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Strong maximum and minimum principles for parabolic functional-differential problems with initial inequalities $u(t_0, x) \underset{(\geqslant)}{\leqslant} K$

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Abstract. The aim of the paper is to give strong maximum and minimum principles for parabolic functional-differential problems with initial inequalities in relatively arbitrary $(n+1)$-dimensional time-space sets more general than the cylindrical domain.

1. Introduction. In this paper we consider diagonal systems of non-linear parabolic functional-differential inequalities of the form

$$u_t^i(t, x) \underset{(\geqslant)}{\leqslant} f^i(t, x, u(t, x), u_x^i(t, x), u_{xx}^i(t, x), u) \quad (i = 1, \ldots, m) \tag{1.1}$$

for $(t, x) = (t, x_1, \ldots, x_n) \in D$, where $D \subset (t_0, t_0+T] \times \boldsymbol{R}^n$ is one of three relatively arbitrary sets more general than the cylindrical domain $(t_0, t_0+T] \times D_0 \subset \boldsymbol{R}^{n+1}$ The symbol u denotes the mapping

$$u\colon \tilde{D} \ni (t,x) \to u(t, x) = \big(u^1(t, x), \ldots, u^m(t, x)\big) \in \boldsymbol{R}^m,$$

where $\tilde{D}$ is an arbitrary set contained in $(-\infty, t_0+T] \times \boldsymbol{R}^n$ such that $\bar{D} \subset \tilde{D}$. The right-hand sides f^i $(i = 1, \ldots, m)$ of systems (1.1) are functionals of u; $u_x^i(t, x) = \operatorname{grad}_x u^i(t, x)$ $(i = 1, \ldots, m)$ and $u_{xx}^i(t, x)$ $(i = 1, \ldots, m)$ denote the matrices of second order derivatives with respect to x of $u^i(t, x)$ $(i = 1, \ldots, m)$. We give two theorems on strong maximum and minimum principles for problems with inequalities (1.1) and with the initial inequalities

$$u(t_0, x) \underset{(\geqslant)}{\leqslant} K \quad \text{for } x \in S_{t_0},$$

respectively, where $K = (K^1, \ldots, K^m)$ is a constant function and

$$S_{t_0} := \operatorname{int}\{x \in \boldsymbol{R}^n\colon (t_0, x) \in \bar{D}\}.$$

The results obtained are a generalization of those given by Redheffer and Walter [3], by Szarski [4] and [5], by Besala [1], by Walter [7] and by the author [2] and base on those results. To prove the results of this paper we use the theorem on strong maximum principle from [2].

2. Preliminaries. The notation and definitions given in this section are valid throughout the paper. Some of them are similar to those applied by Szarski ([6] and [5]), Redheffer and by Walter [3], Besala [1] and by the author [2].

We use the following notation: $\boldsymbol{R} = (-\infty, \infty)$, $\boldsymbol{N} = \{1, 2, \ldots\}$, $x = (x_1, \ldots, x_n)$ $(n \in \boldsymbol{N})$.

For any vectors $z = (z^1, \ldots, z^m) \in \boldsymbol{R}^m$, $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^m) \in \boldsymbol{R}^m$ we write

$$z \leqslant \tilde{z} \quad \text{if} \quad z^i \leqslant \tilde{z}^i \quad (i = 1, \ldots, m).$$

Let t_0 be a real finite number and let $0 < T < \infty$. A set $D \subset \{(t, x)\colon t > t_0,\ x \in \boldsymbol{R}^n\}$ (bounded or unbounded) is called a *set of type* (P) if:

(a) The projection of the interior of D on the t-axis is the interval $(t_0, t_0 + T)$.

