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On the stability of difference schemes for nonlinear elliptic differential equations with boundary conditions of Dirichlet type

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Abstract. We consider the stability of the difference scheme for a nonlinear elliptic differential equation with boundary conditions of Dirichlet type, investigated by M. Malec in [3]. In this paper we give a stability theorem and a criterion for the existence and uniqueness of a solution of this scheme.

1. Introduction. The present paper may be considered as a continuation of [2] and [3] dealing with nonlinear elliptic equations with boundary conditions of Dirichlet type.

In [2] M. Malec proved two theorems on difference inequalities corresponding to the differential equation

$$f(x, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \ldots, u_{x_1x_n}, \ldots, u_{x_nx_n}) = 0,$$

which contains mixed partial derivatives of the second order of the function $u(x)$, with the boundary conditions

$$u(x) = \varphi_i(x) \quad \text{for} \quad x_i = 0 \quad (i = 1, 2, \ldots, n),$$
$$u(x) = \psi_i(x) \quad \text{for} \quad x_i = \sigma \quad (i = 1, 2, \ldots, n).$$

Now we will use the results of [2] to prove a stability theorem.

In [3] the author used the results of [2] to derive an estimate for the rate of convergence of a difference scheme. But, since the scheme is not explicit, the existence of a solution of the scheme is assumed. The problem has arisen whether that existence and uniqueness follow from the remaining assumptions accepted in [3] and how to compute this solution. Our work gives – under some additional assumptions concerning the function $f(x, u, q, w)$ – an affirmative answer to the above question. It is shown that, if we make all the assumptions of [3] relevant to the function $f(x, u, q, w)$ and assume that the inequalities

$$(1.1) \qquad L_1 \leqslant f_u, \quad f_{w_{ii}} \leqslant G \quad (i = 1, 2, \ldots, n),$$

hold true in a set D (see (i)), then there is one and only one solution of the difference problem and there exists an effective method of computing this solution (with an estimation of the error). The question whether this result can be reached without the assumption (1.1) remains open.

At the end of this paper we give a theorem which is a consequence of the preceding theorems, but its assumptions are easy to check.

2. Notation, assumptions, and the difference scheme. Let us consider the nodal points of the Euclidean space R^n whose coordinates are

$$x_i^{m_i} = m_i \cdot h \quad (i = 1, 2, \ldots, n), \tag{2.1}$$

where $m_i = 0, 1, \ldots, N$ $(i = 1, 2, \ldots, n)$, $0 < h = \sigma/N$, and N is a natural number.

Let us denote the nodal point $(x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n})$ by x^M, where $M = (m_1, m_2, \ldots, m_n)$, and write

$$\begin{aligned} Z &= \{M:\ 0 \leqslant m_i \leqslant N,\ i = 1, 2, \ldots, n\}, \\ Z^- &= \{M:\ 0 < m_i < N,\ i = 1, 2, \ldots, n\}, \end{aligned} \tag{2.2}$$

$$\left.\begin{aligned} i(M) &= (m_1, \ldots, m_{i-1}, m_i+1, m_{i+1}, \ldots, m_n), \\ -i(M) &= (m_1, \ldots, m_{i-1}, m_i-1, m_{i+1}, \ldots, m_n), \end{aligned}\right. \quad (i = 1, 2, \ldots, n). \tag{2.3}$$

Let us suppose that

(i) the function $f(x, u, q, w)$, where

$$x \in R^n, \quad u \in R, \quad q = (q_1, q_2, \ldots, q_n) \in R^n, \quad w = [w_{ij}]_{i,j=1}^n \in R^{n^2},$$

is of the class C^1 in the domain $D = [0, \sigma]^n \times R^{1+n+n^2}$;

(ii) the derivatives of this function satisfy in the set D the inequalities

$$f_u \leqslant L < 0, \quad |f_{q_i}| \leqslant \Gamma,$$

$$0 < g \leqslant f_{w_{ii}} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |f_{w_{ij}}| \quad (i = 1, 2, \ldots, n),$$

where L, Γ, g are some constants;

(iii) for arbitrary i, j $(i \neq j,\ i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n)$ the derivative $f_{w_{ij}}$ is nonpositive in D or nonnegative in D and

$$f_{w_{ij}} = f_{w_{ji}}$$

in the set D;

