denote the mapping defined on the Cartesian product $M \times N$ by the formula

$$(f \times g)(p, q) = (f(p), g(q)) \quad \text{for } p \in M,\ q \in N,$$

having its values in the Cartesian product $M' \times N'$. The mapping

$$f \times g\colon (M \times N, \mathscr{C} \times \mathscr{D}) \to (M' \times N', \mathscr{C}' \times \mathscr{D}')$$

is smooth if and only if the mappings

$$f\colon (M, \mathscr{C}) \to (M', \mathscr{C}') \quad \text{and} \quad g\colon (N, \mathscr{D}) \to (N', \mathscr{D}')$$

are smooth.

**2. A weak coregular division of a differential space by an equivalence relation.**

LEMMA 1. *If a mapping* (1) *is weak coregular and* $R$ *is the set of all pairs* $(x, y) \in M \times M$ *such that* $f(x) = f(y)$, *then the mapping*

$$\mathrm{pr}_1|R\colon (R, (\mathscr{C} \times \mathscr{C})_R) \to (M, \mathscr{C}) \tag{2}$$

*is weak coregular.*

Proof. Let (1) be a weak coregular mapping at a point $y_0$. It follows that there exist neighbourhoods $U$ and $V$ of points $y_0$ and $f(y_0)$ open in the topologies $\tau_{\mathscr{C}}$ and $\tau_{\mathscr{D}}$, respectively, such that $f[U] \subset V$, and a smooth mapping

$$\sigma\colon (V, \mathscr{D}_V) \to (U, \mathscr{C}_U)$$

satisfying the conditions

$$f \circ \sigma = \mathrm{id}_V, \quad \sigma(f(y_0)) = y_0. \tag{3}$$

Setting $U_0 = f^{-1}[V]$, we see that $U_0$ is an open set in $\tau_{\mathscr{C}}$. Denote by $\sigma_0$ the mapping

$$\sigma_0\colon U_0 \to U_0 \times U$$

given by the formula

$$\sigma_0(x) = \big(x, \sigma(f(x))\big) \quad \text{for } x \in U_0.$$

From the above definition it follows that $\sigma_0$ is smooth, and for any $x \in U_0$ we get $f(x) \in V$, so that $f\big(\sigma(f(x))\big) = f(x)$, whence $\big(x, \sigma(f(x))\big) \in R$ and $\sigma_0[U_0] \subset R$. Then we have

$$\sigma_0\colon U_0 \to (M \times M) \cap R.$$

Let us take any $(x_0, y_0) \in R$. The set $U_0 \times U$ is a neighbourhood of $(x_0, y_0)$ open in the topology $\tau_{\mathscr{C}\times\mathscr{C}}$. For any $x \in U_0$ we have $f(x) \in V$ and $\sigma(f(x)) \in U$, and so

$$\sigma_0\colon (U_0, \mathscr{C}_{U_0}) \to (U_0 \times U, \mathscr{C}_{U_0} \times \mathscr{C}_U). \tag{4}$$

Moreover, $\mathrm{pr}_1(\sigma_0(x)) = \mathrm{pr}_1\big(x, \sigma(f(x))\big) = x$ for $x \in U_0$, making use of (3), we get

$$\sigma_0(\mathrm{pr}_1(x_0, y_0)) = \sigma_0(x_0) = \big(x_0, \sigma(f(x_0))\big) = \big(x_0, \sigma(f(y_0))\big) = (x_0, y_0). \tag{5}$$

The following diagrams commute:

$$\begin{tikzcd}
(U_0, \mathscr{C}_{U_0}) \arrow[rr, "f|U_0"] \arrow[dr, "\sigma_0 \circ f|U_0"'] & & (V, \mathscr{D}_V) \arrow[dl, "\sigma"] \\
& (U, \mathscr{C}_U) &
\end{tikzcd}$$

$$\begin{tikzcd}
(U_0 \times U, \mathscr{C}_{U_0} \times \mathscr{C}_U) \arrow[rr, "\mathrm{pr}_1"] & & (U_0, \mathscr{C}_{U_0}) \\
& (U_0, \mathscr{C}_{U_0}) \arrow[ul, "\sigma_0"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

From (4) and (5) it follows that $\mathrm{pr}_1$ is weak coregular at the point $(x_0, y_0)$, which implies the weak coregularity of mapping (2).

In the theory of differential manifolds the notion of a Whitney product is of great importance. Consider smooth mappings

$$f_i\colon (M_i, \mathscr{C}_i) \to (N, \mathscr{D}) \quad \text{for } i = 1, 2 \tag{6}$$

and the set

$$M_1 \times_{f_1 \times f_2} M_2 = \{(p_1, p_2)\colon (p_1, p_2) \in M_1 \times M_2, f_1(p_1) = f_2(p_2)\},$$

where $(f_1 \times f_2)(p_1, p_2) = (f_1(p_1), f_2(p_2))$ for $(p_1, p_2) \in M_1 \times M_2$. It is proved that if mappings (6) are transversal (see [4]), then the set $M_1 \times_{f_1 \times f_2} M_2$ determinates a differential submanifolds of the manifold $(M_1, \mathscr{C}_1) \times (M_2, \mathscr{C}_2)$.

If one of the mappings (6) is coregular, then these mappings are transversal. Thus if a mapping (1) is coregular at a point $y_0$ and if $f(x_0) = f(y_0)$, then this mapping is self-transversal at the point $(x_0, y_0)$. Hence the set $M \times_{f \times f} M$ determinates a differential submanifold of the manifold $(M, \mathscr{C}) \times (M, \mathscr{C})$. It is easy to verify that the differential space $(R, (\mathscr{C} \times \mathscr{C})_R)$ from Lemma 1 is the submanifold considered.

