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On expansions of Meijer's functions III

A problem of the changed parameters and particular cases

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§ 4. A problem of the changed parameters. Let us introduce the notations of the first part of this paper (see [3]) and further let

(63)
$$A_h(s) = \Big\{\prod_{\substack{j=1\\ j\neq h}}^{n} \Gamma(a_j - s) \prod_{j=1}^{m} \Gamma(1+s-b_j)\Big\} \Big/ \Big\{\prod_{j=n+1}^{p} \Gamma(1+s-a_j) \prod_{j=m+1}^{q} \Gamma(b_j - s)\Big\} \quad (h = 1, \dots, n),$$

(64)
$$a(s) = \prod_{j=n+1}^{p} \Gamma(1+s-a_j),$$

(47)
$$D_h(s) = \Big\{\prod_{\substack{j=1\\ j\neq h}}^{\mu} \Gamma(d_j - s) \prod_{j=1}^{\nu} \Gamma(1+s-c_j)\Big\} \Big/ \Big\{\prod_{j=\mu+1}^{\tau} \Gamma(1+s-d_j) \prod_{j=\nu+1}^{\sigma} \Gamma(c_j - s)\Big\} \quad (h = 1, \dots, \mu),$$

(48)
$$\delta(s) = \prod_{j=\mu+1}^{\tau} \Gamma(1+s-d_j).$$

Finally let

(65)
$$G_{2A}(x) = \left|\exp(-tx)\, G_{\sigma,\tau}^{\mu,\nu}\left(\omega/x \,\middle|\, \begin{matrix} c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau \end{matrix}\right)\right| \sum_{r=0}^{\infty} \left|\sum_{h=1}^{n} (1/r!)\, A_h(a_h)(\eta x)^{a_h} \times \right.$$
$$\left. \times (tx)^r {}_{q+1}F_{p-1}\left(\begin{matrix} -r, 1+a_h-b_1, \dots, 1+a_h-b_q; \\ 1+a_h-a_1, \dots * \dots, 1+a_h-a_p; (-1)^{q-m-n+1}\eta t \end{matrix}\right)\right| (1/x),$$

(66)
$$G_{2B}(x) = \left|\exp(-t/x)\, G_{q,p}^{n,m}\left(\eta x \,\middle|\, \begin{matrix} b_1, \dots, b_q \\ a_1, \dots, a_p \end{matrix}\right)\right| \sum_{r=0}^{\infty} \left|\sum_{h=1}^{\mu} (1/r!)\, D_h(d_h)(\omega/x)^{d_h} \times \right.$$
$$\left. \times (t/x)^r {}_{\sigma+1}F_{\tau-1}\left(\begin{matrix} -r, 1+d_h-c_1, \dots, 1+d_h-c_\sigma; \\ 1+d_h-d_1, \dots * \dots, 1+d_h-d_\tau; (-1)^{\sigma-\mu-\nu+1}\omega/t \end{matrix}\right)\right| (1/x),$$

where the asterisk * in the first formula denotes that the number $1+a_h-a_h$ is to be omitted in the sequence $1+a_h-a_1, \ldots, 1+a_h-a_p$ and, analogously, the asterisk * in the second formula denotes that the number $1+d_h-d_h$ is to be omitted in the sequence $1+d_h-d_1, \ldots, 1+d_h-d_\tau$. In formulae (65) and (66) we assume that $x>0$ and t fulfils the condition

$$(28) \qquad t \neq 0, \quad |\arg t| < \tfrac{1}{2}\pi .$$

The functions A_h and D_h exist if the Gamma functions appearing in the numerators have no poles at the given points. Analogously, the functions α and δ exist if the Gamma functions appearing in the respective formulae exist. The functions G_{2A} and G_{2B} exist in each of the cases (VI), (VII), (VIII), (IX); this will be proved below in Theorems 2A and 2B, respectively. In the case where some of the numbers

$$(67) \qquad a_j-a_h \quad (j = n+1, \ldots, p;\ h = 1, \ldots, n),$$

$$(52) \qquad d_j-d_h \quad (j = \mu+1, \ldots, \tau;\ h = 1, \ldots, \mu)$$

are natural, formulae (65) and (66) must be understood in the sense of Remarks 9 and 6, respectively.

Remark 9. In the case where some of the numbers (67) are natural, the respective coefficients $A_h(a_h)$ are to be replaced by the limit of the products $A_h(a_h^*)\alpha(a_h^*)$ as $a_h^* \to a_h$, and the respective functions ${}_{q+1}F_{p-1}(1+a_h-b_1, \ldots)$ by the limit of the quotients

$${}_{q+1}F_{p-1}(1+a_h^*-b_1, \ldots)/\alpha(a_h^*) \quad \text{as} \quad a_h^* \to a_h .$$

Remark 6, analogous to Remark 9, was formulated in the second part of this paper (see [3], § 3).

Now we shall formulate and prove two more theorems. To this end, in the considerations of the previous paragraph, we replace formula (7) by formula (32), as has been announced in § 1 of this paper.

THEOREM 2A. *Let $m, n, p, q, \mu, \nu, \sigma, \tau$ be integers, let t fulfil* (28) *and let one of the cases* (VI), (VII), (VIII), (IX) *take place. If for large x_0*

$$(68) \qquad \int_{x_0}^{\infty} G_{2A}(x)\,dx \quad \textit{converges},$$

then

(69)
$$G_{q+\sigma,p+\tau}^{n+\mu,m+\nu}\left(\eta\omega\left|\begin{matrix} b_1, \ldots, b_m, c_1, \ldots, c_\sigma, b_{m+1}, \ldots, b_q \\ a_1, \ldots, a_p, d_1, \ldots, d_\tau, a_{n+1}, \ldots, a_p\end{matrix}\right.\right) = \sum_{r=0}^{\infty}\sum_{h=1}^{n}(1/r!)A_h(a_h)(\eta/t)^{a_h} \times$$
$$\times {}_{q+1}F_{p-1}\left(\begin{matrix}-r, 1+a_h-b_1, \ldots, 1+a_h-b_q; \\ 1+a_h-a_1, \ldots * \ldots, 1+a_h-a_p; (-1)^{q-m-n+1}\eta/t\end{matrix}\right) \times$$
$$\times G_{\sigma,\tau+1}^{\mu+1,\nu}\left(\omega t\left|\begin{matrix} c_1, \ldots, c_\sigma \\ a_h+r, d_1, \ldots, d_\tau\end{matrix}\right.\right),$$

*where the asterisk * denotes that the number $1+a_h-a_h$ is to be omitted in the sequence $1+a_h-a_1, \dots, 1+a_h-a_p$, and in the case where some of the numbers* (67) *are natural, formulae* (69) *and* (65) *must be understood in the sense of Remark* 9. *The connection between the branches of $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}$ and $G^{\mu+1,\nu}_{\sigma,\tau+1}$ is determined by Remark* 3.

