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Asymptotic stability of dynamical systems with multiplicative perturbations

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Abstract. Sufficient conditions for the asymptotic stability and for periodicity of dynamical systems with multiplicative perturbations are given. These criteria are applied to the Watkinson–MacDonald model of annual plants with a seedbank.

1. Introduction. In the last few years dynamical systems with stochastic perturbations were intensively studied ([5], [8], [9], [14], [15]). In this paper we consider dynamical systems with multiplicative perturbations. From the applied point of view this type of perturbations seems to be of great utility. For example the stochastical model of the brightness of Milky-Way proposed by Chandrasekhar and Münch ([3], [4]) leads to the same type of kernel operators as our multiplicative model described by formula (4). Moreover, the multiplicative perturbations in a natural way appear in biological systems when the stimulating factor is described by a deterministic transformation and the restraining factor is stochastic [11].

The organization of this paper goes as follows. In the next section we specify the class of stochastically perturbed systems to be considered. In Section 3, it is proved that the Markov operators governing the evolution of densities corresponding to our systems are weakly constrictive. In Section 4 it is shown that, under additional assumptions, these operators are asymptotically stable. Finally Section 5 contains an application to population dynamics. Namely using our technique we may theoretically explain (in some cases) the behaviour of the Watkinson–MacDonald "bottleneck" model of annual plants with a seedbank. This model was recently studied by Ellner [6].

2. Formulation of the problem. Consider a stochastically perturbed discrete time dynamical system of the form

$$x_{n+1} = S(x_n)\xi_n \quad \text{for } n = 0, 1, 2, \ldots, \tag{1}$$

where S is a Borel measurable transformation of $\boldsymbol{R}_+ = [0, +\infty)$ into itself, and ξ_n are independent random variables with values in $\boldsymbol{R}_+$.

In our study of the behaviour of (1) we shall admit the following assumptions:

(i) The random variables ξ_n are independent and all have the same distribution with a density function g.

(ii) The density g has a finite first moment, i.e.,

$$m = \int_0^{+\infty} xg(x)\,dx < +\infty. \tag{2}$$

(iii) There are positive constants a_0 and b such that the restriction S_1 of S to the interval $[0, a_0)$ is a C^2, increasing function and

$$S(x) \geqslant b \quad \text{for } x \geqslant a_0.$$

(iv) $S(0) = 0$ and $S'(0) > 0$.

(v) $S(x) \leqslant \eta x + \beta$, where η, β are nonnegative constants and $\eta m < 1$.

Finally, in addition to conditions (i)–(v), we assume that the initial condition x_0 is independent of the sequence of perturbation $\{\xi_n\}$.

Our goal is to study the asymptotic behaviour of the sequence $\{x_n\}$. Since the ξ_n are random, we adopt the strategy of studing the sequence of distributions of x_n. If we let the density of the distributions of x_n be denoted by f_n, then (cf. [7]) the relation between f_{n+1} and f_n is given by the formula

$$f_{n+1}(x) = \int_0^{+\infty} f_n(y)\, g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}\,dy. \tag{3}$$

Thus, given an arbitrary initial density f_0, the evolution of densities corresponding to the system (1) is described by the sequence of iterates $\{P^n f_0\}$, where

$$Pf(x) = \int_0^{+\infty} f(y)\, g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}\,dy \tag{4}$$

is a linear (Markov) operator from L^1 into itself.

3. Weak constrictiveness. By D we shall denote the set of all nonnegative elements $f \in L^1$ such that $\|f\| = 1$, where $\|\cdot\|$ stands for the norm in L^1. Our first step in the study of the operator (4) is to find sufficient conditions for its weak constrictiveness. A Markov operator P is called *weakly constrictive* if there is a weakly precompact set $\mathcal{F} \subset L^1$ such that

$$\lim_{n\to+\infty} \varrho(P^n f, \mathcal{F}) = 0 \quad \text{for } f \in D, \tag{5}$$

where $\varrho(f, \mathcal{F})$ denotes the distance between the function f and the set $\mathcal{F}$ in L^1 norm.

