ANNALES
POLONICI MATHEMATICI
XLII (1983)

Analytic structure on locally compact spaces determined by algebras of continuous functions

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Dedicated to the memory of Jacek Szarski

Abstract. In the present note we give a multidimensional analogue of Aupetit–Wermer's theorem on analytic structure on locally compact spaces (Theorem 3.3). The method of our proof is in fact a standard modification of the ideas of Wermer [10] and Basener [4].

I. Introduction. Let X denote a locally compact Hausdorff space and $C(X)$ the algebra of all continuous complex-valued functions defined on X. The following problem arises naturally in many situations, especially in the theory of function algebras: to find a non-trivial class $\mathscr{A}_n$ of subalgebras of $C(X)$ such that for every $A \in \mathscr{A}_n$ there exists a uniquely determined structure of n-dimensional complex analytic space on X with $A \subset \mathcal{O}(X)$ ($\mathcal{O}(X)$ denotes the space of all holomorphic functions on the analytic space X).

There exists a class of pairs $(X, \mathscr{A}_1)$, easy to describe such that each subalgebra $A \in \mathscr{A}_1$ determines a one-dimensional analytic structure on X. This class of algebras was examined by Wermer in [9], [10]. Let us recall the fundamental definition from [9].

Definition 1.1. A subalgebra $A \subset C(X)$ is called a *maximum modulus algebra* (*m.m.a.*) on X iff

(i) A separates points on X and contains the constants,

(ii) for every $g \in A$ and every compact set $K \subset X$ we have

$$\|g\|_K \leqslant \|g\|_{\partial K}$$

(∂K denotes the topological boundary of K relative to X).

The following theorem, generalizing the classical Bishop's analytic structure theorem, was proved by Aupetit and Wermer in [3] (see also [10], Theorem 1).

Theorem 1.2. *Let A be a m.m.a. on X and let $f \in A$ be a proper mapping of X onto a domain Ω in $\mathbf{C}$. Assume that there exists a subset E of Ω*

of positive logarithmic capacity and $k \in \boldsymbol{N}$ *such that* $\# f^{-1}(\lambda) \leqslant k$ *for* $\lambda \in E$. *Then*

(1) $\# f^{-1}(\lambda) \leqslant k$ *for every* $\lambda \in \Omega$;

(2) *there exists a discrete subset* Γ *of* Ω *such that* $f^{-1}(\Omega \setminus \Gamma)$ *can be equipped with a uniquely determined structure of Riemann surface and* $A \subset \mathcal{O}\big(f^{-1}(\Omega \setminus \Gamma)\big)$.

In the present note we propose a certain generalization of the notion of a maximal modulus algebra and we modify Wermer's potential theory method in order to obtain a result valid in the higher-dimensional case. Such a way seems to be natural in view of the original Basener's theorem [4] on multi-dimensional analytic structure and its generalizations due to Aupetit ([2], Theorem 2.13).

II. Preliminaries. Let A be a subalgebra of $C(X)$ containing the constants. Suppose that there exists a proper mapping $F = (f_1, \ldots, f_n) \in A^n = A \times \ldots \times A$ of X onto some domain in $\boldsymbol{C}^n$. Let L be an affine complex line in $\boldsymbol{C}^n$ with $\Omega \cap L \neq \emptyset$. Denote $X_L = F^{-1}(\Omega \cap L)$, $A_L = A|_{X_L} = \{g|_{X_L}: g \in A\}$. If $\lambda: \boldsymbol{C} \ni t \to a + bt \in \boldsymbol{C}^n$, where $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \neq 0$ and $L = \lambda(\boldsymbol{C})$, then the inverse of λ is of the form

$$\lambda^{-1}: L \ni (z_1, \ldots, z_n) \to \Big(\sum_1^n (z_i - a_i)\bar{b}_i\Big)/|b|^2 \in \boldsymbol{C}.$$

Therefore,

$$F_L = \lambda^{-1} \circ F|_{X_L} = \lambda^{-1} \circ (f_1|_{X_L}, \ldots, f_n|_{X_L}) = |b|^{-2} \sum_1^n \bar{b}_i (f_i|_{X_L} - a_i) \in A_L$$

and the mapping $F_L: X_L \to \Omega_L = \lambda^{-1}(\Omega \cap L)$ is proper.

DEFINITION 2.1. We call $\{A, X \xrightarrow{F} \Omega\}$ a *structural system* of order n if

(i) A is a subalgebra of $C(X)$ separating the points of X and containing the constants;

(ii) Ω is a domain in $\boldsymbol{C}^n$ and $F \in A^n$ is a proper mapping of X onto Ω;

(iii) for every affine complex line L in $\boldsymbol{C}^n$ with $\Omega \cap L \neq \emptyset$, A_L is a m.m.a. on X_L.

As standard models of a structural system of order n we propose

EXAMPLE 2.2. Let Ω be a domain in $\boldsymbol{C}^n$ and let $A \subset \mathcal{O}(\Omega)$ be a subalgebra which contains the constants and the coordinate functions $z_1, \ldots, z_n$. Then $\{A, \Omega \xrightarrow{\text{id}} \Omega\}$ is a structural system of order n.