Proof. As in the proofs of Theorems 1A and 1B, we first state that for any complex $t \neq 0$ in each of the cases (VI), (VII), (VIII), (IX) we have

$$(70)\quad J = \int_0^\infty \sum_{h=1}^{n} A_h(a_h)(\eta x)^{a_h}\exp(-tx)\sum_{r=0}^{\infty}(1/r!)(tx)^r \times \\ \times {}_{q+1}F_{p-1}\left(\begin{matrix}-r, 1+a_h-b_1, \dots, 1+a_h-b_q; \\ 1+a_h-a_1, \dots * \dots, 1+a_h-a_p; (-1)^{q-m-n+1}\eta/t\end{matrix}\right) \times \\ \times G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x\,\middle|\,\begin{matrix}c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau\end{matrix}\right)(1/x)\,dx.$$

Let us consider first the cases (VI) and (VII). Write the integrand of (70) in the form

$$(71)\quad (\eta x)^{-\delta_{2A}-1}\exp(-tx)\,G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x\,\middle|\,\begin{matrix}c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau\end{matrix}\right)(1/x) \times \\ \times \sum_{h=1}^{n}\sum_{r=0}^{\infty} A_h(a_h)(1/r!)(\eta x)^{\delta_{2A}+a_h+1}(tx)^r \times \\ \times {}_{q+1}F_{p-1}\left(\begin{matrix}-r, 1+a_h-b_1, \dots, 1+a_h-b_q; \\ 1+a_h-a_1, \dots * \dots, 1+a_h-a_p; (-1)^{q-m-n+1}\eta/t\end{matrix}\right),$$

where

$$\delta_{2A} = -\max_{j=1,\dots,\nu} \operatorname{re} c_j - \varepsilon^* \qquad (\varepsilon^* > 0 \text{ sufficiently small})$$

and introduce the notations

$$\Phi^{(2A)}(x) = (\eta x)^{-\delta_{2A}-1}\exp(-tx)\,G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x\,\middle|\,\begin{matrix}c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau\end{matrix}\right)(1/x),$$

$$f^{(2A)}_{r,h}(x) = (1/r!)A_h(a_h)\eta^{\delta_{2A}+a_h+1}(tx)^r \times \\ \times {}_{q+1}F_{p-1}\left(\begin{matrix}-r, 1+a_h-b_1, \dots, 1+a_h-b_q; \\ 1+a_h-a_1, \dots * \dots, 1+a_h-a_p; (-1)^{q-m-n+1}\eta/t\end{matrix}\right) \\ (r = 0, 1, \dots;\ h = 1, \dots, n),$$

$$f^{(2A)}_r(x) = \sum_{h=1}^{n} x^{\delta_{2A}+a_h+1} f^{(2A)}_{r,h}(x).$$

Applying, as in the proof of Theorems 1A and 1B, the Tests 1 and 2 from § 1, we state when it is possible to perform integration in (70) term by term with the restriction that the integrals will be summed first with respect to h and only then with respect to r. Moreover, we state when it is possible to evaluate these integrals from the formula (43) with $r = 0, 1, \ldots$ The required conditions are (28) and (68) in each of the cases (VI) and (VII). Parts (i) and (iii) of the proof are analogous to the corresponding parts of the proof of Theorems 1A and 1B. In part (ii) the estimation of $|\Phi^{(2A)}|$ in the sector $0 < x \leqslant \beta$, where β $(0 < \beta < 1)$ is sufficiently small, runs in a different manner.

Here, for $\nu > 0$ we apply a well-known asymptotic formula

$$G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x\left|\begin{matrix} c_1, \ldots, c_\sigma \\ d_1, \ldots, d_\tau\end{matrix}\right.\right) \sim \sum_{j=1}^{\nu} (\omega/x)^{c_j-1} \left[B_{j,0} + B_{j,1}(\omega/x)^{-1} + B_{j,2}(\omega/x)^{-2} + \ldots\right],$$

where $B_{j,h}$ $(j = 1, \ldots, \nu;\ h = 1, 2, \ldots)$ are constants (see [4], I, formula (32), p. 87). Hence, if β $(0 < \beta < 1)$ is sufficiently small, there exists such a constant $M > 0$ that

$$|\Phi^{(2A)}(x)| \leqslant M|(\eta x)^{-\delta_{2A}-1}\exp(-tx)| \sum_{j=1}^{\nu} |B_{j,0}(\omega/x)^{c_j-1}|(1/x)$$

$$\leqslant M|\eta^{-\delta_{2A}-1}| \max\{1, \exp(-\beta \operatorname{re} t)\} \left\{\sum_{h=1}^{\nu} |B_{j,0}\,\omega^{c_h-1}|\right\} x^{-1+\varepsilon^*},$$

and consequently the integral

$$\int_0^{\beta} |\Phi^{(2A)}(x)|\,dx \tag{72}$$

converges. Similarly, for $\nu = 0$ we apply a well-known formula

$$G^{\mu,0}_{\sigma,\tau}(\omega/x) = \sum_{h=0}^{\tau-\mu} B'_h G^{\tau,0}_{\sigma,\tau}\left((\omega/x)\, e^{(\tau-\mu-2h)\pi i}\right),$$

where the function $G^{\tau,0}_{\sigma,\tau}$ has an asymptotic expansion

$$G^{\tau,0}_{\sigma,\tau}\left(\omega/x\left|\begin{matrix} c_1, \ldots, c_\sigma \\ d_1, \ldots, d_\tau\end{matrix}\right.\right) \sim (\omega/x)^{\vartheta'} \exp\left((\sigma-\tau)(\omega/x)^{1/(\tau-\sigma)}\right) \times$$

$$\times \left[B''_0 + B''_1(\omega/x)^{-1/(\tau-\sigma)} + B''_2(\omega/x)^{-2/(\tau-\sigma)} + \ldots\right];$$

here B'_h $(h = 0, \ldots, \tau-\mu)$, ϑ', B''_h $(h = 1, 2, \ldots)$ are constants and

$$\vartheta' = (\tau-\sigma)^{-1}\left[\tfrac{1}{2}(\sigma-\tau+1) - \sum_{h=1}^{\sigma} c_h + \sum_{h=1}^{\tau} d_h\right]$$

(see [4], I, formulae (34) and (33), pp. 87-88). Hence, if β $(0<\beta<1)$ is sufficiently small, there exists such a constant $M'>0$ that

$$\begin{aligned}|\Phi^{(2A)}(x)| &\leqslant M'|(\eta x)^{-\delta_{2A}-1}\exp(-tx)|\sum_{h=0}^{\tau-\mu}\Big|B'_h B''_0(\omega/x)^{\vartheta'}\times\\ &\qquad\times\exp\big(-(\tau-\sigma)[(\omega/x)e^{(\tau-\mu-2h)\pi i}]^{1/(\tau-\sigma)}\big)\Big|(1/x)\\ &\leqslant M'|\eta^{-\delta_{2A}-1}\omega^{\vartheta'}|\max\{1,\exp(-\beta\operatorname{re}t)\}\Big\{\sum_{h=0}^{\tau-\mu}|B'_h B''_0|\times\\ &\qquad\times\exp\big(-(\tau-\sigma)\operatorname{re}[(\omega/x)e^{(\tau-\mu-2h)\pi i}]^{1/(\tau-\sigma)}\big)\Big\}x^{-\delta_{2A}-2-\operatorname{re}\vartheta'}.\end{aligned}$$