An answer to this problem is given by the following

THEOREM 1. *Assume that the transformation* $S\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ *and the density* g *satisfy conditions* (i)–(v). *Further assume that there exists a positive constant* λ *such that*

$$\int_0^{+\infty} \frac{g(x)}{(\gamma x)^\lambda}\,dx < 1, \quad \textit{where } \gamma = S'(0). \tag{6}$$

Then the Markov operator P *defined by equation* (4) *is weakly constrictive.*

Proof of Theorem 1. Define

$$M(f) = \int_0^{+\infty} x f(x)\,dx$$

and consider the sequence $\{M(P^n f)\}$ for an $f \in D$. From equation (4) and assumption (v) it follows immediately that

$$\begin{aligned} M(P^{n+1} f) &= \int_0^{+\infty}\int_0^{+\infty} x P^n f(y)\, g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}\,dy\,dx \\ &= \int_0^{+\infty} P^n f(y) S(y) \int_0^{+\infty} z g(z)\,dz\,dy \leqslant \eta m \int_0^{+\infty} y P^n f(y)\,dy + \beta m \\ &= \eta m M(P^n f) + \beta m. \end{aligned}$$

As a consequence

$$M(P^n f) \leqslant \frac{\beta m}{1-\eta m} + \eta^n m^n M(f).$$

Choose an arbitrary $M_* > \beta m/(1-\eta m)$. If $M(f) < +\infty$, then there is an integer $n_0 = n_0(f)$ such that

$$M(P^n f) \leqslant M_* \quad \text{for } n \geqslant n_0. \tag{7}$$

Using this inequality and a classical Chebyshev type argument we obtain

$$\int_{x \geqslant r} P^n f(x)\,dx \leqslant M_*/r \quad \text{for } r > 0 \text{ and } n \geqslant n_0. \tag{8}$$

For an arbitrary function $\varphi\colon (0, +\infty) \to (0, +\infty)$, denote by $\mathscr{F}\varphi$ the set of densities f satisfying the following two conditions:

$$\int_{x \geqslant r} f(x)\,dx \leqslant M_*/r \quad \text{for } r > 0 \tag{a}$$

and

$$\int_A f(x)\,dx \leqslant \varepsilon \quad \text{for every } \varepsilon > 0 \text{ if } \mu(A) \leqslant \varphi(\varepsilon), \tag{b}$$

where μ denotes the standard Lebesgue measure on $\boldsymbol{R}_+$.

Evidently $\mathscr{F}\varphi$ is a weakly precompact set. We are going to find a φ such that condition (5) holds for $\mathscr{F} = \mathscr{F}\varphi$. Fix an $\lambda > 0$ for which (6) holds true and denote by D_0 the subset of D consisting of all functions with $M(f) < +\infty$ such that

$$\int_0^{+\infty} \frac{1}{x^\lambda} f(x)\,dx < +\infty.$$

Choose an $f \in D_0$ and set $f_n = P^n f$, $n = 0, 1, 2, \ldots$ Further, let $A \subset \boldsymbol{R}_+$ and $\varepsilon > 0$ be given. Then, from the definition of P in equation (4), we have

$$\begin{aligned}\int_A Pf_n(x)\,dx &= \int_A \int_0^{+\infty} f_n(y)\, g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}\,dy\,dx \\ &= \int_0^{+\infty} f_n(y) \int_0^{+\infty} 1_A\big(S(y)z\big) g(z)\,dz\,dy,\end{aligned}$$

where 1_A is the characteristic function of the set A. Taking an arbitrary $c \in (0, a_0)$ we immediately obtain

$$\begin{aligned}\int_A Pf_n(x)\,dx &= \int_0^c f_n(y) \int_0^{+\infty} 1_A\big(S(y)z\big) g(z)\,dz\,dy + \\ &+ \int_c^{+\infty} f_n(y) \int_0^{+\infty} 1_A\big(S(y)z\big) g(z)\,dz\,dy.\end{aligned} \tag{9}$$

