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Structure of certain closed flows

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Abstract. The classification and characterization of closed planar flows is our objective. Our major result separates such flows into three mutually exclusive and exhaustive classes, namely, flows with critical sets which have empty components, only nonempty compact components, and at least one noncompact component. For each class we specify the bilateral orbital stability properties of the compact orbits and we apply the classification schemes devised by R. McCann for planar flows without critical points to the regular orbits.

1. Introduction. Continuous flows with closed orbits form a central class of dynamical systems. Our objective here is to classify closed planar flows in terms of their critical points and to characterize them in terms of the bilateral stability properties of the compact orbits and divergence of the regular trajectories. The major result, Theorem 1, separates closed planar flows into three mutually exclusive and exhaustive classes, namely, flows with critical sets which have empty components, only nonempty compact components, and at least one noncompact component. For each case we specify the bilateral stability properties of the compact orbits and we apply the classification schemes of [10] to the components of the regular set. The topological structure of the critical set is analyzed in [1] without making use of the stability properties of the compact orbits.

The trajectory (orbit) through x, orbit closure of x, limit set of x, prolongation of x, prolongational limit set of x, and region of attraction of x will be denoted by $C(x)$, $K(x)$, $L(x)$, $D(x)$, $J(x)$, and $A(x)$, respectively. The positive and negative versions of these concepts carry the appropriate plus and minus superscripts. The reader is referred to [2], [4], and [5] for basic properties of dynamical system theory.

2. Closed planar flows. A flow is called *closed* (*compact*) whenever each of its orbits is closed (compact). In this section we analyze the structure of such planar flows completely classifying them in Theorem 1. A series of lemmas many of which are straight forward observations are used to prove Theorem 1. The section concludes with characterizations of closed flows.

Given a flow (R^2, π) we shall denote the extended flow by (R^{2*}, π^*). Items relative to R^{2*} are identified by the superscript $*$. We shall denote the sets of regular, periodic, and critical points of a flow by T, P, and S, respectively. The component of S^* containing ∞ is denoted by S^*_γ and S_γ is $S^*_\gamma - \{\infty\}$. For convenience we denote the neighborhood $\bigcup\{\text{int}\, C(y)\colon y \in P$ and $x \in \text{int}\, C(y)\}$ of x by N_x.

For the convenience of the reader we summarize results obtained by McCann [10] for planar flows having no critical points. Every trajectory has a maximal open, connected, parallelizable neighborhood. The boundary trajectories of these neighborhoods are called *separatrices* and consist of the nondispersive wandering points. Even though the cardinalities of the separatrix sets for two flows are the same the flows may be different. The set M of separatrices is endowed with the induced topology from the quotient space R^{2*}/C.

The following two relations are defined:

(1) two trajectories in M are T-connected if and only if there is a transversal curve intersecting each and

(2) two trajectories are J-related if and only if the first is in the prolongational limit set of the second.

Furthermore, $\bar{M}$ of M in R^{2*}/C is endowed with the above structures.

Let (R^2, π_i) be such flows for $i = 1, 2$ and denote the items above for each flow with the appropriate subscript i. The primary results are:

(a) If a homeomorphism $f\colon M_1 \approx M_2$ preserves relations (1) and (2), then for any maximal subsystem (U, π_1) of (R^2, π_1) there exists a unique maximal dynamical subsystem (V, π_2) of (R^2, π_2) such that $f(M_1 \cap U/C_1) = M_2 \cap V/C_2$ and $f(\partial U/C_1) = \partial V/C_2$.

(b) R^{2*}/C_1 is homeomorphic to R^{2*}/C_2 if and only if there exists a homeomorphism $g\colon \bar{M}_1 \approx \bar{M}_2$ which preserves (1) and (2).

We now state our principal result.