But inequalities (8) and (10) yield for $h=0,\ldots,\tau-\mu$

$$\begin{aligned}|\arg(\omega e^{(\tau-\mu-2h)\pi i})| &\leqslant |\arg\omega|+|\tau-\mu-2h|\pi\\ &<(\mu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi+(\tau-\mu)\pi=\tfrac{1}{2}(\tau-\sigma)\pi.\end{aligned}$$

Thus, in view of (33) in case (VI) and (34) in case (VII), we obtain

$$-(\tau-\sigma)\operatorname{re}[(\omega/x)e^{(\tau-\mu-2h)\pi i}]^{1/(\tau-\sigma)}\leqslant-\varepsilon' x^{-\varepsilon''}\qquad(h=0,\ldots,\tau-\mu),$$

where ε' and $\varepsilon''=1/(\tau-\sigma)$ are positive. In consequence (72) converges.

In cases (VIII) and (IX) we write the integrand of (70) also in form (71), but now we admit

$$\hat{\delta}_{2A}=(\tau-\sigma)^{-1}\Big[\tfrac{1}{2}(\sigma-\tau+1)+\sum_{h=1}^{\sigma}\operatorname{re}c_h-\sum_{h=1}^{\tau}\operatorname{re}d_h\Big].$$

The proof is quite analogous and the required conditions are also (28) and (68); only for the estimation of the modulus of

$$\hat{\Phi}^{(2A)}(x)=(\eta x)^{-\hat{\delta}_{2A}-1}\exp(-tx)\,G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x\,\middle|\,\begin{matrix}c_1,\ldots,c_\sigma\\ d_1,\ldots,d_\tau\end{matrix}\right)(1/x)$$

we must apply, besides the asymptotic expansions quoted above, the well-known formulae

$$\begin{aligned}G^{\mu,\nu}_{\sigma,\tau}(\omega/x) &= k^* G^{\tau,0}_{\sigma,\tau}\big((\omega/x)e^{(\tau-\mu-\nu)\pi i}\big)+\sum_{h=0}^{\nu-1}\gamma^*_h G^{\tau,\nu-h}_{\sigma,\tau}\big((\omega/x)e^{(\tau-\mu-h-2)\pi i}\big)+\\ &\qquad+\sum_{h=1}^{\tau-\mu}\varkappa^*_h G^{\tau,\nu}_{\sigma,\tau}\big((\omega/x)e^{(\tau-\mu-2h)\pi i}\big),\\ G^{\mu,\nu}_{\sigma,\tau}(\omega/x) &= l^* G^{\tau,0}_{\sigma,\tau}\big((\omega/x)e^{(\mu+\nu-\tau)\pi i}\big)+\sum_{h=0}^{\nu-1}\delta^*_h G^{\tau,\nu-h}_{\sigma,\tau}\big((\omega/x)e^{(\mu+h+2-\tau)\pi i}\big)+\\ &\qquad+\sum_{h=1}^{\tau-\mu}\lambda^*_h G^{\tau,\nu}_{\sigma,\tau}\big((\omega/x)e^{(\mu+2h-\tau)\pi i}\big),\end{aligned}$$

$$G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x\,\middle|\,\begin{matrix}c_1,\dots,c_\sigma\\ d_1,\dots,d_\tau\end{matrix}\right)=-(1/2\pi i)\left\{e^{-\pi i d_{\mu+1}}G^{\mu+1,\nu}_{\sigma,\tau}\left((\omega/x)\,e^{\pi i}\,\middle|\,\begin{matrix}c_1,\dots,c_\sigma\\ d_1,\dots,d_\tau\end{matrix}\right)-\right.$$
$$\left.-e^{\pi i d_{\mu+1}}G^{\mu+1,\nu}_{\sigma,\tau}\left((\omega/x)\,e^{-\pi i}\,\middle|\,\begin{matrix}c_1,\dots,c_\sigma\\ d_1,\dots,d_\tau\end{matrix}\right)\right\},$$

$$G^{\mu,\nu}_{\sigma,\tau}\left(\omega/x\,\middle|\,\begin{matrix}c_1,\dots,c_\sigma\\ d_1,\dots,d_\tau\end{matrix}\right)=-(1/2\pi i)\left\{e^{-\pi i c_{\nu+1}}G^{\mu,\nu+1}_{\sigma,\tau}\left((\omega/x)\,e^{\pi i}\,\middle|\,\begin{matrix}c_1,\dots,c_\sigma\\ d_1,\dots,d_\tau\end{matrix}\right)-\right.$$
$$\left.-e^{\pi i c_{\nu+1}}G^{\mu,\nu+1}_{\sigma,\tau}\left((\omega/x)\,e^{-\pi i}\,\middle|\,\begin{matrix}c_1,\dots,c_\sigma\\ d_1,\dots,d_\tau\end{matrix}\right)\right\},$$

where k^*, γ_h^* $(h=0,\dots,\tau-1)$, $\varkappa_h^*$ $(h=1,\dots,\tau-\mu)$, l^*, δ_h^* $(h=0,\dots,\tau-1)$, λ_h^* $(h=1,\dots,\tau-\mu)$ are constants (see [4] I, formulae (39), (40), (41), (42), p. 90).

Assuming that all the conditions mentioned in the reasoning are fulfilled, we obtain from (70) and (43)

$$J=\sum_{r=0}^{\infty}\sum_{h=1}^{n}(1/r!)\,A_h(a_h)\,(\eta/t)^{a_h}\times$$
$$\times{}_{q+1}F_{p-1}\left(\begin{matrix}-r,\,1+a_h-b_1,\,\dots,\,1+a_h-b_q;\\ 1+a_h-a_1,\,\dots\,*\,\dots,\,1+a_h-a_p;\,(-1)^{q-m-n+1}\eta/t\end{matrix}\right)\times$$
$$\times\,G^{\mu+1,\nu}_{\sigma,\tau+1}\left(\omega t\,\middle|\,\begin{matrix}c_1,\dots,c_\sigma\\ a_h+r,\,d_1,\dots,d_\tau\end{matrix}\right),$$

i.e. formula (69), and thus the proof is ended.