LEMMA 2. *Let $(M, \mathscr{C})$ be a differential space, $f\colon M \to N$, and let $\mathcal{O}$ be a covering of $M$ open in $\tau_{\mathscr{C}}$. If all the mappings*

$$f|U\colon (U, \mathscr{C}_U) \to \big(f|U, ((f|U)^{*-1}[\mathscr{C}_U])_{f[U]}\big)$$

*are weak coregular, $f[U]$ is open in $\tau_{f^{*-1}[\mathscr{C}]}$ and*

$$f^{-1}[f[U]] = U,$$

*where $U \in \mathcal{O}$, then*

(7) $$f\colon (M, \mathscr{C}) \to \big(N, (f^{*-1}[\mathscr{C}])_N\big)$$

*is weak coregular.*

Proof. For any $p \in M$ there exists $U \in \mathcal{O}$ such that $p \in U$. From the weak coregularity of $f|U$ it follows that there exist sets $U_1$, $V_1$ open in the topologies $\tau_{\mathscr{C}_U}$ and $\tau_{((f|U)^{*-1}[\mathscr{C}_U])_{f[U]}}$, respectively, and a smooth mapping

$$\sigma\colon \big(V_1, ((f|U)^{*-1}[\mathscr{C}_U])_{V_1}\big) \to (U_1, \mathscr{C}_{U_1})$$

such that

(8) $$p \in U_1, \quad f(p) \in V_1, \quad f|U_1 \circ \sigma = \mathrm{id}_{V_1}, \quad \sigma(f(p)) = p.$$

Making use of Lemma 1.6 from [7], we get

$$\big(V_1, ((f|U)^{*-1}[\mathscr{C}_U])_{V_1}\big) = \big(V_1, (f^{*-1}[\mathscr{C}])_{V_1}\big).$$

Thus

(9) $$\sigma\colon \big(V_1, (f^{*-1}[\mathscr{C}])_{V_1}\big) \to (U_1, \mathscr{C}_{U_1}).$$

From (8) and (9) it follows that mapping (7) is weak coregular.

THEOREM 3. *The mapping*

(10) $$p_R\colon (M, \mathscr{C}) \to (M, \mathscr{C})/R$$

*is weak coregular if and only if the following conditions are fulfilled:*

(a) *the mapping* $\mathrm{pr}_1|R\colon (R, (\mathscr{C} \times \mathscr{C})_R) \to (M, \mathscr{C})$ *is weak coregular,*

(b) *for any point* $x_0 \in M$ *there exist a neighbourhood* $U$ *of* $x_0$, *a subset* $W$ *of* $M$ *and a weak coregular mapping*

$$s: (U, \mathscr{C}_U) \to (W, \mathscr{C}_W) \tag{11}$$

*such that*

$$W \cap p_R(u) = \{s(u)\} \quad \textit{for } u \in U. \tag{12}$$

Proof. *Necessity.* Making use of Lemma 1 and taking the natural mapping $p_R$ in place of $f$, we get condition (a).

Let us take any $x_0 \in M$. From the weak coregularity of the mapping $p_R$ it follows that there exist a neighbourhood $U$ of $x_0$ open in $\tau_{\mathscr{C}}$, a neighbourhood $V$ of the point $p_R(x_0)$ open in $\tau_{\mathscr{C}}/R$ with $p_R[U] \subset V$, and a smooth mapping

$$\sigma: \big(V, (\mathscr{C}/R)_V\big) \to (U, \mathscr{C}_U)$$

satisfying the conditions

$$p_R|U \circ \sigma = \mathrm{id}_V, \quad \sigma\big(p_R(x_0)\big) = x_0.$$

Les us set $W = \sigma\big[p_R[U]\big]$. Denote by $s$ the mapping

$$s: (U, \mathscr{C}_U) \to (W, \mathscr{C}_W) \tag{13}$$

given by the formula

$$s(u) = \sigma\big(p_R(u)\big) \quad \text{for } u \in U.$$

It is easy to see that $s(w) = w$ for $w \in W$ and

$$p_R\big(s(u)\big) = p_R\big(\sigma\big(p_R(u)\big)\big) = p_R(u).$$

Let us take any $u \in U \subset M$. By the weak coregularity of $p_R$ at the point $u$ there exist a neighbourhood $U'$ of $u$ open in $\tau_{\mathscr{C}}$, a neighbourhood $V'$ of the point $p_R(u)$ with $p_R[U'] \subset V'$, and a smooth mapping

$$\sigma': \big(V, (\mathscr{C}/R)_{V'}\big) \to (U', \mathscr{C}_{U'})$$

satisfying the conditions

$$p_R|U' \circ \sigma' = \mathrm{id}_{V'}, \quad \sigma'\big(p_R(u)\big) = u.$$

Let $U_0 = U \cap U'$, $W_1 = (\sigma' \circ p_R)^{-1}[U_0]$, $W_0 = W_1 \cap W$. The set $U_0$ is open in the topology $\tau_{\mathscr{C}_U}$, $W_0$ is open in $\tau_{\mathscr{C}_W}$. Setting

$$\sigma_0 = \sigma' \circ p_R|W_0, \tag{14}$$

we have

$$\begin{aligned}\sigma_0[W_0] &= \sigma'[p_R[W_0]] = \sigma'[p_R[W_1 \cap W]] \subset \sigma'[p_R[W_1]] \cap \sigma'[p_R[W]] \\ &= \sigma'\big[p_R\big[(\sigma' \circ p_R)^{-1}[U_0]\big]\big] \cap \sigma'[p_R[W]] \\ &= U_0 \cap \sigma'[p_R[W]] \subset U_0 \cap U' = U_0.\end{aligned}$$

Mapping (14) is smooth, because it is the composition of the smooth mappings $\sigma'$ and $p_R$, and so

$$\sigma_0\colon (W_0, \mathscr{C}_{W_0}) \to (U_0, \mathscr{C}_{U_0}).$$

Moreover,

$$s|U_0 \circ \sigma_0 = \sigma \circ p_R|U_0 \circ \sigma' \circ p_R|W_0 = \sigma \circ p_R|W_0 = s|W_0 = \mathrm{id}_{W_0},$$
$$\sigma_0(s(u)) = \sigma'\big(p_R(\sigma(p_R(u)))\big) = \sigma'(p_R(u)) = u.$$

Mapping (11) is weak coregular.