(b) For every $(\tilde{t}, \tilde{x}) \in D$ there is a positive r such that

$$\{(t, x)\colon (t-\tilde{t})^2 + \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 < r,\ t < \tilde{t}\} \subset D.$$

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t = \begin{cases} \operatorname{int}\{x \in \boldsymbol{R}^n\colon (t_0, x) \in \bar{D}\} & \text{for } t = t_0, \\ \{x \in \boldsymbol{R}^n\colon (t, x) \in D\} & \text{for } t \neq t_0, \end{cases}$$

$$\sigma_t = \begin{cases} \operatorname{int}[\bar{D} \cap (\{t_0\} \times \boldsymbol{R}^n)] & \text{for } t = t_0, \\ D \cap (\{t\} \times \boldsymbol{R}^n) & \text{for } t \neq t_0. \end{cases}$$

It is obvious that S_t and σ_t are open sets in $\boldsymbol{R}^n$ and $\boldsymbol{R}^{n+1}$, respectively.

Let $\tilde{D}$ be a set contained in $(-\infty, t_0 + T] \times \boldsymbol{R}^n$ such that $\bar{D} \subset \tilde{D}$. We introduce the following sets:

$$\partial_p D := \tilde{D} \backslash D \quad \text{and} \quad \Gamma := \partial_p D \backslash \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{t}, \tilde{x}) \in D$ we denote by $S^-(\tilde{t}, \tilde{x})$ the set of points $(t, x) \in D$ that can be joined to $(\tilde{t}, \tilde{x})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (t, x) to $(\tilde{t}, \tilde{x})$.

Let $Z_m(\tilde{D})$ denote the space of mappings

$$w\colon \tilde{D} \ni (t, x) \to w(t, x) = \big(w^1(t, x), \ldots, w^m(t, x)\big) \in \boldsymbol{R}^m$$

continuous in $\tilde{D}$.

In the set of mappings bounded from above in $\tilde{D}$ and belonging to $Z_m(\tilde{D})$ we define the functional

$$[w]_t = \max_{i=1,\ldots,m} \sup\{0,\ w^i(\tilde{t}, x)\colon (\tilde{t}, x) \in \tilde{D},\ \tilde{t} \leqslant t\}, \quad \text{where } t \leqslant t_0 + T$$

By X we denote a fixed subset (not necessarily a linear subspace) of $Z_m(\tilde{D})$ and by $M_{n \times n}(\boldsymbol{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n \times n}$.

A mapping $u \in X$ is called *regular* in D if $\dot{u}_t^i$, $u_x^i = \mathrm{grad}_x u^i$, $u_{xx}^i = [u^i_{x_j x_k}]_{n\times n}$ $(i = 1, \ldots, m)$ are continuous in D.

Let the mappings

$$f^i\colon D \times R^m \times R^n \times M_{n\times n}(R) \times X \ni (t, x, z, q, r, w) \to f^i(t, x, z, q, r, w) \in R \qquad (i = 1, \ldots, m)$$

be given and let the operators P_i $(i = 1, \ldots, m)$ be defined by the formulae

$$P_i u(t, x) = u_t^i(t, x) - f^i(t, x, u(t, x), u_x^i(t, x), u_{xx}^i(t, x), u), \qquad u \in X,\ (t, x) \in D\ (i = 1, \ldots, m).$$

A regular mapping u $[v]$ in D is called a *solution* of the system of the functional-differential inequalities

$$P_i u(t, x) \leqslant 0, \quad (t, x) \in D\ (i = 1, \ldots, m) \tag{2.1}$$

$$[P_i v(t, x) \geqslant 0, \quad (t, x) \in D\ (i = 1, \ldots, m)] \tag{[(2.1')}$$

in D if (2.1) [(2.1'), respectively] is satisfied.

For a given regular mapping u in D and for an arbitrary fixed index $i \in \{1, \ldots, m\}$ the mapping f^i is called *uniformly parabolic* with respect to u in a subset $E \subset D$ if there is a constant $\varkappa > 0$ (depending on E) such that for any two matrices $\tilde{r} = [\tilde{r}_{jk}]$, $\hat{r} = [\hat{r}_{jk}] \in M_{n\times n}(R)$ and for $(t, x) \in E$ we have

$$\tilde{r} \leqslant \hat{r} \Rightarrow f^i(t, x, u(t, x), u_x^i(t, x), \hat{r}, u) - f^i(t, x, u(t, x), u_x^i(t, x), \tilde{r}, u) \geqslant \varkappa \sum_{j=1}^{n} (\hat{r}_{jj} - \tilde{r}_{jj}), \tag{2.2}$$

where $\tilde{r} \leqslant \hat{r}$ means that $\sum_{j,k=1}^{n} (\tilde{r}_{jk} - \hat{r}_{jk}) \lambda_j \lambda_k \leqslant 0$ for every $(\lambda_1, \ldots, \lambda_n) \in R^n$.