THEOREM 2B. *Let m, n, p, q, μ, ν, σ, τ be integers, let t fulfil* (28) *and let one of the cases* (VI), (VII), (VIII), (IX) *take place. If for small x_0*

$$\int_0^{x_0}G_{2B}(x)\,dx\quad\textit{converges}\,,\tag{73}$$

then

$$G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}\left(\eta\omega\,\middle|\,\begin{matrix}b_1,\dots,b_m,\,c_1,\dots,c_\sigma,\,b_{m+1},\dots,b_q\\ a_1,\dots,a_n,\,d_1,\dots,d_\tau,\,a_{n+1},\dots,a_p\end{matrix}\right)\tag{74}$$
$$=\sum_{r=0}^{\infty}\sum_{h=1}^{\mu}(1/r!)\,D_h(d_h)\,(\omega/t)^{d_h}\times$$
$$\times{}_{\sigma+1}F_{\tau-1}\left(\begin{matrix}-r,\,1+d_h-c_1,\,\dots,\,1+d_h-c_\sigma;\\ 1+d_h-d_1,\,\dots\,*\,\dots,\,1+d_h-d_\tau;\,(-1)^{\sigma-\mu-\nu+1}\omega/t\end{matrix}\right)\times$$
$$\times\,G^{n+1,m}_{q,p+1}\left(\eta t\,\middle|\,\begin{matrix}b_1,\dots,b_q\\ d_h+r,\,a_1,\dots,a_p\end{matrix}\right),$$

*where the asterisk * denotes that the number* $1+d_h-d_h$ *is to be omitted in the sequence* $1+d_h-d_1, \ldots, 1+d_h-d_\tau$, *and in the case where some of the numbers* (52) *are natural, formulae* (74) *and* (66) *must be understood in the sense of Remark* 6. *The connection between the branches of* $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}$ *and* $G^{n+1,m}_{q,p+1}$ *is determined by Remark* 4.

As is easily seen, this theorem is equivalent to Theorem 2A.

Investigation of the integrals

$$\int\limits_0^\infty G^{m,n}_{p,q}(1/\eta x)\,G^{\mu,\nu}_{\sigma,\tau}(\omega/x)\,dx\,, \qquad \int\limits_0^\infty G^{n,m}_{q,p}(\eta/x)\,G^{\mu,\nu}_{\sigma,\tau}(\omega x)\,(1/x)\,dx$$

gives nothing new, which may be easily verified.

The author poses the problem of finding the sets of validity of (53), (59), (68) and (73).

§ 5. Particular cases. We shall confine ourselves to an analysis of the case $m=1$ for Theorem 1A, of the case $n=1$ for Theorem 2A, and of the case $\mu=1$ for Theorems 1B and 2B. The cases will be numbered so as to correspond to the theorems discussed; e.g. case B(VII) will correspond to case (VII) from Theorem 2B. Since the cases in Theorems 1A and 2A are numbered from (I) to (V), and in Theorems 1B and 2B from (VI) to (IX), such notation excludes ambiguity. Note moreover that since Theorems 2A and 2B are equivalent, we may consider Theorem 2B alone.

Case A(I). Here we have $n=\frac{1}{2}p+\frac{1}{2}q-\frac{1}{2}$, whence by an easy calculation $q=p+1$ and $n=p$. Then formula (54) takes the form

$$(75)\quad \Big[1\Big/\prod_{j=1}^{p}\Gamma(1-b_1+a_j)\Big]\,G^{\mu+p,\nu+1}_{\sigma+p+1,\tau+p}\left(\eta\omega\,\middle|\,\begin{matrix}b_1, c_1, \ldots, c_\sigma, b_2, \ldots, b_{p+1}\\ a_1, \ldots, a_p, d_1, \ldots, d_\tau\end{matrix}\right)$$

$$=\Big[1\Big/\prod_{j=2}^{p+1}\Gamma(1-b_1+b_j)\Big]\,(t/\eta)^{1-b_1}\times$$

$$\times\sum_{r=0}^{\infty}(1/r!)\,{}_{p+1}F_p\left(\begin{matrix}-r, 1-b_1+a_1, \ldots, 1-b_1+a_p;\\ 1-b_1+b_2, \ldots, 1-b_1+b_{p+1};\ t/\eta\end{matrix}\right)\times$$

$$\times\,G^{\mu,\nu+1}_{\sigma+1,\tau}\left(\omega t\,\middle|\,\begin{matrix}b_1-r, c_1, \ldots, c_\sigma\\ d_1, \ldots, d_\tau\end{matrix}\right),$$

whence for $b_1=d_\tau=1$, in view of (see [5], II, formula (39), p. 486)

$$G^{\mu+p,\nu+1}_{\sigma+p+1,\tau+p}\left(\eta\omega\,\middle|\,\begin{matrix}1, c_1, \ldots, c_\sigma, b_2, \ldots, b_{p+1}\\ a_1, \ldots, a_p, d_1, \ldots, d_{\tau-1}, 1\end{matrix}\right)$$

$$= G^{\mu+p,\nu}_{\sigma+p,\tau+p-1}\left(\eta\omega\,\middle|\,\begin{matrix}c_1, \ldots, c_\sigma, b_2, \ldots, b_{p+1}\\ a_1, \ldots, a_p, d_1, \ldots, d_{\tau-1}\end{matrix}\right)$$

$$(\mu\leqslant\tau-1,\ \nu\geqslant 0)\,,$$

we receive, after a suitable change of notations, formula (1) (6). In view of Remark 1 the system of assumptions (I), (28) and (53) must be completed by the condition $|\arg(\eta/t)| < \frac{1}{2}\pi$. Now, if we replace the condition $|\arg\omega| < (\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau)\pi$ by $|\arg(\omega t)| < (\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau)\pi$, then the conditions $|\arg\eta| < \frac{1}{2}\pi$ and $|\arg t| < \frac{1}{2}\pi$ may be omitted (7). Finally, as is easily seen, formula (75) is satisfied if

$$(76)\qquad \begin{gathered} \eta \neq 0\,, \quad \omega \neq 0\,, \quad t \neq 0\,, \\ |\arg(\eta/t)| < \tfrac{1}{2}\pi\,, \quad |\arg(\omega t)| < (\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi\,, \end{gathered}$$

$$(77)\qquad p \geqslant 0\,, \quad 0 < \mu \leqslant \tau\,, \quad 0 \leqslant \nu \leqslant \sigma < \tau-1\,, \quad \tfrac{1}{2}\sigma+\tfrac{1}{2}\tau < \mu+\nu$$

and if the conditions (11), (12), (13), (14), (53) with $m = 1$, $n = p$, $q = p+1$ are fulfilled. The result obtained corresponds to Theorem 4.1 from [5],IV (pp. 189-190) (8).

Case A(II)). Here we have two possibilities: either $n = \frac{1}{2}p+\frac{1}{2}q-\frac{1}{2}$ or $n = \frac{1}{2}p+\frac{1}{2}q-1$. In the first case we state that $n = p$, $q = p+1$ and thus nothing new is obtained: it is included in A(I). In the second case, however, we have $n = p$, $q = p+2$ and in consequence we find, as previously, that the conditions

$$(78)\qquad \begin{gathered} \eta \neq 0\,, \quad \omega \neq 0\,, \quad t \neq 0\,, \quad |\arg(\eta/t)| = 0\,, \\ |\arg(\omega t)| < (\mu+\nu-\tfrac{1}{2}\sigma-\tfrac{1}{2}\tau)\pi\,, \end{gathered}$$

$$(79)\qquad p \geqslant 0\,, \quad 0 < \mu \leqslant \tau\,, \quad 0 \leqslant \nu \leqslant \sigma < \tau-2\,, \quad \tfrac{1}{2}\sigma+\tfrac{1}{2}\tau < \mu+\tau\,,$$

$$(80)\qquad \sum_{h=1}^{p+2} \operatorname{re} b_h + \sum_{h=1}^{p} \operatorname{re} a_h + \tfrac{3}{2} < 2 \max_{j=1,\ldots,\nu} \operatorname{re} c_j$$

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(6) It may easily be verified that formula (75) is only apparently more general than (1). An analogous remark concerns also formulae (81), (90), (100) and (109) below.