Now we are going to evaluate the first term in (9). Since

$$\frac{1}{\gamma^\lambda} \int_0^{+\infty} \frac{g(x)}{x^\lambda}\,dx < 1$$

we may choose $\sigma > 0$ so small that

$$K = \frac{1}{(\gamma-\sigma)^\lambda} \int_0^{+\infty} \frac{g(x)}{x^\lambda}\,dx < 1.$$

From the continuity of the function $y \to S'_1(y)$ and assumption (iv) it follows the existence of a positive constant $\delta < a_0$ such that

$$S(y) \geqslant (\gamma - \sigma)\,y \quad \text{for } y \in [0, \delta]. \tag{10}$$

The mathematical expectation of

$$1/(S(y)\,\xi_n)^\lambda$$

is evidently given by

$$E\left(\frac{1}{(S(y)\,\xi_n)^\lambda}\right) = \frac{1}{S(y)^\lambda} \int_0^{+\infty} \frac{g(x)}{x^\lambda}\,dx.$$

Using inequality (10) we obtain

$$E\left(\frac{1}{(S(y)\xi_n)^\lambda}\right) \leqslant \frac{K}{y^\lambda} + d \quad \text{for } y \in (0, +\infty), \tag{11}$$

where

$$d = \max\left(\frac{1}{S(\delta)^\lambda} \int_0^{+\infty} \frac{g(x)}{x^\lambda} dx, \frac{1}{b^\lambda} \int_0^{+\infty} \frac{g(x)}{x^\lambda} dx\right).$$

Now let

$$V_n(f) = \int_0^{+\infty} \frac{1}{x^\lambda} f_n(x)\, dx$$

so from inequality (11) we have

$$\begin{aligned} V_n(f) &= \int_0^{+\infty} f_{n-1}(y) \int_0^{+\infty} \frac{1}{x^\lambda} g\left(\frac{x}{S(y)}\right) \frac{1}{S(y)} dx\, dy \\ &= \int_0^{+\infty} f_{n-1}(y) E\left(\frac{1}{(S(y)\xi_n)^\lambda}\right) dy \leqslant K V_{n-1}(f) + d. \end{aligned}$$

By an induction argument we obtain

$$V_n(f) \leqslant K^n V_0(f) + \frac{d}{1-K}.$$

Since $V_0(f) < +\infty$ there is some integer $n_1 = n_1(f)$ such that

$$V_n(f) \leqslant \frac{d}{1-K} + 1 \quad \text{for } n \geqslant n_1.$$

As a consequence

$$\begin{aligned} \int_0^c f_n(x)\, dx &= c^\lambda \frac{1}{c^\lambda} \int_0^c f_n(x)\, dx \leqslant c^\lambda \int_0^c \frac{1}{x^\lambda} f_n(x)\, dx \\ &\leqslant c^\lambda V_n(f) \leqslant c^\lambda \left(\frac{d}{1-K} + 1\right) \quad \text{for } n \geqslant n_1. \end{aligned}$$

Fixing c so small that

$$c^\lambda \left(\frac{d}{1-K} + 1\right) \leqslant \frac{\varepsilon}{2}$$

and using the previous inequality we conclude that

$$\int_0^c f_n(x)\, dx \leqslant \varepsilon/2 \quad \text{for } n \geqslant n_1. \tag{12}$$

Finally we estimate the second term in the sum (9). Set

$$c_1 = \min(S_1(c), b)$$

and denote by $\varphi_1(\varepsilon)$ a positive number such that

$$\int_B g(z)\,dz \leqslant \varepsilon/2 \quad \text{whenever } \mu(B) \leqslant \varphi_1(\varepsilon).$$

