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On differentiable solutions of a functional equation

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Dedicated to Professor Stanisław Gołąb to celebrate his 60-th birthday

In the present paper I deal with the problem of the existence of solutions $\varphi(x)$ of the functional equation

$$\varphi[f(x)] = G\big(x, \varphi(x)\big)\,, \tag{1}$$

which are of class C^r in an open interval (a, b). $f(x)$ and $G(x, y)$ denote here known, real-valued functions of real variables.

This problem has been solved for $r = 0$ by J. Kordylewski and M. Kuczma in paper [3]. The authors have proved that there exist an infinite number of solutions of equation (1) that are continuous in the interval (a, b).

I shall prove in § 2 that under suitable assumptions equation (1) possesses also infinitely many solutions of class C^r in the interval (a, b), where r may be infinite.

U. T. Bödewadt has considered in [1] the Abel equation

$$\varphi[f(x)] = \varphi(x) + 1$$

and has proved, the assumptions on the function $f(x)$ being similar to those formulated in (I) below, that there exist infinitely many solutions of the Abel equation of class C^r ($r \leqslant \infty$).

§ 1. Definition. We shall denote by $C^r[E]$ a class of functions defined and of class C^r in a set E.

We suppose (cf. [5]) that

(I) The function $f(x)$ belongs to class $C^r[(a, b)]$, $1 \leqslant r \leqslant \infty$, and fulfils the following conditions:

$$\lim_{x\to a+0} f(x) = a\,, \quad \lim_{x\to b-0} f(x) = b\,, \quad f(x) > x \quad \text{for} \quad x \in (a, b)\,,$$

$$f'(x) > 0 \quad \text{for} \quad x \in (a, b)\,. \tag{2}$$

(The values a and b may be infinite.)

(II) The function $G(x,y)$ belongs to class $C^r[\Omega]$ in an open region Ω, normal with respect to the x-axis. Moreover, the inequality

$$\frac{\partial G(x,y)}{\partial y} \neq 0 \tag{3}$$

holds for $(x,y) \in \Omega$.

Let us denote by Ω_x the x-section of the region Ω, i.e.

$$\Omega_x \overset{\mathrm{df}}{=} \{y;\ (x,y) \in \Omega\},$$

and by Γ_x the set of values assumed by the function $G(x,y)$ for $(x,y) \in \{x\} \times \Omega_x$, i.e.

$$\Gamma_x \overset{\mathrm{df}}{=} \Big\{z;\ \sum_y [y \in \Omega_x,\ z = G(x,y)]\Big\}.$$

We suppose that

(III) $\Omega_x \neq 0$, $\Gamma_x = \Omega_{f(x)}$ for $x \in (a,b)$.

Inequality (3) guarantees the existence of the function $H(x,z)$ inverse to the function $G(x,y)$ with respect to the variable y, i.e. we have

$$z = G(x,y) \quad \equiv \quad y = H(x,z). \tag{4}$$

The function $H(x,z) \in C^r[\Omega']$, where

$$\Omega' \overset{\mathrm{df}}{=} \{(x,z);\ x \in (a,b),\ z \in \Gamma_x\}.$$

Let us introduce the following notation:

$$\begin{aligned} G_1(s,\varphi,\varphi') &\overset{\mathrm{df}}{=} \frac{1}{f'(s)}\Big(G'_x\big(s,\varphi(s)\big) + G'_y\big(s,\varphi(s)\big)\varphi'(s)\Big), \\ G_{k+1}(s,\varphi,\ldots,\varphi^{(k+1)}) &\overset{\mathrm{df}}{=} \frac{1}{f'(s)} \cdot \frac{d}{ds} G_k(s,\varphi,\ldots,\varphi^{(k)}), \\ & \qquad\qquad k = 1,\ldots,r-1. \end{aligned} \tag{5}$$

Finally, we note the following

LEMMA. *Suppose that the sequence $\{a_n\}_0^\infty$ is strictly increasing, $g_n(x) \in C^r[\langle a_n, a_{n+1})]$, $n \geqslant 0$, $0 \leqslant r \leqslant \infty$, and that we have*

$$\lim_{x \to a_{n+1}-0} g_n^{(k)}(x) = g_{n+1}^{(k)}(a_{n+1}), \qquad k = 0,1,\ldots,r.$$

Then the function

$$g(x) \overset{\mathrm{df}}{=} g_n(x) \quad \text{for} \quad x \in \langle a_n, a_{n+1})$$

belongs to class $C^r[\bigcup_{n=0}^{\infty}\langle a_n, a_{n+1})]$.