Since $s(u) \in W_0 \subset W$ and $p_R(u) = p_R(s(u))$, we have the inclusion

$$\{s(u)\} \subset W \cap p_R(u).$$

Let us take any point $p \in W \cap p_R(u)$; then $p \in W$ and $p \in p_R(u)$, and so there exists $x \in U$ such that $p = \sigma(p_R(x)) = s(x)$,

$$p \in p_R(p) = p_R(s(x)) = p_R(\sigma(p_R(x))) = p_R(x).$$

Since $p \in p_R(x)$ and $p \in p_R(u)$, we have $p_R(u) = p_R(x)$ and $p = \sigma(p_R(x)) = \sigma(p_R(u)) = s(u)$. Hence follows the inclusion

$$W \cap p_R(u) \subset \{s(u)\}.$$

Condition (b) is thus satisfied.

The proof of sufficiency of conditions (a) and (b) will be preceded by three lemmas.

LEMMA 4. *Let* $(M, \mathscr{C})$, $(N, \mathscr{D})$, $(P, F)$ *be differential spaces. If the mapping*

$$g\colon (M, \mathscr{C}) \to (N, \mathscr{D})$$

*is onto and weak coregular, the mapping*

$$f\colon (M, \mathscr{C}) \to (P, F)$$

*is weak coregular and there exists a mapping*

$$k\colon (N, \mathscr{D}) \to (P, F) \tag{15}$$

*such that*

$$k \circ g = f;$$

*then mapping* (15) *is weak coregular.*

Proof. For any point $y \in N$ there exists $x \in M$ such that $g(x) = y$. From the weak coregularity of the mapping $f$ at the point $x$ it follows that there exist a set $U$ open in $\tau_{\mathscr{C}}$, a set $W$ open in $\tau_F$, such that $x \in U, f(x) \in W, f[U] \subset W$, and a smooth mapping $\sigma_0\colon (W, F_W) \to (U, \mathscr{C}_U)$ satisfying the following conditions:

$$f|U \circ \sigma_0 = \mathrm{id}_W, \quad \sigma_0(f(x)) = x.$$

Let us set $V = g[U]$. Since the mapping $g$ is weak coregular, it is open, and thus $V$ is open in $\tau_{\mathscr{D}}$ and $y = g(x) \in V$, $k(y) = k(g(x)) = f(x) \in W$, $k[V] = k[g[U]] = f[U] \subset W$.

The following diagram commutes:

$$\begin{tikzcd}
(V, \mathscr{D}_V) \arrow[rr, "k|V"] & & (W, F_W) \\
& (U, \mathscr{C}_U) \arrow[ul, "g|U"'] \arrow[ur, "f|U"] & \\
& (W, F_W) \arrow[u, "\sigma_0"'] \arrow[uul, "\sigma"] \arrow[uur, "\mathrm{id}_W"'] &
\end{tikzcd}$$

Denote by $s$ the mapping

$$s\colon (W, F_W) \to (V, \mathscr{D}_V)$$

given by the formula $\sigma = (g|U) \circ \sigma_0$. Since $\sigma$ is smooth and the conditions

$$k \circ \sigma = k \circ (g|U) \circ \sigma_0 = f|U \circ \sigma_0 = \mathrm{id}_W,$$

$$\sigma(k(y)) = g\big(\sigma_0(k(g(x)))\big) = g\big(\sigma_0(f(x))\big) = g(x) = y$$

are true, the mapping (15) is weak coregular.

LEMMA 5. *Let $R$ be an equivalence relation on $M$. If the mapping*

$$\mathrm{pr}_1|R\colon (R, (\mathscr{C} \times \mathscr{C})_R) \to (M, \mathscr{C})$$

*is weak coregular if for any point $x_0$ of $M$ there exist a neighbourhood $U$ of $x_0$ open in $\tau_{\mathscr{C}}$ such that*

$$p_R^{-1}[p_R[U]] = M \tag{16}$$

*and a weak coregular mapping $s$ satisfying condition* (b) *of Theorem* 3, *and if the natural mapping*

$$p_{R_U}\colon (U, \mathscr{C}_U) \to (U, \mathscr{C}_U)/R_U, \tag{17}$$

*where $R_U = R \cap (U \times U)$, is weak coregular, then mapping* (10) *is weak coregular.*

Proof. For any $u, u' \in U$, if $p_{R_U}(u) = p_{R_U}(u')$, then $p_R(u) = p_R(u')$, and so there exists exactly one mapping

$$h\colon U/R_U \to M/R \tag{18}$$

such that

$$h \circ p_{R_U} = p_R|U. \tag{19}$$

It is easy to see that mapping (18) is one-to-one and onto. For any point $z \in M/R$ there exists $u_0 \in M$ such that $z = p_R(u_0)$. In view of (16) there exists $u \in U$ such that $p_R(u) = p_R(u_0)$ and $z = p_R(u) = (p_R|U)(u) = h(p_{R_U}(u))$. Let $h(x) = h(x')$ for any $x, x' \in U/R_U$; then there exist $u, u' \in U$ such that $x = p_R(u)$, $x' = p_R(u')$, and so $h(p_{R_U}(u)) = h(p_{R_U}(u'))$. Hence and from (19) we get $(p_R|U)(u) = (p_R|U)(u')$. Since for $u \in U$ we have $(p_R|U)(u) = p_{R_U}(u)$, it follows that $p_{R_U}(u) = p_{R_U}(u')$ and $x = x'$.

Let $(M/R, \mathscr{D})$ be a differential space coinduced in the set $M/R$ from the space $(U, \mathscr{C}_U)/R_U$ by mapping (18); then $\mathscr{D} = h^{*-1}[p_R^{*-1}[\mathscr{C}_U]]$ and (18) is smooth. The mapping $h$ is also a diffeomorphism.