It is clear that if $\mu(A) \leqslant c_1\,\varphi_1(\varepsilon)$ then

$$\int_c^{+\infty} f_n(y) \int_0^{+\infty} 1_A\big(S(y)z\big) g(z)\,dz\,dy \leqslant \varepsilon/2.$$

This inequality in conjunction with inequality (12) and equality (9) gives

$$\int_A P^n f(x)\,dx \leqslant \varepsilon \quad \text{whenever } \mu(A) \leqslant c_1\,\varphi_1(\varepsilon) \text{ and } n \geqslant n_1.$$

As a consequence $P^n f \in \mathscr{F}\varphi$ with $\varphi = c_1\,\varphi_1$ for $n \geqslant \max(n_0, n_1)$. Since D_0 is dense in D, this implies (5), and the proof is complete. □

Remark. The importance of weak constrictiveness is a consequence of the following theorem of Komornik [10]:

SPECTRAL DECOMPOSITION THEOREM. *Let P be a weakly constrictive Markov operator. Then there is an integer r, two sequences of nonnegative functions $h_i \in D$ and $w_i \in L^\infty$, $i = 1, 2, \ldots, r$, and an operator Q: $L^1 \to L^1$ such that for all $f \in L^1$, Pf may be written in the form*

$$Pf(x) = \sum_{i=1}^{r} \lambda_i(f)\,h_i(x) + Qf(x), \tag{13}$$

where

$$\lambda_i(f) = \int_0^{+\infty} f(x)\,w_i(x)\,dx.$$

The functions h_i and the operator Q have the following properties:

(a) $h_i(x)\,h_j(x) = 0$ for all $i \neq j$, so that the density h_i have disjoint supports.

(b) For each integer i there exists a unique integer $\omega(i)$ such that $Ph_i = h_{\omega(i)}$. Further, $\omega(i) \neq \omega(j)$ for $i \neq j$ and thus the operator P just serves to permute the functions h_i.

(c) $\|P^n Qf\| \to 0$ as $n \to +\infty$ for every $f \in L^1$.

From equation (13) it is clear that $P^n f$ may be written as

$$P^n f = \sum_{i=1}^{r} \lambda_i(f)\,h_{\omega^n(i)} + Q_n(f), \tag{14}$$

where $Q_n = P^{n-1}Q$, and $\omega^n(i) = \omega(\omega^{n-1}(i)) = \ldots$, and $\|Q_n f\| \to 0$ as $n \to +\infty$. The terms in the summation in equation (14) are permuted with each application of P. Hence, the summation portion of (14) is periodic with period less than $r!$. Furthermore, since $\|Q_n f\| \to 0$ as $n \to +\infty$ we say that for a weakly constrictive Markov operator the sequence $\{P^n f\}$ is *asymptotically periodic*.

4. Asymptotic stability. A Markov operator P is said to be *asymptotically stable* if there is a unique $f_* \in D$ such that $Pf_* = f_*$ and

$$\lim_{n \to +\infty} \|P^n f - f_*\| = 0 \quad \text{for all } f \in D.$$

Using Theorem 1 we may prove the following result concerning the appearance of asymptotic stability of the Markov operator P defined by equation (4):

THEOREM 2. *Let the transformation S: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ and the density g satisfy conditions* (i)–(v) *and inequality* (6). *Assume that there exists a nonnegative constant u such that*

$$g(x) > 0 \quad \text{a.e. for } x \geqslant u. \tag{15}$$

Then the Markov operator defined by equation (4) *is asymptotically stable.*

Proof of Theorem 2. To prove this theorem, we employ the following

LEMMA. *Let P be a weakly constrictive Markov operator. Assume there is a set $E \subset \boldsymbol{R}_+$ of nonzero measure, $\mu(E) > 0$, with the property that for every $f \in D$ there is an integer $n_2(f)$ such that*

$$P^n f(x) > 0 \tag{16}$$

for almost all $x \in E$ and all $n \geqslant n_2(f)$. Then P is asymptotically stable.