THEOREM 1. *Let (R^2, π) be a closed planar flow. Then exactly one of the subsequent three conditions is necessary.*

(1) *S is empty and each of the following hold.*

(a) *T is locally parallelizable.*

(b) *T is classified by the McCann classification scheme.*

(c) $A^*(\infty) = R^{2*}$.

(2) *Each component of S is compact ($S_\infty = \emptyset$) and the following hold.*

(a) *S and each of its components are bilaterally stable.*

(b) *P consists of countably many open annular regions each surrounding at least one component of S.*

(c) *Each component of S is interior to an open annular region of periodic orbits.*

(d) $\{N_x\colon x \in S\}$ *is a countable collection of unbounded open pairwise disjoint sets homeomorphic to* R^2 *and covering* S. *Moreover,* $P \cup S = \bigcup \{N_x\colon x \in S\}$.

(e) T *is locally parallelizable relative to* $\pi | R^2 - \partial P \cap T$.

(f) T *is closed and each component is classified by the McCann scheme as a subflow of a planar flow without critical points.*

(g) *Each component of* T^0 *is homeomorphic to* R^2.

(h) $A^*(\infty) = T \cup \{\infty\}$.

(i) $\partial T \subset \partial P$.

(3) *At least one component of* S *is noncompact* ($S_\infty \neq \emptyset$) *and the following hold.*

(a) *Each compact component of* S *is bilaterally stable.*

(b) P *consists of countably many open annular regions each surrounding at least one component of* S.

(c) *Each compact component of* S *is interior to an open annular region of periodic orbits.*

(d) $\{N_x\colon x \in S\}$ *is a countable collection of open pairwise disjoint sets homeomorphic to* R^2 *and covering* $S - S_\infty$. *Moreover,* $P \cup S - S_\infty = \bigcup \{N_x\colon x \in S\}$.

(e) T *is locally parallelizable relative to* $\pi | R^2 - \partial P \cap T$.

(f) *Each component of* T *is classified by the McCann scheme as a subflow of a planar flow without critical points.*

(g) *Each component of* T^0 *is homeomorphic to* R^2.

(h) $A^*(\infty) = T \cup \{\infty\}$.

(i) $\partial T - T \subset \partial S_\infty$ *and* $\partial T \cap T \subset \partial P$.

In each of the following lemmas (R^2, π) is a closed planar flow.

LEMMA 1. $L(x) = \emptyset$ *if and only if* x *is regular.*

Proof. If $L(x) \neq \emptyset$ for some point x, then $L(x) \subset C(x)$ implying that $x \in L(x)$. Hence, $x \in P \cup S$ (see [11]). Thus, if x is regular, $L(x) = \emptyset$. The converse is obvious.

LEMMA 2. *Each periodic orbit is bilaterally stable.*

Proof. According to the Cycle Stability Theorem (3.3, p. 196, [2]) each periodic orbit is bilaterally stable since no point is attracted to a periodic orbit.

LEMMA 3. $J(T \cup S) \subset T \cup S$, $D(T \cup S) = T \cup S$, *and* $D(P) = P$.

Proof. Suppose that y is in $J(x) \cap P$ for some point x of $T \cup S$. Since $C(y)$ is bilaterally stable $D(y) = C(y)$ (7.6, p. 77, [4]), and hence, $J(y) = C(y)$. Thus, $x \in J(y) = C(y)$ implying that x is periodic which is absurd. The lemma follows.

LEMMA 4. *The set P is open; T is closed if and only if T and S are separated; and T is open if and only if $\partial T \subset S$.*

Proof. Since S is closed and every orbit in P is bilaterally stable, each periodic orbit has a compact invariant neighborhood disjoint from S. In view of Lemma 1 such neighborhoods must consist of periodic points. Hence, P is open. The other properties evidently follow.

LEMMA 5. *$P \cup S$ is open whenever S is compact.*

Proof. The set $T \cup S$ is closed, and hence, locally compact. According to Ura's alternatives (9.1, p. 94, [4]) S is either asymptotically stable or negatively asymptotically stable in $T \cup S$. Since $A^+(S) = A^-(S) = S$ we have S open in $T \cup S$. Thus, T and S are components of $T \cup S$ and the result follows.