The following diagram commutes:

$$\begin{tikzcd}
(U, \mathscr{C}_U)/R_U = (U/R_U, p_{R_U}^{*-1}[\mathscr{C}_U]) \arrow[rr, shift left, "h"] & & (M/R, h^{*-1}[p_R^{*-1}[\mathscr{C}_U]]) \arrow[ll, shift left, dashed, "h^{-1}"] \\
& (U, \mathscr{C}_U) \arrow[ul, "p_{R_U}"] \arrow[ur, "p_R|U"'] &
\end{tikzcd}$$

and there exists the inverse mapping $h^{-1}\colon M/R \to U/R_U$. We shall show that $h^{-1}$ is smooth.

Let us consider a function $\beta \in p_{R_U}^{*-1}[\mathscr{C}_U]$; then $\beta \circ p_{R_U} \in \mathscr{C}_U$. Since $h^{-1} \circ h = \mathrm{id}_{U/R_U}$, we have

$$\beta \circ p_{R_U} = \beta \circ h^{-1} \circ h \circ p_{R_U} \in \mathscr{C}_U.$$

Hence $\beta \circ h^{-1} \circ h \in p_{R_U}^{*-1}[\mathscr{C}_U]$ and $\beta \circ h^{-1} \in h^{*-1}[p_{R_U}^{*-1}[\mathscr{C}_U]]$. It follows that $h^{-1}$ is smooth.

Let us set $k = h^{-1} \circ p_R$. For any point $z = (x, y) \in (U \times M) \cap R$ we have $x \in U$, $y \in M$ and $p_R(x) = p_R(y)$,

$$\begin{aligned} k(\mathrm{pr}_2(x, y)) = k(y) &= h^{-1}(p_R(y)) = h^{-1}(p_R(x)) = h^{-1}(p_R|U(x)) \\ &= h^{-1}(h(p_{R_U}(x))) = p_{R_U}(x) = p_{R_U}(\mathrm{pr}_1(x, y)). \end{aligned}$$

Thus the following equality is true:

(20) $$k \circ \mathrm{pr}_2|(U \times M) \cap R = p_{R_U} \circ \mathrm{pr}_1|(U \times M) \cap R.$$

The weak coregularity of the mappings $\mathrm{pr}_1|R$, (17) and (20) yields the weak coregularity of the mapping $k \circ \mathrm{pr}_2$. The mapping $\mathrm{pr}_2$ is also weak coregular, because $R$ is a symmetric relation and $\mathrm{pr}_1$ is weak coregular.

Applying Lemma 4, we get the weak coregularity of the mapping $k$. The following diagram commutes:

$$\begin{tikzcd}
(U \times M) \cap R \arrow[rr, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr] & & M \arrow[dl, "k"'] \arrow[dr, "p_R"] & \\
U \arrow[r, "p_{R_U}"] & U/R_U \arrow[rr, "h"] & & M/R
\end{tikzcd}$$

Since the mapping $k$ is weak coregular and $h$ is a diffeomorphism, $p_R$ is weak coregular.

According to Lemma 3.5 of [6], we get $\mathscr{D} = p_R^{*-1}[\mathscr{C}]$ and the mapping (10) is weak coregular.

LEMMA 6. *If for a set $U$ open in $\tau_{\mathscr{C}}$ there exist a set $W \subset U$ and a weak coregular mapping* (11) *satisfying* (12), *then the natural mapping* (17) *is weak coregular.*

Proof. For any $u, u' \in U$, if $s(u) = s(u')$, then (by (12)) $W \cap p_R(u) = W \cap p_R(u')$; therefore $p_R(u) = p_R(u')$ and $p_{R_U}(u) = p_{R_U}(u')$.

Let us define a mapping

$$l\colon W \to U/R_U \tag{21}$$

such that

$$l \circ s = p_{R_U}. \tag{22}$$

Using (12) again, we see that if $w, w' \in W$, then $l(w) = l(w')$, and so $w = s(w)$, $w' = s(w')$, $l(w) = l(s(w)) = l(s(w'))$.

According to (22), we have

$$p_{R_U}(w) = p_{R_U}(w'), \quad p_{R_U}(w) \cap W = p_{R_U}(w') \cap W$$

and

$$w = s(w) = s(w') = w'.$$

For any $z \in U/R_U$ there exists $u \in U$ such that $z = p_R(u)$. If $w = s(u)$, then $l(w) = l(s(u)) = p_{R_U}(u) = z$. Thus the mapping $l$ is one-to-one.

Let $(U/R_U, \mathscr{D})$ be a differential space coinduced from $(W, \mathscr{C}_W)$ by mapping (21); then $\mathscr{D} = l^{*-1}[\mathscr{C}_W]$ and the mapping

$$l\colon (W, \mathscr{C}_W) \to (U/R_U, l^{*-1}[\mathscr{C}_W])$$

is smooth. Since, for any function $\alpha \in \mathscr{C}_W$, we have $\alpha \circ l^{-1} \in l^{*-1}[\mathscr{C}_W]$, the mapping $l^{-1}$ is smooth.

The weak coregularity of the mapping $s$ and the diffeomorphism $l$ imply the weak coregularity of the mapping

$$p_{R_U} = l \circ s\colon (U, \mathscr{C}_U) \to (U/R_U, \mathscr{D}).$$

According to Lemma 3.5 of [6], we get $\mathscr{D} = p_{R_U}^{*-1}[\mathscr{C}_U]$, and so

$$p_{R_U}\colon (U, \mathscr{C}_U) \to (U/R_U, p_R^{*-1}[\mathscr{C}_U]) = (U, \mathscr{C}_U)/R_U.$$

Proof of sufficiency of conditions (a) and (b). Suppose that (a) and (b) hold. In view of Lemma 6 the mapping $p_{R_{U'_x}}$ is weak coregular, where $U'_x$ is a neighbourhood of any point $x \in M$.