We omit the simple inductive proof of this lemma.

§ 2. Let us take an arbitrary $x_0 \in (a, b)$. Denoting by $f^{-1}(x)$ the function inverse to the function $f(x)$, let us put

$$(6) \qquad \begin{array}{lll} x_1 = f(x_0)\,, & x_{n+1} = f(x_n)\,, & n > 0\,, \\ x_{-1} = f^{-1}(x_0)\,, & x_{-n-1} = f^{-1}(x_{-n})\,, & n > 0\,. \end{array}$$

The sequence $\{x_n\}$ is strictly increasing and converges to b, and the sequence $\{x_{-n}\}$ is strictly decreasing and converges to a.

We shall prove the following

THEOREM 1. *Suppose that hypotheses* (I)-(III) *are fulfilled and a function* $\psi(x)$ *ix defined in the interval* $\langle x_0, x_1\rangle$, $\psi(x) \in C^r[\langle x_0, x_1\rangle]$ $(1 \leqslant r \leqslant \infty)$, *and fulfils the following conditions*:

$$(7) \qquad \psi(x) \in \Omega_x \quad \text{for} \quad x \in \langle x_0, x_1\rangle\,,$$

$$(8) \qquad \psi(x_1) = G\big(x_0, \psi(x_0)\big)\,,$$

$$(9) \qquad \psi^{(k)}(x_1) = G_k(x_0, \psi, \psi', \ldots, \psi^{(k)})\,, \quad k = 1, \ldots, r\,.$$

Then there exists exactly one solution $\varphi(x) \in C^r[(a, b)]$ *of equation* (1) *which is an extension of the function* ψ. *This solution is given by the formulae*

$$(10) \qquad \varphi(x) = \begin{cases} \varphi_n(x) & \text{for} \quad x \in \langle x_n, x_{n+1})\,, \quad n \geqslant 0\,, \\ \varphi_{-n}(x) & \text{for} \quad x \in \langle x_{-n}, x_{-n+1})\,, \quad n \geqslant 1\,, \end{cases}$$

where the functions $\varphi_n(x)$ *and* $\varphi_{-n}(x)$ *are defined in the intervals* $\langle x_n, x_{n+1})$ *and* $\langle x_{-n}, x_{-n+1})$ *respectively* (*cf.* (6)) *by the formulae*

$$(11) \qquad \varphi_0(x) = \psi(x)\,, \quad \varphi_{n+1}(x) = G\big(f^{-1}(x), \varphi_n[f^{-1}(x)]\big)\,, \quad n \geqslant 1\,,$$

$$(12) \qquad \varphi_{-n}(x) = H\big(x, \varphi_{-n+1}[f(x)]\big)\,, \quad n \geqslant 1\,,$$

and the function H *is defined by* (4).

Proof [1]. From a theorem proved in [3] it follows in particular that the function $\varphi(x)$ defined by formulae (10)-(12) is a continuous solution of equation (1) for $x \in (a, b)$ already when the function $\psi(x)$ is continuous in the interval $\langle x_0, x_1\rangle$ and fulfils conditions (7) and (8). Consequently it remains to prove that, if the function $\psi(x)$ fulfils the r of conditions (9), then formulae (10)-(12) define the function of class C^r in the whole interval (a, b).

At first we shall prove that

$$(*) \qquad \varphi_n(x) \in C^r[\langle x_n, x_{n+1})]\,, \quad n \geqslant 0\,, \quad 0 \leqslant r \leqslant \infty\,,$$

$$(**) \qquad \lim_{x \to x_{n+1}-0} \varphi_n^{(k)}(x) = \varphi_{n+1}^{(k)}(x_{n+1})\,, \quad n \geqslant 0\,, \quad k = 0, 1, \ldots, r\,.$$

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[1] I should like to express my thanks to Dr St. Balcerzyk for his valuable remarks concerning the proof of this theorem.