(7) Putting it in a different way formula (75) is valid if $|\arg\eta| < \frac{1}{2}\pi$, $|\arg\omega| < (\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau)\pi$, $|\arg t| < \frac{1}{2}\pi$, $|\arg(\eta/t)| < \frac{1}{2}\pi$ and as well if $|\arg(\eta/t)| < \frac{1}{2}\pi$, $|\arg(\omega t)| < (\mu+\nu-\frac{1}{2}\sigma-\frac{1}{2}\tau)\pi$ with the restriction that in both cases all the other conditions given in the assumption of the theorem considered are fulfilled. A similar remark concerns also formulae (81), (90), (96), (100) and (109) below. Wishing to compare the results obtained here with C. S. Meijer's results the author will give in the sequel only the systems of assumptions analogous to the second one of those presented here; to obtain systems analogous to the first one gives no difficulty at all.

(8) From the above consideration it follows that formula (1) requires, in the notations of this paragraph, additionally $\mu \neq \tau$. Moreover, conditions (11) and (13) may be rejected in the case $b_1 = d_\tau = 1$ (but *not* in the general case) by analytic continuation of the series on the right-hand side of formula (75), as is easily seen on the basis on the completed definition of Meijer's functions (see e.g. § 1 of this paper). An analogous remark concerns also the next cases.

and (11), (12), (13), (14), (53), where $m=1$, $n=p$, $q=p+2$, must be fulfilled. If this takes place, one obtains

$$\text{(81)}\quad \Big[1\Big/\sum_{j=1}^{p}\Gamma(1-b_1+a_j)\Big]\,G^{\mu+p,\tau+1}_{\sigma+p+2,\nu+p}\left(\eta\omega\,\middle|\,\begin{matrix}b_1,\,c_1,\,\dots,\,c_\sigma,\,b_2,\,\dots,\,b_{p+2}\\ a_1,\,\dots,\,a_p,\,d_1,\,\dots,\,d_\tau\end{matrix}\right)$$
$$=\Big[1\Big/\sum_{j=1}^{p+2}\Gamma(1-b_1+b_j)\Big](t/\eta)^{1-b_1}\times$$
$$\times\sum_{r=0}^{\infty}(1/r!)\,{}_{p+1}F_{p+1}\left(\begin{matrix}-r,\,1-b_1+a_1,\,\dots,\,1-b_1+a_p;\\ 1-b_1+b_2,\,\dots,\,1-b_1+b_{p+2};\;t/\eta\end{matrix}\right)\times$$
$$\times\,G^{\mu,\nu+1}_{\sigma+1,\tau}\left(\omega t\,\middle|\,\begin{matrix}b_1-r,\,c_1,\,\dots,\,c_\sigma\\ d_1,\,\dots,\,d_\tau\end{matrix}\right),$$

whence for $b_1=d_\tau=1$, analogously to the previous case, we receive, after a suitable change of notations, formula (2). The result obtained corresponds to Theorem 7A.1 from [5], VII (pp. 84-85) for $q\neq p+1$ (according to our notations for $\tau-1\neq\sigma+1$). If one extends this result also to the case $q=p+1$ (i.e. $\tau=\sigma+2$), then, as is shown by C. S. Meijer in [5], all the given conditions remain unchanged. Hence it is easy to infer Theorem 6B.1 from [5], VII (p. 84) and thus formula (6).

Case A(III). In this case we receive $n=p$, $q=p+1$ and in consequence the conditions

$$\text{(82)}\quad \begin{gathered}\eta\neq 0\,,\quad \omega\neq 0\,,\quad t\neq 0\,,\quad |\arg(\eta/t)|<\tfrac12\pi\,,\\ |\arg(\omega t)|\leqslant(\mu+\nu-\tfrac12\sigma-\tfrac12\tau)\pi\,,\end{gathered}$$

$$\text{(83)}\quad p\geqslant 0\,,\quad 0<\mu\leqslant\tau\,,\quad 0<\mu\leqslant\sigma<\tau-1\,,\quad \tfrac12\sigma+\tfrac12\tau\leqslant\mu+\nu\,,$$

$$\text{(84)}\quad \sum_{h=1}^{\tau}\operatorname{re}d_h-\sum_{h=1}^{\sigma}\operatorname{re}c_h-\tfrac12(\tau-\sigma+1)<(\tau-\sigma)\min_{j=1,\dots,n}\operatorname{re}a_j$$

and (11), (12), (13), (14), (53), where $m=1$, $n=p$, $q=p+1$, by which one obtains formula (75). This corresponds to Theorem 4.2 from [5], IV (p. 190).

Case A(IV). Here we have, as in case A(II), two possibilities: either $n=\frac12p+\frac12q-\frac12$ or $n=\frac12p+\frac12q-1$. If $n=\frac12p+\frac12q-\frac12$, then $n=p$, $q=p+1$, i.e. nothing new is obtained: it is included in A(I). However, if $n=\frac12p+\frac12q-1$, then $n=p$, $q=p+2$ and the conditions

$$\text{(85)}\quad \begin{gathered}\eta\neq 0\,,\quad \omega\neq 0\,,\quad t\neq 0\,,\quad |\arg(\eta/t)|=0\,,\\ |\arg(\omega t)|\leqslant(\mu+\nu-\tfrac12\sigma-\tfrac12\tau)\pi\,,\end{gathered}$$

$$\text{(86)}\quad p\geqslant 0\,,\quad 0<\mu\leqslant\tau\,,\quad 0\leqslant\nu\leqslant\sigma<\tau-2\,,\quad \tfrac12\sigma+\tfrac12\tau\leqslant\mu+\nu\,,$$

$$\text{(87)}\quad \tfrac12+\sum_{h=1}^{p+2}\operatorname{re}b_h-\sum_{h=1}^{p}\operatorname{re}a_h>\frac{2}{\tau-\sigma}\Big(\tfrac12+\sum_{h=1}^{\tau}\operatorname{re}d_h-\sum_{h=1}^{\sigma}\operatorname{re}c_h\Big)$$

and (11), (12), (13), (14), (80), (84), (53), where $m=1$, $n=p$, $q=p+2$, are obtained; under these conditions formula (81) holds. The result obtained corresponds to Theorems 7A.2 (for $q\neq p+1$, i.e. $\tau-1\neq\sigma+1$ according to our notations) and 7A.3 from [5], VII (p. 85). If one extends this result also to the case $q=p+1$ (i.e. $\tau=\sigma+2$), then, as is shown by C. S. Meijer in [5], the constant $\frac{3}{2}$ in (80) must be replaced by $\frac{1}{2}$ and, moreover, one must assume $\frac{1}{2}\sigma+\frac{1}{2}\tau<\mu+\nu$; the remaining conditions are unchanged. Hence it is easy to infer Theorem 6B.2 from [5], VII (p. 84).