Let us write

$$U_x = p_R^{-1}[p_R[U'_x]] \quad \text{for } x \in M. \tag{23}$$

For any point $p \in U_x$ we have $p_R(p) \in p_R[U'_x]$, and so there exists $q \in U'_x$ such that $p_R(p) = p_R(q)$; hence $p \in \mathrm{pr}_2[R \cap (U'_x \times M)]$. Moreover, if $p \in \mathrm{pr}_2[R \cap (U'_x \times M)]$, then there exists $q \in U'_x$ such that $(q, p) \in U'_x \times M$ and $p_R(q) = p_R(p)$; hence $p \in p_R^{-1}[p_R[U'_x]]$.

We get the equality

$$U_x = \mathrm{pr}_2[R \cap (U'_x \times M)].$$

From (a) and the fact that the relation $R$ is symmetric it follows that the mapping $\mathrm{pr}_2$ is weak coregular, so it is an open mapping. Hence $U_x$ is an open set in $\tau_{\mathscr{C}}$.

Since $U'_x \subset U_x$, condition (23) leads to the equality

$$U_x = p_{R_{U_x}}^{-1}[p_{R_{U_x}}[U'_x]].$$

Condition (a) gives the weak coregularity of the mapping

$$\mathrm{pr}_1|R_{U_x}\colon (R_{U_x}, (\mathscr{C}_{U_x} \times \mathscr{C}_{U_x})_{R_{U_x}}) \to (U_x, \mathscr{C}_{U_x}).$$

Equality (23) implies $U_x = p_R^{-1}[p_R[U_x]]$. Since $p_R|U_x = p_{R_{U_x}}$, applying Lemma 5 we get the weak coregularity of the mapping

$$p_R|U_x\colon (U_x, \mathscr{C}_{U_x}) \to (U_x, \mathscr{C}_{U_x})/R_{U_x}.$$

Let $\mathcal{O} = \{U_x\colon x \in M\}$; the family $\mathcal{O}$ is a covering of $M$ open in $\tau_{\mathscr{C}}$. From Lemma 2 it follows that mapping (10) is weak coregular. The proof of Theorem 3 is complete.

**3. A weak coregular division of a generalized Lie group by a Lie subgroup.** Let $\mathscr{G} = (G, \mathscr{C}, \cdot)$ be a generalized Lie group. For any set $U \subset G$ and $a \in G$ we denote by $a \cdot U$ the set $\{au; u \in U\}$, and by $U \cdot a$ the set $\{ua; u \in U\}$. For any $a \in G$ we denote by $L_a$ and $R_a$ the left and right-translations, i.e., the mappings

$$L_a\colon G \to G, \quad R_a\colon G \to G$$

given by the formulas

$$L_a(g) = a \cdot g \quad \text{for } g \in G, \quad R_a(g) = g \cdot a \quad \text{for } g \in G.$$

From the definition it follows that the mappings $L_a$ and $R_a$ are smooth.

We shall write $\varphi$ instead of the multiplication; thus

$$L_a = \varphi(a, \cdot), \quad R_a = \varphi(\cdot, a),$$
$$L_a^{-1} = \varphi(a^{-1}, \cdot), \quad R_a^{-1} = \varphi(\cdot, a^{-1});$$

$L_a$ and $R_a$ are diffeomorphisms.

Since $a \cdot U$ and $U \cdot a$ are the images of $U$ under $L_a$ and $R_a$, it follows that if $U$ is an open set in $\tau_{\mathscr{C}}$, then $a \cdot U$, $U \cdot a$ are also open sets in $\tau_{\mathscr{C}}$.

EXAMPLE. Let $(G, \cdot)$ be a group, let $\mathscr{C}$ be the smallest differential structure generated by the set $\{c_G; c \in R\}$, where $c_G(g) = c$ for $g \in G$.

It is easy to verify that $(G, \mathscr{C})$ is a differential space and $\tau_{\mathscr{C}} = \{\Phi, G\}$.

We shall prove that the mappings

$$\varphi: (G, \mathscr{C}) \times (G, \mathscr{C}) \to (G, \mathscr{C})$$

and

$$\psi: (G, \mathscr{C}) \to (G, \mathscr{C}),$$

given by

$$\varphi(p, q) = p \cdot q \quad \text{for } p, q \in G,$$
$$\psi(p) = p^{-1} \quad \text{for } p \in G,$$

are smooth.

From the definition of the product group we have

$$(G, \mathscr{C}) \times (G, \mathscr{C}) = \big(G \times G, (\mathscr{C} \circ \mathrm{pr}_1)_{G \times G}\big),$$

where

$$\mathscr{C} \circ \mathrm{pr}_1 = \{\alpha \circ \mathrm{pr}_1; \alpha \in \mathscr{C}\}.$$

For any $c_G \in \mathscr{C}$, $p, q \in G$ we have

$$c_G\big(\varphi(p, q)\big) = c_G(p \cdot q) = c = c_G\big(\mathrm{pr}_1(p, q)\big)$$

and $c_G \circ \psi = c_G \in \mathscr{C}$ and so $(G, \mathscr{C}, \cdot)$ is a generalized Lie group.

The product of generalized Lie groups is a generalized Lie group, similarly to the theory of manifolds. More exactly, we have

THEOREM 7. *If $(G_1, \mathscr{C}_1, \varphi_1)$, $(G_2, \mathscr{C}_2, \varphi_2)$ are generalized Lie groups, then*

$$(G_1, \mathscr{C}_1, \varphi_1) \times (G_2, \mathscr{C}_2, \varphi_2) = (G_1 \times G_2, \mathscr{C}_1 \times \mathscr{C}_2, \varphi),$$

*with the mapping*

$$\varphi: (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$$

*given by*

$$\varphi\big((p_1, q_1), (p_2, q_2)\big) = \big(\varphi_1(p_1, p_2), \varphi_2(q_1, q_2)\big) \quad \textit{for } p_1, p_2 \in G_1,\ q_1, q_2 \in G_2,$$

*is a generalized Lie group.*

Proof. It is easy to see that $(G_1 \times G_2, \varphi)$ is a group. The identity of this group is $e = (e_1, e_2)$, where $e_i$ are the identities of $G_i$ for $i = 1, 2$. The inverse point to $(g_1, g_2)$ is $(g_1^{-1}, g_2^{-1})$. Let $\psi_1$, $\psi_2$, and $\psi$ be the mappings $\psi_i: G_i \to G_i$, $\psi: G \to G$ given by the formulae

$$\psi_i(g_i) = g_i^{-1} \quad \text{for } g_i \in G_i,$$
$$\psi\big((g_1, g_2)\big) = \big(\psi_1(g_1), \psi_2(g_2)\big) \quad \text{for } g_1 \in G_1,\ g_2 \in G_2.$$

We shall prove that the mappings $\varphi$ and $\psi$ are smooth.