We shall prove assertion (*) by induction. Let $n = 0$. Then, according to (11), $\varphi_0(x) = \psi(x)$ for $x \in \langle x_0, x_1)$ and (*) holds as a consequence of the assumptions regarding the function $\psi(x)$. Further, if $\varphi_m(x) \in C^r[\langle x_m, x_{m+1})]$, $m \geqslant 0$, then by (11) we have

$$\varphi_{m+1}(x) = G\big(f^{-1}(x), \varphi_m[f^{-1}(x)]\big) , \qquad x \in \langle x_{m+1}, x_{m+2}) .$$

But now $f^{-1}(x) \in \langle x_m, x_{m+1})$ and $\varphi_m[f^{-1}(x)] \in C^r[\langle x_{m+1}, x_{m+2})]$, and consequently, on account of hypotheses (I) and (II) we have $\varphi_{m+1}(x) \in C^r[\langle x_{m+1}, x_{m+2})]$.

Equalities (**) follow for $k = 0$ from the fact that the function $\varphi(x)$ defined by formulae (10)-(12) is continuous in (a, b). Let us fix a number k, $1 \leqslant k \leqslant r$. Let us note that from definitions (5) we have

$$\text{(13)} \qquad \frac{d^k}{dx^k} G\big(f^{-1}(x), \varphi[f^{-1}(x)]\big) = G_k(s, \varphi, \ldots, \varphi^{(k)}) , \qquad s = f^{-1}(x) .$$

Since we have $\varphi_{n+1}(x) \in C^r[\langle x_{n+1}, x_{n+2})]$, the above formula yields for every natural n (cf. (11))

$$\varphi_{n+1}^{(k)}(x) = G(s, \varphi_n, \ldots, \varphi_n^{(k)}) , \qquad s = f^{-1}(x) .$$

In particular, we have for $x = x_{n+1}$

$$\text{(14)} \qquad \varphi_{n+1}^{(k)}(x_{n+1}) = G_k(x_n, \varphi_n, \ldots, \varphi_n^{(k)}) .$$

It follows from (9), (11) and (14) that (**) holds for $n = 0$. We now assume that

$$\text{(15)} \qquad \lim_{x \to x_{m+1}-0} \varphi_m^{(l)}(x) = \varphi_{m+1}^{(l)}(x_{m+1}) , \qquad m \geqslant 0 , \qquad l = 0, 1, \ldots, k .$$

We have by (11) and (13)

$$\lim_{x \to x_{m+2}-0} \varphi_{m+1}^{(k)}(x) = \lim_{x \to x_{m+2}-0} \frac{d^k}{dx^k} G\big(f^{-1}(x), \varphi_m[f^{-1}(x)]\big)$$
$$= \lim_{s \to x_{m+1}-0} G_k(s, \varphi_m, \ldots, \varphi_m^{(k)}) .$$

Finally, we get on account of (15) and (14) (for $n = m+1$)

$$\lim_{x \to x_{m+2}-0} \varphi_{m+1}^{(k)}(x) = \varphi_{m+2}^{(k)}(x_{m+2}) , \qquad k = 1, \ldots, r .$$

Assertions (*) and (**) are valid for every n. Now let us put in lemma $a_n = x_n$, $g_n(x) = \varphi_n(x)$, $n \geqslant 0$. We infer that the function

$$\text{(16)} \qquad \varphi^*(x) \stackrel{\text{df}}{=} \varphi_n(x) \qquad \text{for} \qquad x \in \langle x_n, x_{n+1}) , \qquad n \geqslant 0$$

belongs to class $C^r[\langle x_0, b)]$, since $\bigcup\limits_{n=0}^{\infty} \langle x_n, x_{n+1}) = \langle x_0, b)$ (cf. (6)).

Now we define the functions

$$\bar{\varphi}_0(x) = \varphi^*(x) \quad \text{for} \quad x \in \langle x_0, b),$$
$$\bar{\varphi}_{-n-1}(x) = H\big(x, \bar{\varphi}_{-n}[f(x)]\big) \quad \text{for} \quad x \in \langle x_{-n-1}, b).$$