(iv) the step h satisfies the condition

$$g/h - \Gamma/2 \geqslant 0.$$

We shall deal with the stability of the following difference scheme:

$$\text{(2.4)}\qquad \begin{aligned} f(x^M, v^M, v^{MI}, v^{MIJ}) &= 0 && \text{for} \quad M \in Z^-, \\ v^M &= \varphi_i^M && \text{for} \quad m_i = 0 \ (i = 1, 2, \ldots, n, \ M \in Z), \\ v^M &= \psi_i^M && \text{for} \quad m_i = N \ (i = 1, 2, \ldots, n, \ M \in Z), \end{aligned}$$

where

$$v^{MI} = (v^{M1}, v^{M2}, \ldots, v^{Mn}), \qquad v^{MIJ} = (v^{M11}, \ldots, v^{M1n}, \ldots, v^{Mn1}, \ldots, v^{Mnn}),$$

$$v^{Mi} = \frac{1}{2h}(v^{i(M)} - v^{-i(M)}),$$

$$v^{-Mij} = \frac{1}{2h^2}(v^{i(M)} + v^{j(M)} + v^{-i(M)} + v^{-j(M)} - 2v^M - v^{i(-j(M))} - v^{-i(j(M))}),$$

$$v^{+Mij} = \frac{1}{2h^2}(-v^{i(M)} - v^{j(M)} - v^{-i(M)} - v^{-j(M)} + 2v^M + v^{i(j(M))} + v^{-i(-j(M))}),$$

$$v^{Mij} = \begin{cases} v^{-Mij} & \text{for} \quad i = j \quad \text{or} \quad f_{w_{ij}} \leqslant 0, \\ v^{+Mij} & \text{for} \quad i \neq j \quad \text{and} \quad f_{w_{ij}} \geqslant 0, \end{cases}$$

$$(i = 1, 2, \ldots, n, j = 1, 2, \ldots, n, \ M \in Z^-).$$

3. Stability of the difference scheme. Consider the following two difference problems for a settled h:

$$\text{(3.1)}\qquad \begin{aligned} f(x^M, v_1^M, v_1^{MI}, v_1^{MIJ}) &= \varepsilon_1^M && \text{for} \quad M \in Z^-, \\ v_1^M &= \eta_1^M && \text{for} \quad M \in \gamma, \end{aligned}$$

$$\text{(3.2)}\qquad \begin{aligned} f(x^M, v_2^M, v_2^{MI}, v_2^{MIJ}) &= \varepsilon_2^M && \text{for} \quad M \in Z^-, \\ v_2^M &= \eta_2^M && \text{for} \quad M \in \gamma, \end{aligned}$$

where $\gamma \overset{\text{df}}{=} Z - Z^-$, η_1, η_2 are given functions, $\eta_i\colon \partial E \to R$ for $i = 1, 2$, where $E = [0, \sigma]^n$ and ∂E is the boundary of E. The stability of problem (2.4) results immediately from the following

THEOREM 1. *If assumptions* (i)–(iv) *are satisfied and there exist solutions* v_1^M, v_2^M $(M \in Z)$ *of problems* (3.1) *and* (3.2), *respectively, then*

$$\text{(3.3)}\qquad \max_{M \in Z} |v_1^M - v_2^M| \leqslant \max_{M \in \gamma} |\eta_1^M - \eta_2^M| - L^{-1} \max_{M \in Z^-} |\varepsilon_1^M - \varepsilon_2^M|.$$

Proof. According to (3.1), (3.2) and using the mean value theorem, we have

$$\text{(3.4)}\qquad \begin{aligned} \varepsilon_1^M - \varepsilon_2^M &= f_u \cdot r^M + \sum_{i=1}^{n} f_{q_i} r^{Mi} + \sum_{i,j=1}^{n} f_{w_{ij}} \cdot r^{Mij} && \text{for} \quad M \in Z^-, \\ r^M &= \eta_1^M - \eta_2^M && \text{for} \quad M \in \gamma, \end{aligned}$$

where $r^M \overset{\mathrm{df}}{=} v_1^M - v_2^M$ $(M \in Z)$, $f_u = f_u(-)$, $f_{q_i} = f_{q_i}(-)$, $f_{w_{ij}} = f_{w_{ij}}(-)$ $(i, j = 1, 2, \ldots, n)$ and $(-)$ are some intermediate points.