For any $\beta \in \mathscr{C}_1 \times \mathscr{C}_2$ we have either $\beta = \alpha \circ \mathrm{pr}_1$, where $\alpha \in \mathscr{C}_1$, or $\beta = \alpha' \circ \mathrm{pr}_2$, where $\alpha' \in \mathscr{C}_2$. If $\beta = \alpha \circ \mathrm{pr}_1$, then, for any $(g_1, g_2) \in G_1 \times G_2$,

$$\beta\big(\psi((g_1, g_2))\big) = \alpha\big(\mathrm{pr}_1(\psi_1(g_1), \psi_2(g_2))\big) = \alpha(\psi_1(g_1)) = \alpha\Big(\psi_1\big(\mathrm{pr}_1((g_1, g_2))\big)\Big);$$

hence $\beta \circ \psi = \alpha \circ \psi_1 \circ \mathrm{pr}_1$.

The function $\alpha \circ \psi_1$ belongs to $\mathscr{C}_1$ because $\psi_1$ is smooth, and so $\alpha \circ \psi_1 \circ \mathrm{pr}_1 \in \mathscr{C}_1 \times \mathscr{C}_2$.

Similarly, if $\beta = \alpha' \circ \mathrm{pr}_2$, then $\beta \circ \psi = \alpha' \circ \psi_2 \circ \mathrm{pr}_2 \in \mathscr{C}_1 \times \mathscr{C}_2$; thus

$$\psi\colon (G_1 \times G_2, \mathscr{C}_1 \times \mathscr{C}_2) \to (G_1 \times G_2, \mathscr{C}_1 \times \mathscr{C}_2).$$

Now let $\beta = \alpha \circ \mathrm{pr}_1$ for $\alpha \in \mathscr{C}_1$. For any $((g_1, g_2), (g'_1, g'_2)) \in (G_1 \times G_2) \times \times (G_1 \times G_2)$ we have

$$\beta\big(\varphi((g_1, g_2), (g'_1, g'_2))\big) = \beta(\varphi_1(g_1, g'_1), \varphi_2(g_2, g'_2)) = \alpha\big(\mathrm{pr}_1(\varphi_1(g_1, g'_1), \varphi_2(g_2, g'_2))\big) = \alpha\big(\varphi_1(g_1, g'_1)\big).$$

Since $\varphi_1$ is smooth, $\alpha \circ \varphi_1 \in \mathscr{C}_1 \times \mathscr{C}_1$. There exists a mapping $\gamma \in \mathscr{C}_1$ such that $\alpha \circ \varphi_1 = \gamma \circ \mathrm{pr}_1$; hence

$$\alpha(\varphi_1(g_1, g'_1)) = \gamma(\mathrm{pr}_1(g_1, g'_1)) = \gamma(g_1) = \gamma(\mathrm{pr}_1(g_1, g_2)) = \gamma\Big(\mathrm{pr}_1\big(\mathrm{pr}_1((g_1, g_2), (g'_1, g'_2))\big)\Big).$$

Thus

$$\beta \circ \varphi = \gamma \circ \mathrm{pr}_1|G_1 \times G_2 \circ \mathrm{pr}_1|(G_1 \times G_2) \times (G_1 \times G_2).$$

Since $\gamma \circ \mathrm{pr}_1|G_1 \times G_2 \in \mathscr{C}_1 \times \mathscr{C}_2$, the mapping $\beta \circ \varphi$ belongs to $(\mathscr{C}_1 \times \mathscr{C}_2) \times \times (\mathscr{C}_1 \times \mathscr{C}_2)$.

Similarly, if $\beta = \alpha' \circ \mathrm{pr}_2$, where $\alpha' \in \mathscr{C}_2$, we get

$$\beta \circ \varphi = \delta \circ \mathrm{pr}_2\,|(G_1 \times G_2) \circ \mathrm{pr}_1\,|(G_1 \times G_2) \times (G_1 \times G_2),$$

where $\delta \in \mathscr{C}_2$; thus $\beta \circ \varphi \in (\mathscr{C}_1 \times \mathscr{C}_2) \times (\mathscr{C}_1 \times \mathscr{C}_2)$ and

$$\varphi\colon (G_1 \times G_2, \mathscr{C}_1 \times \mathscr{C}_2) \times (G_1 \times G_2, \mathscr{C}_1 \times \mathscr{C}_2) \to (G_1 \times G_2, \mathscr{C}_1 \times \mathscr{C}_2).$$

The proof of Theorem 7 is complete.

Let $\mathscr{G} = (G, \mathscr{C}, \cdot)$ be a generalized Lie group and $\mathscr{H} = (H, \mathscr{C}_H, \cdot|H \times H)$ a generalized Lie subgroup of $\mathscr{G}$. Denote by $(G, \mathscr{C})/\mathscr{H}$ the differential space $(\Pi_H[G], \Pi_H^{*-1}[\mathscr{C}])$, where $\Pi_H$ is the canonical mapping given by the formula

$$\Pi_H(g) = g \cdot H \quad \text{for } g \in G.$$

THEOREM 8. *Suppose that the following condition is satisfied*:

($b_0$) *there exist a neighbourhood $U$ of the identity $e$ of the group $\mathcal{G}$ open in $\tau_{\mathcal{C}}$, a set $W \subset U$ and a weak coregular mapping*

$$s_e\colon (U, \mathcal{C}_U) \to (W, \mathcal{C}_W), \tag{24}$$

*such that*

$$uH \cap W = \{s_e(u)\} \quad \text{for } u \in U. \tag{25}$$

*Then the canonical mapping*

$$\Pi_H\colon (G, \mathcal{C}) \to (G, \mathcal{C})/\mathcal{H} \tag{26}$$

*is weak coregular.*

*Moreover, if $\mathcal{H}$ is an invariant subgroup of $\mathcal{G}$ and the condition*

$$(\Pi_H \times \Pi_H)^{*-1}[\mathcal{C} \times \mathcal{C}] \subset \Pi_H^{*-1}[\mathcal{C}] \times \Pi_H^{*-1}[\mathcal{C}] \tag{27}$$

*is satisfied, then there exists exactly one generalized quotient Lie group $\mathcal{G}/\mathcal{H}$ such that the mapping*

$$\Pi_H\colon \mathcal{G} \to \mathcal{G}/\mathcal{H}$$

*is a weak coregular homomorphism of these Lie groups, i.e., mapping* (26) *is weak coregular and the mapping*

$$\Pi_H\colon (G, \cdot) \to (G/H, \cdot)$$

*(where $G/H = \Pi_H[G]$, and*

$$\cdot\colon G/H \times G/H \to G/H$$

*being defined by $\Pi_H(g) \cdot \Pi_H(g') = \Pi_H(g \cdot g')$ for $g, g' \in G$) is a homomorphism of the groups.*

Proof. Let $R$ be an equivalence relation given by the formula $R = \{(x, y); (x, y) \in G \times G, x \cdot y^{-1} \in H\}$. We shall show that conditions (a) and (b) of Theorem 3 are satisfied.

Let us consider the mapping

$$\varphi\colon G \times H \to R \tag{28}$$

given by the formula $\varphi(x, h) = (hx, x)$ for $x \in G$, $h \in H$. For any $(x, y) \in R$ there exists $(y, xy^{-1}) \in G \times H$ such that $\varphi(y, xy^{-1}) = ((xy^{-1})y, y) = (x, y)$, so that mapping (28) is onto.

It is easy to verify that the mapping

$$\mathrm{pr}_1\colon (G \times H, \mathcal{C} \times \mathcal{C}_H) \to (G, \mathcal{C}) \tag{29}$$

is weak coregular.

Indeed, let $x \in G$, $h \in H$, and let $U_1, U_2$ be neighbourhoods of $x, h$ open in $\tau_{\mathcal{C}}, \tau_{\mathcal{C}/H}$, respectively. Putting $V = U_1$, $U = U_1 \times U_2$ and $\sigma(u) = (u, h)$ for $u \in U_1$, we get $\sigma[V] \subset U$, $\mathrm{pr}_1 \circ \sigma = \mathrm{id}_V$, $\sigma(\mathrm{pr}_1(x, h)) = \sigma(x)$

$= (x, h)$. The mapping $\sigma\colon V \to U$ is smooth, because for any $\alpha_1 \in \mathscr{C}_{U_1}$, $\alpha_2 \in (\mathscr{C}_H)_{U_2}$ we have $\alpha_1 \circ \mathrm{pr}_1 \circ \sigma = \alpha_1$, $\alpha_2 \circ \mathrm{pr}_2 \circ \sigma = \mathrm{const}$. The diagram

$$\begin{tikzcd}(G \times H, \mathscr{C} \times \mathscr{C}_H) \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}_1"'] & & (R, (\mathscr{C} \times \mathscr{C})_R) \arrow[dl, "\mathrm{pr}_1"] \\ & (G, C) & \end{tikzcd}$$

commutes. From the above we get the weak coregularity of the mapping

$$\mathrm{pr}_1 \mid R\colon \big(R, (\mathscr{C} \times \mathscr{C})_R\big) \to (G, \mathscr{C}).$$

We can restate condition (b) of Theorem 3 as follows:

(b′) for every $x \in G$ there exist a neighbourhood $U_0$ of $x$ open in $\tau_{\mathscr{C}}$, a set $W_0 \subset U_0$ and a weak coregular mapping

$$s_0\colon (U_0, \mathscr{C}_{U_0}) \to (W_0, \mathscr{C}_{W_0}) \tag{30}$$

such that

$$u \cdot H \cap W_0 = \{s_0(u)\} \quad \text{for } u \in U_0. \tag{31}$$

It is clear that (b′) implies ($\mathrm{b}_0$). We shall prove that also ($\mathrm{b}_0$) implies (b′). For any $x_0 \in G$, putting

$$U_0 = x_0 \cdot U = \{x_0 u;\ u \in U\}, \qquad W_0 = x_0 \cdot W,$$

we see that $U_0$ is a neighbourhood of $x$ open in $\tau_{\mathscr{C}}$ and $W_0 \subset U_0$. Denote by $s_0$ the mapping

$$s_0\colon (U_0, \mathscr{C}_{U_0}) \to (W_0, \mathscr{C}_{W_0})$$

given by $s_0(u) = x_0 \cdot s_e(x_0^{-1} u)$ for $u \in U_0$. Making use of (25) for any $u \in U_0$ we get

$$\{s_0(u)\} = \{x_0 s_e(x_0^{-1} u)\} = x_0(x_0^{-1} uH \cap W) = uH \cap W_0,$$

and consequently (31) is satisfied.