We have $\bar{\varphi}_{-n-1}(x) = \bar{\varphi}_{-n}(x)$ for $x \in \langle x_{-n}, b)$, since $\varphi_{-n}(x)$ fulfil equation (1) in $\langle x_{-n}, b)$. If $\bar{\varphi}_{-n}(x) \in C^r[\langle x_{-n}, b)]$ then $\bar{\varphi}_{-n-1}(x) \in C^r[\langle x_{-n-1}, b)]$, because for $x \in \langle x_{-n-1}, b)$, $\bar{\varphi}_{-n}[f(x)] \in C^r[\langle x_{-n}, b)]$. Consequently

$$\bar{\varphi}(x) \stackrel{\mathrm{df}}{=} \bar{\varphi}_{-n}(x) \quad \text{for} \quad x \in \langle x_{-n}, b), \quad n \geqslant 0$$

is a function of class $C^r[(a, b)]$. But we have

$$\bar{\varphi}_{-n}(x) = \begin{cases} \varphi^*(x) & \text{for} \quad x \in \langle x_0, b), \\ \varphi_{-n}(x) & \text{for} \quad x \in \langle x_{-n}, x_{-n+1}), \quad n \geqslant 1 \end{cases}$$

whence, according to (16), we see that

$$\bar{\varphi}(x) = \varphi(x) \quad \text{for} \quad x \in (a, b).$$

This completes the proof of the theorem.

THEOREM 2. *Suppose that* (I)-(III) *are fulfilled. Then equation* (1) *possesses an infinite number of solutions of class* $C^r[(a, b)]$ $(1 \leqslant r \leqslant \infty)$. *These solutions are given by formulae* (10)-(12), *where* $\psi(x) \in C^r[\langle x_0, x_1\rangle]$ *is an arbitrary function which fulfils conditions* (7)-(9).

Proof. For $r < \infty$ this theorem is an immediate consequence of theorem 1. For $r = \infty$ the question arises whether one can find a function $\psi(x) \in C^\infty[\langle x_0, x_1\rangle]$ which will satisfy an infinite number of conditions (9).

Let us take an arbitrary $\bar{x} \in (x_0, x_1)$. Let $a(x)$ be an arbitrary function of class $C^\infty[\langle x_0, \bar{x}\rangle]$ which fulfils the condition

$$a(x) \in \Omega_x \quad \text{for} \quad x \in \langle x_0, \bar{x}\rangle.$$

Then the function $G\big(f^{-1}(x), a[f^{-1}(x)]\big)$ belongs to class $C^\infty[\langle x_1, f(\bar{x})\rangle]$.

It is known (cf. [6]) that there exists a function $\psi(x) \in C^\infty[\langle x_0, f(\bar{x})\rangle]$ such that

$$\psi(x) = \begin{cases} a(x) & \text{for} \quad x \in \langle x_0, \bar{x}\rangle, \\ G\big(f^{-1}(x), a[f^{-1}(x)]\big) & \text{for} \quad x \in \langle x_1, f(\bar{x})\rangle, \end{cases}$$

and

$$\psi(x) \in \Omega_x \quad \text{for} \quad x \in \langle x_0, f(\bar{x})\rangle.$$

Of course, this function fulfils an infinite number of conditions (9).

The function $a(x)$ may be chosen in infinitely many ways, and thus we obtain an infinite number of functions $\psi(x) \in C^\infty[\langle x_0, x_1\rangle]$ (since we have $\langle x_0, x_1\rangle \subset \langle x_0, f(\bar{x})\rangle$) which fulfil conditions (9) for $r = 1, 2, \ldots$

From theorem 1 we infer that there exist infinitely many solutions $\varphi(x)$ of equation (1) given by formulae (10)-(12), belonging to class $C^\infty[(a, b)]$. This completes the proof.

Remark. Theorems analogous to theorem 2 may be proved for more general equations

$$F\big(x, \varphi(x), \varphi[f(x)], \ldots, \varphi[f^n(x)]\big) = 0\,, \tag{17}$$

$$F\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) = 0\,, \tag{18}$$

which have been considered in [2] and [4]. In these equations $\varphi(x)$ denotes the required function, and the remaining ones are known.

Namely if the assumptions stated in [2] (resp. [4]) are fulfilled, the known functions belong to class C^r in suitable sets, and the function $f(x)$ (resp. $f_n(x)$) in equation (17) (resp. (18)) fulfils condition (2), then equation (17) (resp. (18)) possesses an infinite number of solutions of class C $(1 \leqslant r \leqslant \infty)$ in the interval (a, b).

References

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