Let us now examine the two algebraic problems, in which the coefficients $f_u, f_{q_i}, f_{w_{ij}}$ $(i, j = 1, 2, \ldots, n)$ are the same as those in (3.4):

$$(3.5)\qquad \begin{aligned} &0 \leqslant f_u \cdot z_1^M + \sum_{i=1}^{n} f_{q_i} \cdot z_1^{Mi} + \sum_{i,j=1}^{n} f_{w_{ij}} \cdot z_1^{Mij} && \text{for} \quad M \in Z^-, \\ &z_1^M \leqslant 0 && \text{for} \quad M \in \gamma, \end{aligned}$$

$$(3.6)\qquad \begin{aligned} &0 \geqslant f_u \cdot z_2^M + \sum_{i=1}^{n} f_{q_i} \cdot z_2^{Mi} + \sum_{i,j=1}^{n} f_{w_{ij}} \cdot z_2^{Mij} && \text{for} \quad M \in Z^-, \\ &z_2^M \geqslant 0 && \text{for} \quad M \in \gamma. \end{aligned}$$

LEMMA 1. *If all the assumptions of Theorem* 1 *are fulfilled and if* z_1^M (z_2^M) $(M \in Z)$ *satisfy* (3.5) ((3.6)), *then* $z_1^M \leqslant 0$ $(z_2^M \geqslant 0)$ *for all* $M \in Z$.

Proof. Let $z_1^M (M \in Z)$ satisfy (3.5) and $z_1^A = \max_{M \in Z} z_1^M$. If $A \in \gamma$, then we get the assertion of our lemma because of (3.5) and the definition of z_1^A:

$$(3.7)\qquad z_1^M \leqslant z_1^A \leqslant 0 \quad \text{for} \quad M \in Z.$$

If $A \in Z^-$, then by a theorem of M. Malec ([2], p. 861) we have $z_1^A \leqslant 0$, and hence

$$(3.8)\qquad z_1^M \leqslant 0 \quad \text{for} \quad M \in Z.$$

The reasoning for z_2^M is analogous and this completes the proof of the lemma.

Let us define

$$(3.9)\qquad w_1^M \overset{\mathrm{df}}{=} r^M - \max_{M \in \gamma} |\eta_1^M - \eta_2^M| + L^{-1} \cdot \max_{M \in Z^-} |\varepsilon_1^M - \varepsilon_2^M| \quad \text{for} \quad M \in Z.$$

The quantities $\max_{M \in \gamma} |\eta_1^M - \eta_2^M|$ and $\max_{M \in Z^-} |\varepsilon_1^M - \varepsilon_2^M|$ are independent of the nodal points x^M $(M \in Z)$, and so

$$(3.10)\qquad \begin{aligned} &f_u \cdot w_1^M + \sum_{i=1}^{n} f_{q_i} \cdot w_1^{Mi} + \sum_{i,j=1}^{n} f_{w_{ij}} \cdot w_1^{Mij} \\ &\quad = f_u \cdot r^M - f_u \cdot \max_{M \in \gamma} |\eta_1^M - \eta_2^M| + L^{-1} \cdot f_u \cdot \max_{M \in Z^-} |\varepsilon_1^M - \varepsilon_2^M| + \\ &\qquad\qquad + \sum_{i=1}^{n} f_{q_i} \cdot r^{Mi} + \sum_{i,j=1}^{n} f_{w_{ij}} \cdot r^{Mij} \\ &\quad = -f_u \cdot \max_{M \in \gamma} |\eta_1^M - \eta_2^M| + L^{-1} \cdot f_u \cdot \max_{M \in Z^-} |\varepsilon_1^M - \varepsilon_2^M| + \varepsilon_1^M - \varepsilon_2^M \geqslant 0 \\ &\qquad\qquad \text{for} \quad M \in Z^-, \end{aligned}$$

because of $w_1^{Mi} = r^{Mi}$, $w_1^{Mij} = r^{Mij}$ $(i, j = 1, 2, \ldots, n)$, (3.4) and (ii). The coefficients $f_u, f_{q_i}, f_{w_{ij}}$ $(i, j = 1, 2, \ldots, n)$ are the same as those in (3.4). Further, (3.9) implies

$$(3.11) \qquad w_1^M \leqslant 0 \quad \text{for} \quad M \in \gamma.$$

Taking (3.10), (3.11) into account and applying Lemma 1 we get $w_1^M \leqslant 0$ for all $M \in Z$, and hence

$$(3.12) \qquad r^M \leqslant \max_{M \in \gamma} |\eta_1^M - \eta_2^M| - L^{-1} \cdot \max_{M \in Z^-} |\varepsilon_1^M - \varepsilon_2^M| \quad \text{for} \quad M \in Z.$$