Let $u \in U_0$; there exists $u' \in U$ such that $u = x_0 \cdot u'$. By the weak coregularity of $s_e$ at the point $u'$ there exist a neighbourhood $U'$ of $u'$ open in $\tau_{\mathscr{C}_U}$, a neighbourhood $W'$ of $s_e(u')$ open in $\tau_{\mathscr{C}_W}$ and a smooth mapping

$$\sigma'\colon \big(W', (\mathscr{C}_W)_{W'}\big) \to (U', \mathscr{C}_{U'})$$

such that

$$s_e \cdot \sigma' = \mathrm{id}_{W'}, \qquad \sigma'\big(s_e(u')\big) = u.$$

Putting $W'_0 = x_0 W'$, $U'_0 = x_0 U'$, $\sigma(w) = x_0\sigma'(x_0^{-1}w)$ for $x \in W'_0$, we define a smooth mapping

$$\sigma\colon \left(W'_0, (\mathscr{C}_W)_{W'_0}\right) \to \left(U'_0, (\mathscr{C}_U)_{U'_0}\right)$$

satisfying the conditions

$$\begin{aligned} s_0\left(\sigma(w)\right) &= x_0\cdot s_e\left(x_0^{-1}\sigma(w)\right) = x_0\cdot s_e\left(x_0^{-1}x_0\sigma'(x_0^{-1}w)\right)\\ &= x_0\cdot s_e\left(\sigma'(x_0^{-1}w)\right) = x_0\cdot x_0^{-1}\cdot w = w, \end{aligned}$$

for $w \in W'$, so $s_0\circ\sigma = \mathrm{id}_{W'}$ and

$$\begin{aligned} \sigma\left(s_0(u)\right) &= x_0\cdot\sigma'\left(x_0^{-1}\cdot x_0\cdot s_e(x_0^{-1}u)\right) = x_0\cdot\sigma'\left(s_e(x_0^{-1}u)\right)\\ &= x_0\sigma'\left(s_e(u')\right) = x_0 u' = u. \end{aligned}$$

Thus conditions (b′) and ($b_0$) are equivalent, as clained.

Now we shall prove the second assertion of Theorem 8. It suffices to show that the mappings

$$\Phi\colon G/H\times G/H \to G/H, \tag{32}$$

$$\Psi\colon G/H \to G/H \tag{33}$$

given by

$$\Phi(\Pi_H(g), \Pi_H(g')) = \Pi\left(\varphi(g, g')\right), \qquad \Psi\left(\Pi_H(g)\right) = \Pi_H\left(\psi(g)\right)$$

are smooth.

Let $\alpha \in \Pi_H^{*-1}[\mathscr{C}]$; then $\alpha\circ\Pi_H \in \mathscr{C}$. Since the mapping $\psi$ is smooth, we have $\alpha\circ\Pi_H\circ\psi \in \mathscr{C}$. According to the equality $\Psi\circ\Pi_H = \Pi_H\circ\psi$ we have $\alpha\circ\Psi\circ\Pi_H \in \mathscr{C}$ and hence $\alpha\circ\Psi \in \Pi_H^{*-1}[\mathscr{C}]$; thus the mapping (33) is smooth.

The equality

$$\Pi_H^{*-1}[\mathscr{C}]\times\Pi_H^{*-1}[\mathscr{C}] = (\Pi_H\times\Pi_H)^{*-1}[\mathscr{C}\times\mathscr{C}] \tag{34}$$

is true.

Indeed, for $\alpha \in \Pi_H^{*-1}[\mathscr{C}]\times\Pi_H^{*-1}[\mathscr{C}]$ there exsists $\beta \in \Pi_H^{*-1}[\mathscr{C}]$ such that $\alpha = \beta\circ\mathrm{pr}_1$; hence $\beta\circ\Pi_H \in \mathscr{C}$. For any $\Pi_H(p), \Pi_H(q)$ we have

$$\begin{aligned} \alpha\left((\Pi_H(p), \Pi_H(q))\right) &= \beta\left(\mathrm{pr}_1(\Pi_H(p), \Pi_H(q))\right) = \beta\left(\Pi_H(p)\right)\\ &= \beta\left(\Pi_H(\mathrm{pr}_1(p, q))\right), \end{aligned}$$

i.e.,

$$\alpha\circ(\Pi_H\times\Pi_H) = \beta\circ\Pi_H\circ\mathrm{pr}_1.$$

Since $\beta\circ\Pi_H \in \mathscr{C}$, then $\beta\circ\Pi_H\circ\mathrm{pr}_1 \in \mathscr{C}\times\mathscr{C}$ and $\alpha \in (\Pi_H\times\Pi_H)^{*-1}[\mathscr{C}\times\mathscr{C}]$; hence we get the inclusion

$$\Pi_H^{*-1}[\mathscr{C}]\times\Pi_H^{*-1}[\mathscr{C}] \subset (\Pi_H\times\Pi_H)^{*-1}[\mathscr{C}\times\mathscr{C}].$$

From this and from (27) we obtain (34).

For any function $\gamma \in \Pi_H^{*-1}[\mathscr{C}]$ we have $\gamma \circ \Phi \circ (\Pi_H \times \Pi_H) = \gamma \circ \Pi_H \circ \varphi$ and $\gamma \circ \Pi_H \in \mathscr{C}$. Since the mapping $\varphi$ is smooth, we have $\gamma \circ \Pi_H \circ \varphi \in \in \mathscr{C} \times \mathscr{C}$; hence $\gamma \circ \Phi \in (\Pi_H \times \Pi_H)^{*-1}[\mathscr{C} \times \mathscr{C}]$. According to (34), $\gamma \circ \Phi \in \Pi_H^{*-1}[\mathscr{C}] \times \Pi_H^{*-1}[\mathscr{C}]$ and the mapping

$$\Phi\colon (G/H, \Pi_H^{*-1}[\mathscr{C}]) \times (G/H, \Pi_H^{*-1}[\mathscr{C}]) \to (G/H, \Pi_H^{*-1}[\mathscr{C}])$$

is smooth.

**References**

[1] N. Bourbaki, *Topologie générale*, Moscow 1969 (Russian transl.).
[2] S. MacLane, *Geometrical mechanics*, Chicago 1968.
[3] —, *Differentiable spaces*, Notes for Geometrical Mechanics, Winter 1970, p. 1–9 (unpublished).
[4] J. P. Serre, *Lie algebras and Lie groups*, New York–Amsterdam 1965.
[5] R. Sikorski, *An introduction to differential geometry*, Warszawa 1971 (in Polish).
[6] W. Waliszewski, *Regular and coregular mappings of differential spaces*, Ann. Polon. Math. 30 (1974), p. 263–281.
[7] —, *On coregular division of a differential space by an equivalence relation*, Colloq. Math. 26 (1972), p. 281–291.

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