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A generalization of a polynomial lemma of Leja

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1. Introduction. The main purpose of this paper is to prove the following

THEOREM 1. *Let X be a compact subset of the complex plane C such that* $\operatorname{dia} Y \geqslant 2r$ *for each component Y of X, where* $\operatorname{dia} Y = \sup\{|a-b|: a, b \in Y\}$ *and $r > 0$ does not depend on the component Y. Let F be a locally convex Hausdorff topological vector space over C. Let q be a continuous seminorm defined on F. Let Φ be a family of polynomials of one complex variable with values in F such that* $\sup_{f\in\Phi} q\circ f(z)$ *is finite for every $z \in X$, where* $q\circ f(z) = q\big(f(z)\big)$.

Then for every $\omega > 1$ there exist $\delta > 0$ and $M > 0$ such that

$$q\circ f(z) \leqslant M\omega^{\deg f}, \quad z \in X^{(\delta)},\ f \in \Phi,$$

where $\deg f$ *denotes the degree of f and* $X^{(\delta)} = \{z \in C: \operatorname{dist}(z, X) \leqslant \delta\}$. *Moreover, M depends on ω and q, while δ depends only on ω but not on q.*

In fact Theorem 1 is true for more general compact sets X (see [2]).

If $F = C$ and X is connected, this theorem (called often a *Polynomial Lemma*) is due to Leja [3]. The Polynomial Lemma of Leja appeared to be a useful and effective tool in some important questions of analysis. The stricking point is that its proof is quite elementary.

First of all the Polynomial Lemma has been repeatedly used in the method of extremal points and extremal functions developed by Leja and by his students. This method found applications in the theory of conformal mappings, in the theory of interpolation and approximation, in the theory of Dirichlet problem and in the theory of domains of convergence of series of homogeneous polynomials of several complex variables (for references see [7], [9]).

Secondly, the Polynomial Lemma has been used to give a new proof of the Hartogs theorem on separate analyticity ([5], [6], [7]), as well as of some of its essential generalizations [10].

Alexiewicz and Orlicz [1] applied the Polynomial Lemma (and some other theorems of Leja related to it) in developing a theory of analytic functions in real Banach spaces.

Recently Theorem 1 has been used in [2] as one of basic tools in developing a theory of analytic functions defined in an open subset of a topological vector space E with values in a locally convex space F.

The proof given in [2] was based on a result obtained in [11] by means of a theory of Riesz potentials. The proof we have presented here works for a smaller class of compact sets X than that of [2], but it does not require any knowledge of the potential theory. Moreover, in [2] only the present version of the Polynomial Lemma was used. The present proof is a modification of the original proof due to Leja [3].

Given a compact set $Y \subset C$, we define

$$L(z, Y) = \begin{cases} \exp G(z), & z \in D_\infty, \\ 1, & z \in C \setminus D_\infty, \end{cases} \tag{§}$$

where D_∞ is the unbounded component of $C \setminus Y$ and G is the Green function of D_∞ with pole at ∞ (we put $G(z) \equiv +\infty$, if the transfinite diameter $d(Y) = 0$).

In this paper we also prove the following

THEOREM 2. *Let X have the same meaning as in Theorem 1. Let $\{X_k\}$ be a sequence of compact subsets of C such that $X_k \subset X_{k+1}$ $(k \in N)$ and $X = \bigcup_{k=1}^{\infty} X_k$. Then*

$$L(z, X) = \lim_{k\to\infty} L(z, X_k), \quad z \in C,$$

the convergence being uniform on every compact subset of C.

This theorem has been proved in [11] by means of Riesz potentials theory for a wider class of compact sets X. The present proof is a by-product of the proof of Theorem 1. Let us mention that Theorem 1 may be easily derived from Theorem 2 (see [2]).