Analogously, for $w_2^M \overset{\text{df}}{=} r^M + \max_{M \in \gamma} |\eta_1^M - \eta_2^M| - L^{-1} \cdot \max_{M \in Z^-} |\varepsilon_1^M - \varepsilon_2^M|$ $(M \in Z)$ one can show that

$$(3.13) \qquad r^M \geqslant -\max_{M \in \gamma} |\eta_1^M - \eta_2^M| + L^{-1} \cdot \max_{M \in Z^-} |\varepsilon_1^M - \varepsilon_2^M| \quad \text{for} \quad M \in Z.$$

(3.12) and (3.13) imply (3.3). The proof of Theorem 1 is complete.

4. Existence and uniqueness of solution.

THEOREM 2. *Let us assume that assumptions* (i)–(iv) *are satisfied and for some constants* L_1, G *the inequalities*

$$(4.1) \qquad L_1 \leqslant f_u, \quad f_{w_{ii}} \leqslant G \quad (i = 1, 2, \ldots, n)$$

hold true in the set D. Then there is one and only one solution v^M $(M \in Z)$ *of the difference problem* (2.4).

Proof. Let

$$(4.2) \qquad k(M) \overset{\text{df}}{=} \sum_{i=1}^{n} (N+1)^{i-1} \cdot m_i \quad \text{for} \quad M = (m_1, m_2, \ldots, m_n) \in Z$$

and $\mathscr{I} \overset{\text{df}}{=} \{k(M)\colon M \in Z\}$, $p \overset{\text{df}}{=} (N+1)^n = \# \mathscr{I}$. Let us consider the space R^p with the norm

$$(4.3) \qquad \|V\| \overset{\text{df}}{=} \max_{t \in \mathscr{I}} |v_t| \quad \text{for} \quad V = [v_t]_{t \in \mathscr{I}} \in R^p.$$

Remark 1. Every vector $[v_t]_{t \in \mathscr{I}} \in R^p$ may be written in a unique way as $[v_{k(M)}]_{M \in Z}$ or $[v^M]_{k(M) \in \mathscr{I}}^{M \in Z}$ since the function (4.2) establishes a bijection between $\mathscr{I}$ and Z. Thus $[v_t]_{t \in \mathscr{I}} = [v_{k(M)}]_{M \in Z} = [v^M]_{k(M) \in \mathscr{I}}^{M \in Z} = [v_0, v_1, \ldots, v_{p-1}]$ and $v_t = v_{k(M)} = v^M$ for $t = k(M)$, $t = 0, 1, \ldots, p-1$. This notation will be used.

For any $V = [v_t]_{t \in \mathscr{I}} \in R^p$ we define a vector $F(V) = [c_t]_{t \in \mathscr{I}}$ as follows (here $t = k(M)$):

$$(4.4) \qquad c_{k(M)} \overset{\text{df}}{=} \begin{cases} f(x^M, v^M, v^{MI}, v^{MIJ}) & \text{for } M \in Z^-, \\ Lv^M - L \cdot \varphi_i^M & \text{for } m_i = 0 \ (i = 1, 2, \ldots, n,\ M \in Z), \\ Lv^M - L \cdot \psi_i^M & \text{for } m_i = N \ (i = 1, 2, \ldots, n,\ M \in Z). \end{cases}$$

Let s be a real number such that

(4.5) $\quad 0 < 1+h^s \cdot L,\ h^s(h^{-2}\cdot 4nG - L - L_1) \leqslant 2$ if $1+h^s\cdot L_1 - 2h^{s-2}\cdot nG < 0$

and Φ: $R^p \to R^p$ be a function such that

$$\Phi(V) \overset{\mathrm{df}}{=} h^s \cdot F(V) + V \quad \text{for} \quad V \in R^p. \tag{4.6}$$