LEMMA 6. *Each compact component of S is bilaterally stable and is interior to a region of periodic orbits each of which surrounds the component.*

Proof. The set $G = R^2 - T \cup S_\infty$ is the union of 2-cells so that each compact component of S is in such a 2-cell [1]. Let S_0 be a compact component of S. Each periodic orbit is bilaterally stable, and hence, $D(x) = C(x)$ for each x in P. We have $D(S_0) \subset G$ since $S_0 \subset G$ and G is open. Thus, $D(S - S_\infty) = S - S_\infty$. For a point s in S_0 the set $D(s) \cap S_0$ is compact and contains a compact component of $D(s)$. Thus, $D(s) = D(s) \cap S_0$, and hence, $D(S_0) = S_0$ (6.12.2, p. 68, [4]). The component S_0 is bilaterally stable (7.6, p. 77, [4]).

Although each compact component of S is in the open set G, we must show that each such component is surrounded by an annular region of periodic orbits. Let $V = \bigcup \{\operatorname{int} C(x)\colon x \in P\}$ and $S_0 = S \cap (R^2 - V)$. We shall show that each compact component of S is in V by demonstrating that $S_0 = S_\infty$. Let S_1 be a maximal compact connected subset of S_0. If no such maximal set exists, then S_0 is unbounded and we are done. Either $S_1 = \emptyset$ or S_1 is a component of S_0. In either case S_1 is bilaterally stable and S_1 has a compact connected simply connected invariant neighborhood W. If $S_1 = \emptyset$ we select $W = \emptyset$. No unbounded set N_x meets W since $\infty \notin D^*(S_1)$. Thus, W consists of a compact subset S union the bounded sets N_x. By virtue of the bilateral stability of each periodic orbit, if a periodic orbit were in the boundary of a set N_x the orbit must surround x, and hence, be interior to N_x. Thus, $\partial N_x \subset T \cup S$ for each set N_x and $\partial N_x \subset S$ whenever N_x is bounded. Countably many sets N_{x_i} (indexed by positive integers) meet W. Define $W_k = W - \bigcup \{N_{x_i}\colon i = 1, \ldots, k\}$ for each index k. The set $M = \bigcap \{W_k\colon k \text{ any index}\}$ is a continuum since each W_k is a continuum. No point of M is periodic so that M must be S_1. Since W is simply connected $W = S_1 \cup (\bigcup \{N_x\colon N_x \cap W \neq \emptyset\}) = \bigcup \{W^0 \cup N_x\colon N_x \cap W \neq \emptyset\}$. The compact set W is open yielding $S_1 = \emptyset$. The proof is complete.

Remark. At this point we can make several observations on the structure of closed flows. If S is compact, then S and each of its components are bilaterally stable. Whenever T is empty, each component of S^* is bilaterally stable in R^{2*}. Each isolated point of S is a Poincaré center. Each boundary point of T is contained in T or the boundary of S_∞. If the set N_x is bounded for some x in S, then the boundary of N_x is a subset of the union of the boundaries of S_∞ and T. There are countably many distinct sets N_x each homeomorphic to R^2. Also, S_∞^* is either bilaterally stable or an unstable attractor relative to $M \cup S_\infty^*$ provided M is a component of P or of T, respectively. Either $S \cup P$ is $\{\infty\}$ or ∞ is an accumulation point of $S \cup P$ in R^{2*}. Finally, the point ∞ is an attractor in R^{2*} if and only if S is empty.