2. Proof of Theorem 1. We shall need the following lemma due to Leja [3]:

LEMMA 1. *Let A be a closed subset of the compact interval $[0, r]$ such that the Lebesgue measure $m(A) > 0$. Then for every $n \in N$ there exist $n+1$ points $t_0, \ldots, t_n \in A$ such that*

$$0 \leqslant t_0 < t_1 < \ldots < t_n \leqslant r, \tag{1}$$

$$t_j - t_k \geqslant \frac{j^2 - k^2}{n^2} m(A), \quad j > k,\ j, k = 0, \ldots, n. \tag{2}$$

Proof. Put $t_0 = \inf A$, $s_0 = t_0 + \frac{1^2}{n^2} m(A)$, $A_1 = A \setminus [0, s_0)$. The set A_1 is not empty. Otherwise $A \subset [t_0, s_0)$, whence it follows that $m(A) < s_0 - t_0 = \frac{1}{n^2} m(A) \leqslant m(A)$. We get a contradiction, thus $A_1 \neq \emptyset$.

Put $t_1 = \inf A_1$, $s_1 = t_1 + \frac{2^2-1^2}{n^2} m(A)$, $A_2 = A \setminus [0, s_1)$. It is obvious that $t_1 - t_0 \geqslant s_0 - t_0 = \frac{1^2}{n^2} m(A)$. We claim that $A_2 \neq \emptyset$. Otherwise $A_2 \subset [t_0, s_0] \cup [t_1, s_1)$. Thus $m(A) < s_0 - t_0 + s_1 - t_1 = \frac{2^2}{n^2} m(A) \leqslant \leqslant m(A)$. Again we get a contradiction. So $A_2 \neq \emptyset$.

Put $t_2 = \inf A_2$, $s_2 = t_2 + \frac{3^2-2^2}{n^2} m(A)$, $A_3 = A \setminus [0, s_2)$. We have $t_2 - t_1 \geqslant s_1 - t_1 = \frac{2^2-1^2}{n^2} m(A)$. It is obvious that continuing this procedure we shall construct the required system of points $t_0, t_1, \ldots, t_n \in A$.

Let X have the same meaning as in Theorem 1. Given a point $a \in X$ and a closed subset $B \subset X$, define

$$p_a(B) = \{t \in [0, r]: B \cap C(a, t) \neq \emptyset\}, \tag{3}$$

where $C(a, t) = \{z \in \boldsymbol{C}: |z - a| = t\}$.

Put

$$I(\alpha) = \exp \int_0^1 \log \frac{\alpha^2 + x^2}{x^2}\, dx, \qquad \alpha \geqslant 0.$$

One may easily check that $\log I(\alpha) = \log(1+\alpha^2) + 2\alpha \operatorname{arctg} \frac{1}{\alpha}$. Therefore

$$\log I(\alpha) \leqslant (\pi + \alpha)\alpha, \qquad \alpha \geqslant 0. \tag{4}$$

LEMMA 2. *If g is any complex polynomial and $\delta > 0$, then*

$$|g(z)| \leqslant \|g\|_B \left[I\left(\sqrt{\frac{\delta + r - m(A)}{m(A)}} \right) \right]^{\deg g}, \qquad |z - a| \leqslant \delta, \tag{+}$$

where $\|g\|_B = \sup\{|g(z)|: z \in B\}$ and $A = p_a(B)$.

Proof. Let $t_0, \ldots, t_n$ be the points of A satisfying (1) and (2). Let $T_n = \{z_0, \ldots, z_n\}$ denote a system of $n+1$ points $z_j \in B$ such that $|z_j - a| = t_j$ $(j = 0, \ldots, n)$. Put

$$L^{(j)}(z, T_n) = \prod_{k=0 (k \neq j)}^{n} \frac{z - z_k}{z_j - z_k}, \qquad j = 0, \ldots, n.$$