Let us take two arbitrary vectors $V, W \in R^p$, $V = [v_t]_{t\in\mathscr{I}}$, $W = [w_t]_{t\in\mathscr{I}}$ and write $Z = V - W = [z_t]_{t\in\mathscr{I}}$, $\Phi(V) - \Phi(W) = [d_t]_{t\in\mathscr{I}}$. By (iii) and the mean value theorem (see Remark 1) we get:

$$\begin{aligned} d_{k(M)} = {} & h^{s-1}\cdot \sum_{i=1}^{n} \Big[h^{-1}\Big(f^M_{w_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n} |f^M_{w_{ij}}|\Big) + \tfrac{1}{2} f^M_{q_i}\Big]\cdot z^{i(M)} + \\ & + h^{s-1}\cdot \sum_{i=1}^{n} \Big[h^{-1}\Big(f^M_{w_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n} |f^M_{w_{ij}}|\Big) - \tfrac{1}{2} f^M_{q_i}\Big]\cdot z^{-i(M)} + \\ & + 2^{-1} h^{s-2}\cdot \sum_{\substack{i,j=1\\ i\neq j}}^{n} |f^M_{w_{ij}}|\big(z^{i(s(i,j)j(M))} + z^{-i(-s(i,j)j(M))}\big) + \\ & + \Big[1 + h^s\cdot f^M_u - h^{s-2}\cdot \sum_{\substack{i,j=1\\ i\neq j}}^{n} |f^M_{w_{ij}}| - \\ & \qquad - 2h^{s-2}\cdot \sum_{i=1}^{n} \Big(f^M_{w_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n} |f^M_{w_{ij}}|\Big)\Big]\cdot z^M \quad \text{for} \quad M \in Z^-, \\ d_{k(M)} = {} & (1 + h^s\cdot L)\cdot z^M \quad \text{for} \quad M \in Z - Z^-, \end{aligned} \tag{4.7}$$

where

$$f^M_u = f^M_u(-), \quad f^M_{q_i} = f^M_{q_i}(-), \quad f^M_{w_{ij}} = f^M_{w_{ij}}(-) \quad (i, j = 1, 2, \ldots, n,\ M \in Z^-)$$

and $(-)$ are some intermediate points; the upper index M indicates the dependence on the nodal point x^M. Moreover, let $s(i,j)$ be defined as

$$s(i,j) = \begin{cases} +1 & \text{for} \quad f_{w_{ij}} \geqslant 0, \\ -1 & \text{for} \quad f_{w_{ij}} \leqslant 0 \end{cases} \tag{4.8}$$
$$(i \neq j,\ i, j = 1, 2, \ldots, n,\ M \in Z^-).$$

We shall show that there exists a constant $K (0 < K < 1)$, independent of V and W, such that

$$\|\Phi(V) - \Phi(W)\| \leqslant K\cdot \|V - W\| \quad \text{for} \quad V, W \in R^p. \tag{4.9}$$

To prove this let us write

$$\sum_{i=1}^{n} (f^M_{w_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n} |f^M_{w_{ij}}|) = \alpha^M, \qquad \sum_{\substack{i,j=1\\ i\neq j}}^{n} |f^M_{w_{ij}}| = \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} |f^M_{w_{ij}}| = \beta^M.$$

By the definition of the norm $\|\ \|$ in R^p it follows that there are $A \in Z$ and $l \in \mathcal{I}$ $(l = k(A))$ such that

$$\|\Phi(V) - \Phi(W)\| = |d_l|. \tag{4.10}$$

Let us write $K \overset{\mathrm{df}}{=} 1 + h^s \cdot L$. (4.5) and (ii) imply

$$0 < K < 1. \tag{4.11}$$

If $A \in Z - Z^-$, then (see (4.7))

$$|d_l| = (1 + h^s \cdot L) \cdot |z^A| \leqslant K \cdot \|Z\|. \tag{4.12}$$

If $A \in Z^-$, then (ii) and (iv) imply

$$\sum_{i=1}^{n} [h^{-1}(f^A_{w_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n} |f^A_{w_{ij}}|) + \tfrac{1}{2} f^A_{q_i}] \geqslant g/h - \Gamma/2 \geqslant 0,$$

$$\sum_{i=1}^{n} [h^{-1}(f^A_{w_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n} |f^A_{w_{ij}}|) - \tfrac{1}{2} f^A_{q_i}] \geqslant g/h - \Gamma/2 \geqslant 0,$$

and hence

$$\begin{aligned} |d_l| &\leqslant [h^{s-2} \cdot \alpha^A + h^{s-1} \cdot \sum_{i=1}^{n} \tfrac{1}{2} f^A_{q_i} + h^{s-2} \alpha^A - h^{s-1} \cdot \sum_{i=1}^{n} \tfrac{1}{2} f^A_{q_i} + \\ &\quad + h^{s-2} \cdot \beta^A + |1 + h^s \cdot f^A_u - h^{s-2} \cdot \beta^A - 2h^{s-2} \cdot \alpha^A|] \cdot \|Z\| \\ &= [2h^{s-2} \cdot \alpha^A + h^{s-2} \cdot \beta^A + |1 + h^s \cdot f^A_u - h^{s-2} \cdot \beta^A - 2h^{s-2} \cdot \alpha^A|] \cdot \|Z\|. \end{aligned} \tag{4.13}$$

Now, we shall examine two cases.