Case A(V). In this case we receive $p=0$, $q=1$ and in consequence the conditions

$$(88) \qquad \eta\neq 0\,,\quad \omega\neq 0\,,\quad t\neq 0\,,\quad |\arg(\eta/t)|<\tfrac{1}{2}\pi\,,$$

$$(89) \qquad p=0\,,\quad 0<\mu\leqslant\tau\,,\quad 0\leqslant\nu\leqslant\sigma<\tau-1$$

and (11), (14), (53), where $m=1$, $n=0$, $p=0$, $q=1$, by which one obtains formula (75). This corresponds to Theorem 4.3 from [5],IV (p. 190) (⁹).

Case B(I). In this case, as may easily be verified, the system of inequalities (9) and (10) is contradictory.

Case B(II). Here, as we verify analogously, the system of (16) and (10) is contradictory.

Case B(III). Here we have $\nu=\sigma$, whence $\tau=\sigma+2$, $q=p+1$ and consequently formula (60) takes the form

$$(90) \qquad \Big[1\Big/\prod_{j=1}^{\sigma}\Gamma(1+d_1-c_j)\Big]\times$$
$$\times\, G^{n+1,m+\sigma}_{p+\sigma+1,p+\sigma+2}\left(\eta\omega\,\middle|\,\begin{matrix}b_1,\ldots,b_m,c_1,\ldots,c_\sigma,b_{m+1},\ldots,b_{p+1}\\ a_1,\ldots,a_n,d_1,\ldots,d_{\sigma+2},a_{n+1},\ldots,a_p\end{matrix}\right)$$
$$=\Big[1\Big/\prod_{j=2}^{\sigma+2}\Gamma(1+d_1-d_j)\Big](1/\eta\omega)(\omega/t)^{1+d_1}\times$$
$$\times\sum_{r=0}^{\infty}(1/r!)\,{}_{\sigma+1}F_{\sigma+1}\left(\begin{matrix}-r,1+d_1-c_1,\ldots,1+d_1-c_\sigma;\\ 1+d_1-d_2,\ldots,1+d_1-d_{\sigma+2};\ \omega/t\end{matrix}\right)\times$$
$$\times\, G^{m,n+1}_{p+1,p+1}\left(1/\eta t\,\middle|\,\begin{matrix}-d_1-r,-a_1,\ldots,-a_p\\ -b_1,\ldots,-b_{p+1}\end{matrix}\right),$$

(⁹) It is easy to see that the condition $m+n-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2}\geqslant 0$ is superfluous in this theorem.

whence for $b_{p+1} = d_1 = 0$, analogously to the case A(I), but using moreover (62), after a suitable change of notations, we receive formula (3). Simultaneously we obtain the respective sufficient conditions

$$\text{(91)}\qquad \begin{gathered} \eta \neq 0\,, \quad \omega \neq 0\,, \quad t \neq 0\,, \\ |\arg(\eta t)| < (m+n-p-\tfrac{1}{2})\pi\,, \quad |\arg(\omega/t)| = 0\,, \end{gathered}$$

$$\text{(92)}\qquad 0 < m \leqslant p+1\,, \quad 0 \leqslant n \leqslant p\,, \quad p+\tfrac{1}{2} < m+n, \quad \sigma \geqslant 0\,,$$

$$\text{(93)}\qquad \sum_{h=1}^{\sigma+2} \operatorname{re} d_h - \sum_{h=1}^{\sigma} \operatorname{re} c_h - \tfrac{3}{2} < 2 \min_{j=1,\ldots,n} \operatorname{re} a_j$$

and (11), (12), (13), (14), (59), where $q = p+1$, $\mu = 1$, $\nu = \sigma$, $\tau = \sigma+2$. The above result corresponds to Theorem 7B from [5], VII (p. 85) for $p+\frac{1}{2} < m+n$, $|\arg w| < (m+n-p-\frac{1}{2})\pi$ (according to our notations for $p+\frac{1}{2} < m+n$, $|\arg(\eta t)| < (m+n-p-\frac{1}{2})\pi$).

Case B(IV). Similarly, as in the previous case, we have $\nu = \sigma$, whence $\tau = \sigma+2$, $q = p+1$ and then formula (90) is also valid under the conditions

$$\text{(94)}\qquad \begin{gathered} \eta \neq 0\,, \quad \omega \neq 0\,, \quad t \neq 0\,, \\ |\arg(\eta t)| \leqslant (m+n-p-\tfrac{1}{2})\pi\,, \quad |\arg(\omega/t)| = 0\,, \end{gathered}$$

$$\text{(95)}\qquad 0 < m \leqslant p+1\,, \quad 0 \leqslant n \leqslant p\,, \quad p+\tfrac{1}{2} \leqslant m+n\,, \quad \sigma \geqslant 0$$

and (11), (12), (13), (14), (17), (93), (22), (59), where $q = p+1$, $\mu = 1$, $\nu = \sigma$, $\tau = \sigma+2$ [10]. This corresponds to Theorem 7B from [5], VII (p. 85) in its complete form.

Case B(V). In this case we receive the formula

$$\text{(96)}\qquad \Big[1\Big/\prod_{j=1}^{\nu} \Gamma(1+d_1-c_j)\Big]\, G^{1,m+\nu}_{q+\sigma,p+\tau}\left(\eta\omega \,\middle|\, \begin{matrix} b_1, \ldots, b_m, c_1, \ldots, c_\sigma, b_{m+1}, \ldots, b_q \\ d_1, \ldots, d_\tau, a_1, \ldots, a_p \end{matrix}\right)$$

$$= \Big[1\Big/\prod_{j=\nu+1}^{\sigma} \Gamma(c_j-d_1) \prod_{j=2}^{\tau} \Gamma(1+d_1-d_j)\Big] (1/\eta\omega)(\omega/t)^{1+d_1} \times$$

$$\times \sum_{r=0}^{\infty} (1/r!)\, {}_{\sigma+1}F_{\tau-1}\left(\begin{matrix} -r, 1+d_1-c_1, \ldots, 1+d_1-c_\sigma; \\ 1+d_1-d_2, \ldots, 1+d_1-d_\tau; (-1)^{\sigma-\nu}\omega/t \end{matrix}\right) \times$$

$$\times\, G^{m,1}_{p+1,q}\left(1/\eta t \,\middle|\, \begin{matrix} -d_1-r, -a_1, \ldots, -a_p \\ -b_1, \ldots, -b_q \end{matrix}\right),$$