If (2.2) is satisfied for $\tilde{r} = u_{xx}^i(t, x)$, $\hat{r} = u_{xx}^i(t, x) + r$, $r \geqslant 0$ and $\varkappa = 0$, then f^i is called *parabolic* with respect to u in E.

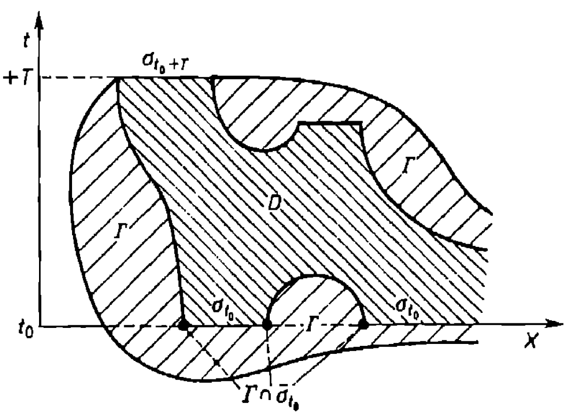

Fig. 1. The set D of type (P_Γ) if $D = (\mathrm{int} D) \cup \sigma_{t_0+T}$

An unbounded set D of type (P) is called a *set of type* (P_Γ) (see Fig. 1) if

$$\Gamma \cap \bar{\sigma}_{t_0} \neq \emptyset. \tag{2.3}$$

A bounded set D of type (P) is called a *set of type* (P_B).

It is easy to see that each set D of type (P_B) satisfies condition (2.3). Moreover, it is obvious that if D_0 is a bounded subset [D_0 is an unbounded proper subset] of $\boldsymbol{R}^n$, then $D = (t_0, t_0 + T] \times D_0$ is a set of type (P_B) [(P_Γ), respectively].

3. Lemma. As a consequence of Theorem 3.1 from [2] we obtain the following:

Lemma 3.1. *Assume that:*

(1) *D is a set of type (P).*

(2) *The mappings f^i $(i = 1, \ldots, m)$ are weakly increasing with respect to $z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^m$ $(i = 1, \ldots, m)$. Moreover, there is a positive constant L such that*

$$f^i(t, x, z, q, r, w) - f^i(t, x, \tilde{z}, \tilde{q}, \tilde{r}, \tilde{w})$$
$$\leqslant L\Big(\max_{k=1,\ldots,m} |z^k - \tilde{z}^k| + |x| \sum_{j=1}^{n} |q^j - \tilde{q}^j| + |x|^2 \sum_{j,k=1}^{n} |r_{jk} - \tilde{r}_{jk}| + [w - \tilde{w}]_t\Big)$$

for all $(t, x) \in D$, $z, \tilde{z} \in \boldsymbol{R}^m$, $q, \tilde{q} \in \boldsymbol{R}^n$, $r, \tilde{r} \in M_{n \times n}(\boldsymbol{R})$, $w, \tilde{w} \in X$, where

$$\sup_{(t,x) \in \tilde{D}} [w(t, x) - \tilde{w}(t, x)] < \infty \qquad (i = 1, \ldots, m).$$

(3) *The mapping u belongs to X, and $\sup_{(t,x) \in D} u(t, x) < \infty$.*

(4) *$u(t, x) \leqslant K$ for $(t, x) \in \partial_p D$, where $K = (K^1, \ldots, K^m)$ is a constant function belonging to X.*