We shall estimate $|L^{(j)}|$ for z in the disc $|z - a| \leqslant \delta$. It is obvious that

$$|z - z_k| \leqslant |z - a| + |z_k - a| \leqslant \delta + t_k \quad (k = 0, \ldots, n),\ |z - a| \leqslant \delta.$$

It follows from (2) that

$$r-t_k \geqslant \frac{n^2-k^2}{n^2}\, m(A),$$

whence

$$t_k \leqslant r-m(A)+\frac{k^2}{n^2}\, m(A).$$

Therefore

$$|z-z_k| \leqslant m(A)\left(a^2+\frac{k^2}{n^2}\right), \quad \text{where } a^2 = [\delta+r-m(A)]/m(A). \tag{5}$$

It is clear that

$$|z_j-z_k| \geqslant |t_j-t_k| \geqslant \frac{|j^2-k^2|}{n^2}\, m(A). \tag{6}$$

By (5) and (6)

$$|L^{(j)}(z,T_n)| \leqslant \prod_{k=0\,(k\neq j)}^{n} \left[\left(a^2+\frac{k^2}{n^2}\right)\Big/\frac{|j^2-k^2|}{n^2}\right] \leqslant 2\prod_{k=1}^{n}\left[\left(a^2+\frac{k^2}{n^2}\right)\Big/\frac{k^2}{n^2}\right],$$

when $|z-a| \leqslant \delta$ and $j = 0, \ldots, n$.
Observe that

$$\frac{1}{n}\sum_{k=1}^{n} \log\frac{a^2+k^2/n^2}{k^2/n^2} \leqslant \int_0^1 \log\frac{a^2+x^2}{x^2}\,dx = \log I(a), \qquad n \geqslant 1.$$

Therefore

$$|L^{(j)}(z,T_n)| \leqslant 2[I(a)]^n, \qquad |z-a| \leqslant \delta;\ j = 0,\ldots,n, \tag{7}$$

where a is given by (5).

Given any complex polynomial g of degree $\leqslant \nu$, it follows from the Lagrange interpolation formula that

$$g^l(z) = \sum_{j=0}^{n} g^l(z_j)L^{(j)}(z,T_n), \qquad z\in C;\ l = 1,2,\ldots;\ n = l\nu.$$

Hence, by virtue of (7),

$$|g(z)|^l \leqslant \|g\|_B^l(l\nu+1)2[I(a)]^{l\nu}, \qquad |z-a| \leqslant \delta,\ l \geqslant 1.$$

By taking the root of order l from both sides of the inequality and letting l tend to infinity we get $(+)$.

As a direct consequence of $(+)$ we obtain the following

COROLLARY 1. *If X has the same meaning as in Theorem 1, then*

$$(++)\qquad |g(z)| \leqslant \|g\|_X \left[I\left(\sqrt{\frac{\delta}{r}}\right)\right]^{\deg g}, \quad z \in X^{(\delta)},$$

where g is any complex polynomial.

Inequality $(++)$ implies that

$$L(z, X) \leqslant I\left(\sqrt{\frac{\delta}{r}}\right), \quad z \in X^{(\delta)},$$

because $L(z, X) = \sup |g(z)|^{1/\deg g}$, where sup is spread over all polynomials g such that $\|g\|_X \leqslant 1$ (see [9]).

LEMMA 3. *Let X and $\{X_k\}$ have the same meaning as in Theorem 2. Then for each $\delta > 0$ there exists a sequence of real numbers $\{m_k(\delta)\}$ such that $0 \leqslant m_k(\delta) \leqslant r$ $(k \in \boldsymbol{N})$, $\lim\limits_{k\to\infty} m_k(\delta) = r$ and*

$$(8)\qquad |g(z)| \leqslant \|g\|_k \left[I\left(\sqrt{\frac{\delta}{r}}\right) I(a_k)\right]^{\deg g}, \quad z \in X^{(\delta)},\ k \in \boldsymbol{N},$$

where g is any complex polynomial, $\|g\|_k = \sup\{|g(z)|: z \in X_k\}$ and $a_k = \left[\left(\delta + r - m_k(\delta)\right)/m_k(\delta)\right]^{1/2}$.