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[10] In the case $b_{p+1} = d_1 = 0$ conditions (11), (14), (17) and (22) may be rejected (comp. [8]).

which for $m = q$, in view of (37), may also be written in the form

$$(97)\quad \Big[\prod_{j=1}^{q} \Gamma(\beta_j)\Big/\prod_{j=1}^{p} \Gamma(a_j)\Big]\,{}_{q+\sigma}F_{p+\vartheta}\left(\begin{matrix}\beta_1, \ldots, \beta_q, \gamma_1, \ldots, \gamma_\sigma;\\ \delta_1, \ldots, \delta_\vartheta, a_1, \ldots, a_p; -v\eta\end{matrix}\right)$$

$$= \sum_{r=0}^{\infty} (1/r!)\,{}_{\sigma+1}F_\vartheta\left(\begin{matrix}-r, \gamma_1, \ldots, \gamma_\sigma;\\ \delta_1, \ldots, \delta_\vartheta; v/t\end{matrix}\right)\times$$

$$\times (\eta t)^r E(\beta_1+r, \ldots, \beta_q+r;\ a_1+r, \ldots, a_p+r;\ 1/\eta t)\,,$$

where $v = (-1)^{\sigma-\nu}\omega$, $\vartheta = \tau-1$, $a_j = 1+d_1-a_j$ $(j = 1, \ldots, p)$, $\beta_j = 1+d_1-b_j$ $(j = 1, \ldots, q)$, $\gamma_j = 1+d_1-c_j$ $(j = 1, \ldots, \sigma)$, $\delta_j = 1+d_1-d_{j+1}$ $(j = 1, \ldots, \vartheta)$ and E denotes MacRobert's function (for the definition, see e.g. [2], II, p. 433). Formula (96) is valid if

$$(98)\quad \begin{gathered}\eta \neq 0\,, \quad \omega \neq 0\,, \quad t \neq 0\,,\\ |\arg(\eta/t)| < (m-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\,, \quad |\arg\{(-1)^{\sigma-\nu}\omega t\}| < \tfrac{1}{2}\pi\,,\end{gathered}$$

$$(99)\quad \begin{gathered}n = 0\,, \quad 0 < m \leqslant q\,, \quad 0 \leqslant p < q < p+\tau-\sigma\,,\\ \tfrac{1}{2}p+\tfrac{1}{2}q < m\,, \quad 0 \leqslant \nu \leqslant \sigma\,,\end{gathered}$$

and if conditions (11), (14) and (59) are fulfilled. It seems to the author that conditions (11), (14) and (59) may be simplified or, on the other hand, weakened by analytic continuation; the problem requires a separate publication.

Case B (VI). In this case $\nu = \frac{1}{2}\sigma+\frac{1}{2}\tau-\frac{1}{2}$, whence $\nu = \sigma$, $\tau = \sigma+1$ and consequently formula (74) takes the form

$$(100)\quad \Big[1\Big/\prod_{j=1}^{\sigma} \Gamma(1+d_1-c_1)\Big]\times$$

$$\times G^{n+1,m+\sigma}_{q+\sigma,p+\sigma+1}\left(\eta\omega\,\middle|\,\begin{matrix}b_1, \ldots, b_m, c_1, \ldots, c_\sigma, b_{m+1}, \ldots, b_q\\ a_1, \ldots, a_n, d_1, \ldots, d_{\sigma+1}, a_{n+1}, \ldots, a_p\end{matrix}\right)$$

$$= \Big[1\Big/\prod_{j=2}^{\sigma+1} \Gamma(1+d_1-d_j)\Big](\omega/t)^{d_1}\times$$

$$\times \sum_{r=0}^{\infty} (1/r!)\,{}_{\sigma+1}F_\sigma\left(\begin{matrix}-r, 1+d_1-c_1, \ldots, 1+d_1-c_\sigma;\\ 1+d_1-d_2, \ldots, 1+d_1-d_{\sigma+1}; \omega/t\end{matrix}\right)\times$$

$$\times G^{n+1,m}_{q,p+1}\left(\eta t\,\middle|\,\begin{matrix}b_1, \ldots, b_q\\ d_1+r, a_1, \ldots, a_p\end{matrix}\right),$$

whence for $b_q = d_1 = 0$, in view of (see [5], II, formulae (40) and (43), p. 486)

$$G^{n+1,m+\sigma}_{q+\sigma,p+\sigma+1}\left(\eta\omega\middle|\begin{matrix} b_1, \dots, b_m, c_1, \dots, c_\sigma, b_{m+1}, \dots, b_{q-1}, 0 \\ a_1, \dots, a_n, 0, d_2, \dots, d_{\sigma+1}, a_{n+1}, \dots, a_p\end{matrix}\right)$$
$$= G^{n,m+\sigma}_{q+\sigma-1,p+\sigma}\left(\eta\omega\middle|\begin{matrix} b_1, \dots, b_m, c_1, \dots, c_\sigma, b_{m+1}, \dots, b_{q-1} \\ a_1, \dots a_n, d_2, \dots, d_{\sigma+1}, a_{n+1}, \dots, a_p\end{matrix}\right)$$
$$(m \leqslant q-1,\ n \geqslant 0)$$

and

$$G^{n+1,m}_{q,p+1}\left(\eta t\middle|\begin{matrix} b_1, \dots, b_{q-1}, 0 \\ r, a_1, \dots, a_p\end{matrix}\right) = (-1)^r G^{n,m+1}_{q,p+1}\left(\eta t\middle|\begin{matrix} 0, b_1, \dots, b_{q-1} \\ a_1, \dots, a_p, r\end{matrix}\right)$$
$$(0 \leqslant m \leqslant q-1 \leqslant p, \quad 0 \leqslant n \leqslant p, \quad r = 0, 1, \dots),$$

we receive, after a suitable change of notations, formula (4), which is valid under the conditions

$$\text{(101)} \qquad \begin{gathered} \eta \neq 0, \quad \omega \neq 0, \quad t \neq 0, \\ |\arg(\eta t)| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi, \quad |\arg(\omega/t)| < \tfrac{1}{2}\pi, \end{gathered}$$

$$\text{(102)} \quad 0 < n \leqslant p, \quad 0 \leqslant m \leqslant q < p, \quad \tfrac{1}{2}p + \tfrac{1}{2}q < m+n, \quad \sigma \geqslant 0$$

and (11), (12), (13), (14), (73), where $\mu = 1$, $\nu = \sigma$, $\tau = \sigma+1$. The result obtained corresponds to Theorem 3.1 from [5], III (p. 43) for $m > 0$, $p < q-1$ and $|\arg w| < (m+n-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2})\pi$ (according to our notations for $n > 0$, $q < p$ and $|\arg(\eta t)| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$). Formula (100), as can be verified after C. S. Meijer in [5], is also valid for $n = 0$ and for $q = p$, provided that

$$\text{(103)} \quad b_j \neq 1, 2, \dots \quad (j = 1, \dots, n), \quad \operatorname{re} c_j < 1 \quad (j = 1, \dots, \sigma),$$

and thus the system of inequalities (102) may be replaced by

$$\text{(104)} \quad 0 \leqslant n \leqslant p, \quad 0 \leqslant m \leqslant q \leqslant p, \quad \tfrac{1}{2}p + \tfrac{1}{2}q < m+n, \quad \sigma \geqslant 0.$$

Remark 10. To shorten the text, in the following cases the author will include among the conditions obtained the cases $n = 0$, $q = p$ with (103), though it does not spring from the above considerations.