(5) *$f^i(t, x, K, 0, 0, K) \leqslant 0$ for $(t, x) \in D$ $(i = 1, \ldots, m)$.*

(6) *The mapping u is a solution of system* (2.1) *in D.*

(7) *The mappings f^i $(i = 1, \ldots, m)$ are parabolic with respect to u in D and uniformly parabolic with respect to K in any compact subset of D.*

Then

$$u(t, x) \leqslant K \quad \text{for } (t, x) \in \tilde{D}.$$

Moreover, if there is a point $(\tilde{t}, \tilde{x}) \in D$ such that $u(\tilde{t}, \tilde{x}) = K$, then

$$u(t, x) = K \quad \text{for } (t, x) \in S^-(\tilde{t}, \tilde{x}).$$

4. Strong maximum and minimum principles with initial inequalities in sets of types (P_Γ) and (P_B). Now we shall prove the following theorem on strong maximum principles with initial inequalities in sets of types (P_Γ) and (P_B):

THEOREM 4.1. *Assume that:*

(i) *D is a set of type (P_Γ) or (P_B) and assumption* (2) *of Lemma* 3.1 *holds.*

(ii) *The mapping u belongs to X and the maximum of u on Γ is attained. Moreover,*

$$K := \max_{(t,x)\in\Gamma} u(t, x) \tag{4.1}$$

and $K \in X$.

(iii) *The inequality*

$$u(t_0, x) \leqslant K \quad \textit{for } x \in S_{t_0} \tag{4.2}$$

is satisfied.

(iv) *The maximum of u in $\tilde{D}$ is attained. Moreover,*

$$M := \max_{(t,x)\in\tilde{D}} u(t, x) \tag{4.3}$$

and $M \in X$.

(v) *$f^i(t, x, M, 0, 0, M) \leqslant 0$ for $(t, x) \in D$ $(i = 1, \ldots, m)$.*

(vi) *The mapping u is a solution of system* (2.1) *in D.*

(vii) *The mappings f^i $(i = 1, \ldots, m)$ are parabolic with respect to u in D and uniformly parabolic with respect to M in any compact subset of D.*

Then

$$\max_{(t,x)\in\tilde{D}} u(t, x) = \max_{(t,x)\in\Gamma} u(t, x). \tag{4.4}$$

Moreover, if there is a point $(\tilde{t}, \tilde{x}) \in D$ such that $u(\tilde{t}, \tilde{x}) = \max_{(t,x)\in\tilde{D}} u(t, x)$, then

$$u(t, x) = \max_{(t,x)\in\Gamma} u(t, x) \quad \textit{for } (t, x) \in S^-(\tilde{t}, \tilde{x}).$$

Proof. We shall prove Theorem 4.1 for a set of type (P_Γ) only since the proof for a set of type (P_B) is analogous.

We shall argue by contradiction. Suppose

$$M \neq K. \tag{4.5}$$

From (4.1) and (4.3) we have

$$K \leqslant M. \tag{4.6}$$

Consequently,

$$K < M. \tag{4.7}$$

Observe, from assumption (iv), that

(4.8) There is $(t^*, x^*)\in\tilde{D}$ such that $u(t^*, x^*) = M := \max_{(t,x)\in\tilde{D}} u(t, x)$.

By (4.8), by assumption (ii) and by (4.7) we have

$$(t^*, x^*)\in\tilde{D}\setminus\Gamma = D\cup\sigma_{t_0}. \tag{4.9}$$

Suppose that

$$(t^*, x^*)\in D. \tag{4.10}$$

From assumptions (v) and (vi) and from (4.8), we get

$$\begin{aligned} &f^i(t, x, M, 0, 0, M)\leqslant 0 \quad \text{for } (t, x)\in D\ (i = 1,\ldots, m),\\ &u\in X \text{ and } u_t^i,\ u_x^i,\ u_{xx}^i\ (i = 1,\ldots, m) \text{ are continuous in } D,\\ &P_i u(t, x)\leqslant 0 \quad \text{for } (t, x)\in D\ (i = 1,\ldots, m),\\ &u(t, x)\leqslant M \quad \text{for } (t, x)\in\tilde{D},\\ &u(t^*, x^*) = M\in X. \end{aligned} \tag{4.11}$$