The proof of this lemma may be found in [10]. ◻

Since, by Theorem 1, we know that P is weakly constrictive, we need only to demonstrate that P satisfies the rest of the assumptions of the lemma. Let $f \in D$ be arbitrary. Since f is integrable there is a bounded subset $B \subset \boldsymbol{R}_+$ such that

$$\int_B f(x)\, dx = \tfrac{1}{2}.$$

Define $\tilde{f}(x) = 2f(x) 1_B(x)$. Clearly, $\tilde{f} \in D$ and $M(\tilde{f}) < +\infty$. From inequality (8) it follows that

$$\int_{x \geqslant r} P^n \tilde{f}(x)\, dx \leqslant M_*/r \quad \text{for } r > 0 \text{ and } n \geqslant n_0(\tilde{f}).$$

Thus

$$\int_{x \leqslant r} P^n f(x)\,dx \geqslant \tfrac{1}{2} \int_{x \leqslant r} P^n \tilde{f}(x)\,dx = \tfrac{1}{2}\{1 - \int_{x \geqslant r} P^n \tilde{f}(x)\,dx\} \tag{17}$$

$$\geqslant \tfrac{1}{2}(1 - M_*/r) > 0 \quad \text{for } r > M_* \text{ and } n \geqslant n_0(\tilde{f}).$$

Now we may write

$$P^n f(x) = \int_0^{+\infty} P^{n-1} f(y) g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}\,dy \tag{18}$$

$$\geqslant \int_{y \leqslant r} P^{n-1} f(y) g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)}\,dy.$$

Fix $r > M_*$ and set $E = [(\eta r + \beta)u, +\infty)$. Using assumption (v) and inequality (15) we conclude that

$$g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)} > 0 \quad \text{for } x \in E \text{ and } y \leqslant r.$$

From this and (17) it follows that for every $x \in E$ the product

$$P^{n-1} f(y) g\left(\frac{x}{S(y)}\right)\frac{1}{S(y)} \quad \text{with } n \geqslant n_0(\tilde{f}) + 1$$

as a function of y, does not vanish in the set defined by $y \leqslant r$. As a consequence, applying inequality (18), we finally obtain

$$P^n f(x) > 0 \quad \text{for } x \in E \text{ and } n \geqslant n_0(\tilde{f}) + 1.$$

Thus, the proof of the theorem is complete. □

5. An application. As an illustration, we now apply our results to a special case of the MacDonald and Watkinson "bottleneck" model of annual plants with a seedbank.

In this model the size x_n of the population in the n-th generation is given by the formula

$$x_{n+1} = \frac{x_n \xi_n}{(1 + (k+p)x)^\alpha (1 + px)^{1-\alpha}}, \tag{19}$$

where k, p, α are positive constants.

It is easy to verify that the transformation

$$S(x) = \frac{x}{(1 + (k+p)x)^\alpha (1 + px)^{1-\alpha}} \tag{20}$$

satisfies conditions (iii)–(v). In particular condition (v) is satisfied with $\eta = 0$

and

$$\beta = \begin{cases} \dfrac{(\alpha-1)^{\alpha-1}}{k\alpha^{\alpha}} & \text{if } k\alpha > k+p, \\ \dfrac{1}{(k+p)^{\alpha}p^{1-\alpha}} & \text{if } k\alpha \leqslant k+p. \end{cases}$$

Thus if the density g of the independent random variables ξ_n has a finite first moment and satisfies the condition

$$\int_0^{+\infty} \frac{g(x)}{x^{\lambda}}\,dx < 1$$

with some constant $\lambda > 0$, then Theorem 1 implies that the Markov operator P corresponding to (19) is weakly constrictive.

If, in addition, the density g satisfies the following condition

$$g(x) > 0 \quad \text{a.e. for } x \geqslant u,$$

where u is a nonnegative constant, according to Theorem 2, then the operator P is asymptotically stable.

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