Case B(VII). Here we have, as previously, $\nu = \frac{1}{2}\sigma + \frac{1}{2}\tau - \frac{1}{2}$, whence $\nu = \sigma$, $\tau = \sigma+1$ and then formula (100) is also valid under the conditions

$$\text{(105)} \qquad \begin{gathered} \eta \neq 0, \quad \omega \neq 0, \quad t \neq 0, \\ |\arg(\eta t)| \leqslant (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi, \quad |\arg(\omega/t)| < \tfrac{1}{2}\pi, \end{gathered}$$

$$\text{(106)} \quad p > 0, \quad 0 \leqslant n \leqslant p, \quad 0 \leqslant m \leqslant q \leqslant p, \quad \tfrac{1}{2}p + \tfrac{1}{2}q \leqslant m+n, \quad \sigma \geqslant 0$$

and (11), (12), (13), (14), (35), (73), where $\mu = 1$, $\nu = \sigma$, $\tau = \sigma+1$ (cf. Remark 10). If (103) is valid, we can get rid of condition (35) with $\mu = 1$

by analytic continuation. Then, in a similar way, we can prove that the inequality $|\arg(\eta t)| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ is superfluous and then the formula (100) holds for any values of $\arg(\eta t)$ (11). This corresponds to Theorem 3.1 from [5], III (p. 43) in its complete form.

Case B(VIII). Here we have two possibilities: either $\nu = \frac{1}{2}\sigma+\frac{1}{2}\tau-\frac{1}{2}$ or $\nu = \frac{1}{2}\sigma+\frac{1}{2}\tau-1$. In the first case we state that $\nu = \sigma$, $\tau = \sigma+1$, and thus nothing new is obtained: it is included in B(VII). In the second case, however, we have $\nu = \sigma$, $\tau = \sigma+2$ and in consequence we find that the conditions

$$\text{(107)}\quad \eta \neq 0,\ \omega \neq 0,\ t \neq 0,\quad |\arg(\eta t)| < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,\quad |\arg(\omega/t)| = 0,$$

$$\text{(108)}\quad p > 0,\quad 0 \leqslant n \leqslant p,\quad 0 \leqslant m \leqslant q \leqslant p,\quad \tfrac{1}{2}p+\tfrac{1}{2}q < m+n,\ \sigma \geqslant 0$$

and (11), (12), (13), (14), (36), (73), where $\mu = 1$, $\nu = \sigma$, $\tau = \sigma+2$, must be fulfilled (cf. Remark 10); if (103) is valid, we can get rid of condition (36) with $\tau = \sigma+2$ by analytic continuation. If the above conditions are fulfilled, one obtains the formula

$$\text{(109)}\quad \Big[1\Big/\prod_{j=1}^{\sigma}\Gamma(1+d_1-c_j)\Big]\times$$
$$\times G_{q+\sigma,p+\sigma+2}^{n+1,m+\sigma}\left(\eta\omega\,\middle|\,\begin{matrix} b_1,\dots,b_m,c_1,\dots,c_\sigma,b_{m+1},\dots,b_q \\ a_1,\dots,a_n,d_1,\dots,d_{\sigma+2},a_{n+1},\dots,a_p\end{matrix}\right)$$
$$=\Big[1\Big/\prod_{j=1}^{\sigma+2}\Gamma(1+d_1-d_j)\Big](\omega/t)^{d_1}\times$$
$$\times\sum_{r=0}^{\infty}(1/r!)\,{}_{\sigma+1}F_{\sigma+1}\left(\begin{matrix}-r,1+d_1-c_1,\dots,1+d_1-c_\sigma;\\ 1+d_1-d_2,\dots,1+d_1-d_{\sigma+2};\ \omega/t\end{matrix}\right)\times$$
$$\times G_{q,p+1}^{n+1,m}\left(\eta t\,\middle|\,\begin{matrix}b_1,\dots,b_q\\ d_1+r,a_1,\dots,a_p\end{matrix}\right),$$

whence for $b_q = d_1 = 0$, analogously to the case B(VI), we receive, after a suitable change of notations, formula (5). The result obtained corresponds to Theorem 6A.1 from [5], VII (p. 83) for $|\arg w| < (m+n-\frac{1}{2}p-\frac{1}{2}q-\frac{1}{2})\pi$ (according to our notations for $|\arg(\eta t)| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$).

Case B(IX). Here we have, as in case B(VIII), two possibilities: either $\nu = \frac{1}{2}\sigma+\frac{1}{2}\tau-\frac{1}{2}$ or $\nu = \frac{1}{2}\sigma+\frac{1}{2}\tau-1$. If $\nu = \frac{1}{2}\sigma+\frac{1}{2}\tau-\frac{1}{2}$, then $\nu = \sigma$, $\tau = \sigma+1$, i.e. nothing new is obtained: it is included in B(VII). However,

(11) The condition $\frac{1}{2}p+\frac{1}{2}q < m+n$ is evidently also superfluous. It seems that these results may be generalized also to the case where (103) is not valid and for $\mu > 1$.

if $\nu = \frac{1}{2}\sigma+\frac{1}{2}\tau-1$, then $\nu = \sigma$, $\tau = \sigma+2$ and we obtain the conditions

$$\begin{gathered}\eta \neq 0\,, \quad \omega \neq 0\,, \quad t \neq 0\,, \\ |\arg(\eta t)| \leqslant (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi\,, \quad |\arg(\omega/t)| = 0\,,\end{gathered} \tag{110}$$

$$p > 0\,, \quad 0 \leqslant n \leqslant p\,, \quad 0 \leqslant m \leqslant q \leqslant p\,, \quad \tfrac{1}{2}p+\tfrac{1}{2}q \leqslant m+n\,, \quad \sigma \geqslant 0 \tag{111}$$

and (11), (12), (13), (14), (35), (36), (73), where $\mu = 1$, $\nu = \sigma$, $\tau = \sigma+2$ (cf. Remark 10), by which formula (109) holds. If (103) is valid, we can get rid of conditions (35) with $\mu = 1$ and (36) with $\tau = \sigma+2$ by analytic continuation. Then, in a similar way, we can prove that the inequality $|\arg(\eta t)| \leqslant (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ is superfluous and then formula (109) holds for any values of $\arg(\eta t)$ (11). This corresponds to Theorem 6A.1 from [5], VII (p. 83) in its complete form.

Other cases of the quoted theorems of the papers [5], which have not been mentioned here, do not immediately result from Theorems 1 and 2 of the present paper.

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