The assumption that D is a set of type (P), assumption (2) (see assumption (i)), relations (4.10) and (4.11) and assumption (vii) imply by Lemma 3.1 the equation

$$u(t, x) = M \quad \text{for } (t, x)\in S^-(t^*, x^*). \tag{4.12}$$

On the other hand, from the definition of a set of type (P_Γ), there is a polygonal line $\gamma\subset S^-(t^*, x^*)$ such that

$$\bar{\gamma}\cap\Gamma\neq\emptyset. \tag{4.13}$$

Since $u\in C(\tilde{D})$, we have a contradiction of formulae (4.12) and (4.13) with formulae (4.1) and (4.7). Therefore, $(t^*, x^*)\notin D$ and, consequently, from (4.9), $(t^*, x^*)\in\sigma_{t_0}$. But this leads, by (4.7), to a contradiction of (4.2) with (4.8). The proof of (4.4) is complete.

The second part of Theorem 4.1 is a consequence of equality (4.4) and of Lemma 3.1. Therefore, the proof of Theorem 4.1 is complete.

Arguing analogously to the proof of Theorem 4.1, we obtain the following theorem on strong minimum principles with initial inequalities in sets of types (P_Γ) and (P_B):

THEOREM 4.2. *Assume that:*

(1) *D is a set of type (P_Γ) or (P_B) and assumption* (2) *of Lemma* 3.1 *holds.*

(2) *The mapping v belongs to X and the minimum of v on Γ is attained. Moreover,*

$$k := \min_{(t,x)\in\Gamma} v(t, x)$$

and $k\in X$.

(3) $v(t_0, x) \geqslant k$ *for* $x \in S_{t_0}$.

(4) *The minimum of* v *in* $\tilde{D}$ *is attained. Moreover,*

$$\mathfrak{m} := \min_{(t,x)\in\tilde{D}} v(t, x)$$

and $\mathfrak{m} \in X$.

(5) $f^i(t, x, \mathfrak{m}, 0, 0, \mathfrak{m}) \geqslant 0$ *for* $(t, x) \in D$ $(i = 1, \ldots, m)$.

(6) *The mapping* v *is a solution of system* (2.1') *in* D.

(7) *The mappings* f^i $(i = 1, \ldots, m)$ *are parabolic with respect to* $\mathfrak{m}$ *in* D *and uniformly parabolic with respect to* v *in any compact subset of* D.

Then

$$\min_{(t,x)\in\tilde{D}} v(t, x) = \min_{(t,x)\in\Gamma} v(t, x).$$

Moreover, if there is a point $(\tilde{t}, \tilde{x}) \in D$ *such that* $v(\tilde{t}, \tilde{x}) = \min_{(t,x)\in\tilde{D}} v(t, x)$, *then*

$$v(t, x) = \min_{(t,x)\in\Gamma} v(t, x) \quad \textit{for } (t, x) \in S^-(\tilde{t}, \tilde{x}).$$

Remark 4.1. If D is a set of type (P_B) and if $\tilde{D} = \bar{D}$, then the first parts of assumptions (ii) and (2) of Theorems 4.1 and 4.2 relative to the maximum of u and the minimum of v and the first parts of assumptions (iv) and (4) of these theorems are trivially satisfied since $u, v \in C(\bar{D})$ and Γ is a bounded and closed set in this case.

Remark 4.2. If the mappings f^i $(i = 1, \ldots, m)$ do not depend on the functional argument w, then Theorems 4.1 and 4.2 reduce to theorems on parabolic differential inequalities of the form

$$u_t^i(t, x) \underset{(\geqslant)}{\leqslant} f^i\big(t, x, u(t, x), u_x^i(t, x), u_{xx}^i(t, x)\big) \quad (i = 1, \ldots, m)$$

and in this case we can put $\tilde{D} = \bar{D}$.

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