Proof. Given $\delta > 0$, let a_j $(j = 1, \ldots, s)$ be points of X such that $X \subset \bigcup\limits_{j=1}^{s} D_j$, where $D_j = \{z \in \boldsymbol{C}: |z - a_j| \leqslant \delta\}$. Put

$$A_{jk} = p_{a_j}(X_k), \quad j = 1, \ldots, s,\ k = 1, 2, \ldots,$$

where $p_{a_j}(X_k)$ is defined according to (3). One may easily prove that A_{jk} is closed and $[0, r] = \bigcup\limits_{k=1}^{\infty} A_{jk}$ $(j = 1, \ldots, s)$. In particular, $\lim\limits_{k\to\infty} m(A_{jk}) = r$ $(j = 1, \ldots, s)$.

Put

$$m_k(\delta) = \min_{1 \leqslant j \leqslant s} m(A_{jk}), \quad k = 1, 2, \ldots$$

It is obvious that $\lim\limits_{k\to\infty} m_k(\delta) = r$. By $(+)$

$$(9)\qquad |g(z)| \leqslant \|g\|_k [I(a_k)]^{\deg g}, \quad z \in \Omega = \bigcup_{j=1}^{s} D_j,\ k \in \boldsymbol{N}.$$

In particular, the last inequality holds for $z \in X$. Therefore by Corollary 1, we get (8).

We are now ready to prove Theorem 1. Put

$$(10)\qquad X_k = \{z \in X: q \circ f(z) \leqslant k, f \in \Phi\}, \quad k \in \boldsymbol{N}.$$

It is obvious that X_k is closed, $X_k \subset X_{k+1}$, $X = \bigcup_{k=1}^{\infty} X_k$. Let

$$S_{kn} = \{x_{k0}, x_{k1}, \ldots, x_{kn}\}, \quad n \in \boldsymbol{N},$$

be a system of $n+1$ Fekete points of X_k, i.e.

$$\prod_{0\leqslant\mu<\nu\leqslant n} |x_{k\mu}-x_{k\nu}| \geqslant \prod_{0\leqslant\mu<\nu\leqslant n} |y_\mu - y_\nu|, \quad y_j \in X_k \ (j = 0, \ldots, n).$$

One may easily check that

$$|L^{(j)}(z, S_{kn})| \leqslant 1, \quad z \in X_k, \ j = 0, \ldots, n, \tag{11}$$

where $L^{(j)}$ denotes the fundamental polynomial of Lagrange corresponding to the system of nodes S_{kn}.

By the polynomial formula of Lagrange

$$f(z) = \sum_{j=0}^{n} f(x_{kj}) L^{(j)}(z, S_{kn}), \quad z \in \boldsymbol{C}, \ f \in \Phi, \ \deg f = n.$$

Hence, by (10), (11) and Lemma 3

$$q\circ f(z) \leqslant k(n+1)\left[I\left(\sqrt{\frac{\delta}{r}}\right) I(a_k)\right]^n, \quad z \in X^{(\delta)}, \ f \in \Phi, \ \deg f = n, \ k \in \boldsymbol{N}.$$

Given $\omega > 1$, let n be so large that $\sqrt[n]{n+1} < \sqrt[3]{\omega}$, let $\delta > 0$ be so small that $I\left(\sqrt{\frac{\delta}{r}}\right) < \sqrt[3]{\omega}$ and finally let k be so large that $I(a_k) < \sqrt[3]{\omega}$. Then

$$q\circ f(z) \leqslant k\omega^{\deg f}, \quad z \in X^{(\delta)}, \ f \in \Phi, \ \deg f \geqslant n. \tag{12}$$