(a) $1 + h^s \cdot f^A_u - h^{s-2} \cdot \beta^A - 2h^{s-2} \cdot \alpha^A \geqslant 0$. Then

$$|d_l| \leqslant (1 + h^s \cdot f^A_u) \cdot \|Z\| \leqslant (1 + h^s \cdot L) \cdot \|Z\| = K \cdot \|Z\|. \tag{4.14}$$

(b) $1 + h^s \cdot f^A_u - h^{s-2} \cdot \beta^A - 2h^{s-2} \cdot \alpha^A < 0$. Then

$$\begin{aligned} |d_l| &\leqslant (-1 - h^s \cdot f^A_u + 4h^{s-2} \cdot \alpha^A + 2h^{s-2} \cdot \beta^A) \cdot \|Z\| \\ &\leqslant (-1 - h^s \cdot f^A_u + 4h^{s-2} \cdot (\alpha^A + \beta^A)) \cdot \|Z\|. \end{aligned}$$

From (4.1) we get

$$\alpha^A + \beta^A = \sum_{i=1}^{n} [f^A_{w_{ii}} - \sum_{\substack{j=1\\ j\neq i}}^{n} |f^A_{w_{ij}}|] + \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} |f^A_{w_{ij}}| = \sum_{i=1}^{n} f^A_{w_{ii}} \leqslant nG. \tag{4.15}$$

In view of (4.1), (4.15) and (4.5) we have

(4.16) $$\|d_t\| \leqslant (-1-h^s\cdot f_u^A+4h^{s-2}\cdot(\alpha^A+\beta^A))\cdot\|Z\|$$
$$\leqslant (-1-h^s\cdot L_1+4h^{s-2}\cdot nG)\cdot\|Z\| \leqslant (1+h^s\cdot L)\cdot\|Z\| = K\cdot\|Z\|.$$

The proof of (4.9) is complete, because the vectors $V, W\in R^p$ are arbitrary.

From the iteration theorem ([1], p. 48) it follows that there exists one and only one vector $V^* = [v_*^M]_{\substack{M\in Z\\ k(M)\in\mathcal{I}}} \in R^p$ (see Remark 1) such that $\Phi(V^*) = V^*$, and hence

(4.17) $$F(V^*) = 0.$$

The coordinates v_*^M $(M\in Z)$ of the vector V^* are a solution of (2.4) and this completes the proof of Theorem 2.

Remark 2. If $1+h^s\cdot L_1-2h^{s-2}\cdot nG \geqslant 0$, then the condition $h^s(h^{-2}\times 4nG-L-L_1)\leqslant 2$ in (4.5) is not necessary because (see (4.13)) $1+h^s\cdot f_u^A - h^{s-2}\cdot\beta^A-2h^{s-2}\cdot\alpha^A \geqslant 1+h^s\cdot f_u^A-2h^{s-2}\cdot(\alpha^A+\beta^A)\geqslant 1+h^s\cdot L_1-2h^{s-2}\cdot nG\geqslant 0$ and case (b) does not occur.

From the iteration theorem and Theorem 2 we derive

THEOREM 3. *If* (i)–(iv), (4.5), *and* (4.1) *hold, then for an arbitrary vector* $V_0\in R^p$ *the sequence*

(4.18) $$V_{m+1} \stackrel{\mathrm{df}}{=} \Phi(V_m), \quad m = 0, 1, 2, \ldots,$$

is convergent to the solution V^* *of* (2.4). *Moreover, the following estimation holds true:*

(4.19) $$\|V^*-V_m\| \leqslant \frac{(1+h^s\cdot L)^m}{-h^s\cdot L}\cdot\|V_1-V_0\| \quad \textit{for} \quad m = 1, 2, \ldots$$

THEOREM 4. *If* (i)–(iv) *and* (4.1) *hold, then*

(A) *there exists one and only one solution of the difference scheme* (2.4);

(B) *this solution* V^* *can be computed by successive approximation* (4.18) *with estimation* (4.19) *holding for the m-th approximation*;

(C) *the difference scheme* (2.4) *is stable* (see (3.3)).

This theorem immediately follows from the previous theorems.

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