LEMMA 7. *Each component of T is homeomorphic to a subflow of a flow on R^2 without compact orbits. Moreover, each component of T^0 is homeomorphic to R^2.*

Proof. Let K be a component of T with nonempty interior. We shall show that K^0 is homeomorphic to R^2. It suffices to show that K^0 is simply connected and connected. Suppose C is a simple closed curve in K^0 which has an interior point x not in K^0. The orbit $C(x)$ is compact since it does not meet C. If $x \in P$, then $\operatorname{int} C(x) \subset \operatorname{int} C$. Thus, if x is either periodic or critical, $S \cap \operatorname{int} C \neq \emptyset$. A component S_0 of $S \cap \operatorname{int} C$ is a compact component of S. The set N_y containing S_0 is bounded by C so that $\partial N_y \subset S_\infty$. But this means S_∞ meets $\operatorname{int} C$ which is clearly impossible. Hence, K^0 is simply connected.

Before showing that K^0 is connected we demonstrate that $T \cap \partial K = T \cap \partial K^0$. Since $\partial K^0 \subset \partial K$ we have $T \cap \partial K^0 \subset T \cap \partial K$. Next, let $x \in T \cap \partial K$ and let H be a transversal arc centered at x. (See [2] for a treatment of transversal theory.) Since S is closed, $x \in \partial P$. Thus, there exists a net (x_i) of periodic points on one side of $C(x)$ converging to x. The transversal H intersects $C(x)$ exactly once since otherwise no periodic orbit could intersect H (4.7, p. 175, [2]). A periodic orbit meets H in at most one point (4.4, p. 173, [2]). Any point x_t on the subarc xx_j connecting x to a point x_j is on a periodic orbit $C(x_t)$ separating $C(x)$ from $C(x_j)$ because each orbit crossing xx_j does so in the same direction. The interior of each orbit $C(x_i)$ is a subset of $P \cup S$ so that $\bigcup \operatorname{int} C(x_i) \subset P \cup S$ with $C(x) \subset \partial(\bigcup \operatorname{int} C(x_i))$. Next, let $x_0 x$ be the subarc of H exterior to $\bigcup \operatorname{int} C(x_i)$. If there is a net (y_i) of periodic points in $x_0 x$ converging to x, then again $C(x) \subset \partial(\bigcup \operatorname{int} C(y_i))$ leaving $C(x)$ separated from $K - C(x)$ which is impossible. Thus, there is a subarc $x_1 x$ of $x_0 x$ with endpoint x such that $x_1 x \subset T$. Now $G = (x_1 x - \{x_1, x\})R$ is an open connected subset of T with $C(x)$ contained in its boundary. Either every orbit in G separates $K - C(x)$ from $C(x)$ which is impossible or else $G \subset K$. Hence, $G \subset K^0$ and $C(x) \subset \partial K^0$. We have shown that $T \cap \partial K = T \cap \partial K^0$.

Finally, we show that K^0 is connected. If $K = K^0$, then the proof is complete. Let $T \cap \partial K \neq \emptyset$ and suppose that K^0 is the union of two separated sets A and B. We use M to denote the set of regular trajectories in ∂K. The sets A and B are invariant so that each orbit of M is in ∂A or ∂B. The set $K^*(x)$ for $x \in M$ is a simple closed curve on R^{2*}. It separates K from a set N_y. Thus, A and B are subsets of the same component of $R^{2*} - K^*(x)$ and the orbit $C(x)$ must be in ∂A or ∂B but not both. Let $M_1 = M \cap \partial A$ and $M_2 = M \cap \partial B$. Then $K = \overline{(A \cup M_1) \cup (B \cup M_2)}$. The connectedness of K implies that either $\overline{A \cup M_1} \cap (B \cup M_2) \neq \emptyset$ or $(A \cup M_1) \cap \overline{B \cup M_2} \neq \emptyset$. But $\overline{A \cup M_1} \cap (B \cup M_2) = \bar{A} \cap (B \cup M_2) = (\bar{A} \cap B) \cup (\bar{A} \cap M_2) = \emptyset$, and similarly, $(A \cup M_1) \cap \overline{B \cup M_2} = \emptyset$ which is absurd. Hence, K^0 is connected and we conclude that K^0 is homeomorphic to R^2.