In order to end the proof let $S_n = \{y_0, \ldots, y_n\}$ be a system of $n+1$ different points of X. Then

$$f(z) = \sum_{j=0}^{n} f(y_j) L^{(j)}(z, S_n), \quad z \in \boldsymbol{C}, \ f \in \Phi, \ \deg f \leqslant n.$$

Hence by the pointwise boundedness of the family $\{q\circ f\}_{f\in\Phi}$, we get

$$q\circ f(z) \leqslant M_1 < +\infty, \quad z \in X^{(\delta)}, \ f \in \Phi, \ \deg f \leqslant n. \tag{13}$$

From (12) and (13) we get

$$q\circ f(z) \leqslant M\omega^{\deg f}, \quad z \in X^{(\delta)}, \ f \in \Phi,$$

where $M = k + M_1$. The proof of Theorem 1 is concluded.

3. Proof of Theorem 2. It is known [9] that the function L defined by (§), p. 150, may also be obtained by

$$L(z, Y) = \sup_{n\geqslant 1}[\sup\{|g(z)|^{1/n}: g\in\Phi_n\}], \quad z\in \boldsymbol{C}, \tag{14}$$

where $\Phi_n = \{g: g \text{ is a complex polynomial of degree} \leqslant n \text{ such that } \|g\|_Y \leqslant 1\}$.

By virtue of (9) we get

$$|g(z)| \leqslant I(a_k)^{\deg g}, \quad z\in\Omega,\ g\in\bigcup_{n=1}^{\infty}\Phi_n,\ k\in \boldsymbol{N}.$$

Hence by (14)

$$L(z, X_k) \leqslant I(a_k), \quad z\in\Omega,\ k\in \boldsymbol{N}.$$

We claim that

$$1 \leqslant L(z, X_k)/L(z, X) \leqslant I(a_k), \quad z\in \boldsymbol{C},\ k\in \boldsymbol{N}. \tag{15}$$

Indeed, the left-hand side inequality follows from the fact that $X_k \subset X$. If k is sufficiently large, then $m_k(\delta) > 0$ and $\log L(z, X_k)$ is harmonic in $\boldsymbol{C}\setminus X$. Moreover, the function $u_k(z) = \log[L(z, X_k)/L(z, X)]$ is harmonic in $(\boldsymbol{C}\setminus X)\cup\{\infty\}$, if we put $u_k(\infty) = \log[d(X)/d(X_k)]$, where d denotes the transfinite diameter of the corresponding set. By (15) $\limsup_{z\to\zeta} u_k(z) \leqslant \log I(a_k)$ for $\zeta\in X$. Therefore by the maximum principle

$$0 \leqslant u_k(z) \leqslant \log I(a_k), \quad z\in \boldsymbol{C},\ k\in \boldsymbol{N}.$$

Given $\varepsilon > 0$, let $\delta > 0$ be so small that $\log I\left(\sqrt{\frac{\delta}{r}}\right) < \varepsilon/2$. Next let k_0 be so large that $\log I(a_k) \leqslant \log I\left(\sqrt{\frac{\delta}{r}}\right) + \varepsilon/2$, for $k > k_0$. Then

$$0 \leqslant u_k(z) \leqslant \varepsilon, \quad z\in \boldsymbol{C},\ k > k_0.$$

It follows that $\lim_{k\to\infty} u_k(z) = 0$, $z\in \boldsymbol{C}\cup\{\infty\}$, the convergence being uniform in $\boldsymbol{C}\cup\{\infty\}$.

Now, Theorem 2 follows from the inequalities

$$0 \leqslant L(z, X_k) - L(z, X) \leqslant L(z, X)\,[e^{u_k(z)} - 1], \quad z\in \boldsymbol{C},\ k\in \boldsymbol{N}.$$

COROLLARY 2. *If X and $\{X_k\}$ satisfy the assumptions of Theorem 2, then* $\lim d(X_k) = d(X)$.

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