Any regular boundary trajectory $C(x)$ of K is in the boundary of an unbounded set N_y. Thus, $C(x)$ is a boundary component of a periodic region and $\pi|K$ can be extended from $C(x)$ into an open strip M in the periodic region in such a way that the flow is parallel on $M \cup C(x)$. Obtaining such an extension at each regular boundary orbit produces an extended flow π' on a set K' homeomorphic to R^2. The flow $(K, \pi|K)$ is a restriction of the flow (K', π') which has no periodic or critical points.

The second statement of the lemma is evident completing the proof.

In light of Lemma 7 we can classify each component of T with nonempty interior as a subflow of a planar flow with empty critical set in such a way that the component contains the set of separatrices. Components of T with empty interior are a single trajectory.

Proof of Theorem 1. The remainder of the proof is an easy consequence of the lemmas.

We conclude this section with characterizations of closed flows. Seibert and Tulley have shown [11] that a point of a planar flow is Poisson stable if and only if it is either periodic or critical. In view of Lemma 1 we have the following characterization.

THEOREM 2. *A planar flow is closed if and only if each regular point is bilaterally divergent.*

The following theorem is not valid in general (example, p. 227, [7]). However, the fact that the minimality condition is necessary for a flow on a Hausdorff phase space to be closed is evident from the proof of Theorem 3.

THEOREM 3. *A planar flow is closed if and only if each point is contained in a minimal set.*

Proof. If (R^2, π) is closed, then $C(x) = K(x)$ for each x in R^2, and hence, each point x is in the minimal set $K(x)$.

Conversely, if $L(x) \neq \emptyset$ for some point x, then $K(x)$ is minimal and for any $y \in L(x)$ we have $K(x) = K(y) \subset L(x)$ (12.2.2, p. 133, [4]) so that x is

Poisson stable and $x \in P \cup S$. For such a point x, $C(x) = K(x)$. Finally, $L(x) = \emptyset$ for some point x implies $C(x) = K(x)$.

THEOREM 4. *A planar flow is closed if and only if $A(S) = S$ and each periodic orbit is bilaterally stable.*

Proof. Necessity follows from Lemma 2 and $L(x) \cap S = \emptyset$ for each x in $T \cup P$. Conversely, $x \in P \cup S$ implies $C(x) = K(x)$. If $x \in T$, then by the hypothesis $L(x) \cap (P \cup S) = \emptyset$ and $L(x) \cap T = \emptyset$ (1.11, p. 184, [2]), and hence, $C(x) = K(x)$.

THEOREM 5. *A planar flow is closed if and only if each periodic orbit is bilaterally stable, each compact component of ∂S (S) is bilaterally stable, and $A(S_\infty) = S_\infty$.*

Proof. The reasoning used in the proof of Theorem 4 for sufficiency is applicable here as well. Necessity of the conditions follows from Theorem 1.

The following corollary was obtained by Knight [8] for certain 2-manifolds.

COROLLARY 5.1. *A planar flow is compact if and only if each component of ∂S^* (S^*) and each periodic orbit are bilaterally stable in R^{2*}.*

The equivalence relation C on a flow (X, π) is defined by xCy provided $x \in C(y)$. The space X/C with the quotient topology is called the *orbit space* of π. The orbit space need not be Hausdorff. In fact X/C is Hausdorff if and only if (X, π) is a closed flow in which $D(x) = C(x)$ for each x in X [9]. In terms of Knight's results [6] R^2/C is Hausdorff if and only if the flow is of characteristic 0. The theorem which follows identifies the structure of X/C, where X is Hausdorff and (X, π) is closed. It follows easily from Lemma 1 of [3].

THEOREM 6. *A flow on a Hausdorff space X is closed if and only if X/C is a T_1 space.*

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