EXAMPLE 2.3. Let A be a uniform algebra defined on a compact space T with maximal ideal space M. If K is a compact subset of M, let A_K be the closure of $\{\hat{f}|_K: f \in A\}$ in $C(K)$ ($\hat{f}$ denotes the Gelfand transform of f). For every nonnegative integer n put $\hat{A}^n = \{\hat{F} = (\hat{f}_1, \ldots, \hat{f}_n):$

$f_i \in A,\ i = 1, \ldots, n\}$. If $\hat{F} \in \hat{A}^n$, let $V(\hat{F}) = \hat{F}^{-1}(0)$. Note that the maximal ideal space of $A_{V(\hat{F})}$ is $V(\hat{F})$. Set $\partial_n A = \overline{\bigcup_{\hat{F} \in \hat{A}^n} \partial_0 A_{V(\hat{F})}}$, where $\partial_0 A_{V(\hat{F})}$ is the usual Shilov boundary of A. We call $\partial_n A$ the *Shilov boundary* of A of order n (see Basener [4]; some examples are given by Sibony [8]). Assume that $\partial_{n-1} \hat{A} \subset T$ for some $n \in N$. Let $\hat{F} \in \hat{A}^n$ and let Ω be a connected component of $C^n \setminus \hat{F}(T)$ such that $\hat{F}(M) \cap \Omega \neq \emptyset$. Put $X = \hat{F}^{-1}(\Omega)$, $A_0 = \hat{A}|_X$, $F = \hat{F}|_X$.

We claim that $\{A_0, X \xrightarrow{F} \Omega\}$ is a structural system of order n. Obviously, the algebra A_0 contains constants and separates points. Since $\hat{F}(M) \cap \Omega = \Omega$ ([4], Lemma 2), we have $F(X) = \Omega$. Obviously, the mapping F is proper. Thus conditions (i) and (ii) of Definition 2.1 are satisfied. Let

$$L = \bigcap_{j=1}^{n-1} \Big\{(z_1, \ldots, z_n) \in C^n: \sum_{i=1}^{n} a_{ij} z_i = b_j\Big\}, \qquad a_{ij}, b_j \in C$$

be a given affine complex line in C^n with $\Omega \cap L \neq \emptyset$. Then we have $X_L = F^{-1}(\Omega \cap L) \subset \hat{G}^{-1}(0) = V(\hat{G})$, where

$$\hat{G} = (\hat{g}_1, \ldots, \hat{g}_{n-1}), \qquad \hat{g}_j = \sum_{i=1}^{n} a_{ij} \hat{f}_i - b_j, \qquad j = 1, \ldots, n-1,$$

i.e. $\hat{G} \in \hat{A}^{n-1}$. Let K be a compact subset of X_L and $g \in A$. Since X_L is an open subset of $\hat{G}^{-1}(0)$, and, by the hypothesis, $X_L \cap \partial_0 A_{V(\hat{G})} = \emptyset$, the local maximum modulus principle ([5], Theorem 8.2) applied to $A_{V(\hat{G})}$ gives the inequality $\|g\|_K \leqslant \|g\|_{\partial_L K}$, where $\partial_L K$ denotes the topological boundary of K relative to X_L. Therefore, condition (iii) is satisfied, so that $\{A_0, X \xrightarrow{F} \Omega\}$ is a structural system of order n, as claimed.

PROPOSITION 2.4. *Let $\{A, X \xrightarrow{F} \Omega\}$ be a structural system of order n and let $\# F^{-1}(y) < \infty$ for $y \in \Omega$. Then the mapping F is open.*

Proof. Fix $x_0 \in X$ and write $y_0 = F(x_0)$. Then $F^{-1}(y_0) = \{x_0\} \cup \{x_1, \ldots, x_l\}$. Fix a compact neighbourhood U of x_0 and a neighbourhood V of $\{x_1, \ldots, x_l\}$ such that $U \cap V = \emptyset$. Since F is proper, there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0} \subset \Omega$ and $F^{-1}(B_{\varepsilon_0}) \subset U \cup V$ (we put $B_a = \{y \in C^n: |y - y_0| \leqslant a\}$, $S_a = \{y \in C^n: |y - y_0| = a\}$ for $a > 0$).

Let $\tilde{U} = U \cap F^{-1}(B_{\varepsilon_0})$. Then $\tilde{U}$ is a compact neighbourhood of x_0. We claim that $F(\tilde{U}) = B_{\varepsilon_0}$. Assume the contrary. Then there exists $0 < \varepsilon \leqslant \varepsilon_0$ and an affine complex line L through y_0 such that $\tilde{\Gamma}_\varepsilon = F(F^{-1}(S_\varepsilon \cap L) \cap U)$ is a proper subset of $\Gamma_\varepsilon = S_\varepsilon \cap L$. Hence there exists a polynomial P in one variable with $|P(y_0)| > \max_{\tilde{\Gamma}_\varepsilon} |P|$. Let K

$= F^{-1}(B_s \cap L) \cap U$. Then $\partial_L K \subset K_0 = F^{-1}(S_s \cap L) \cap U$ and, by Definition 2.1, we have

$$|P(y_0)| = |(P\circ F_L)(x_0)| \leqslant \max_{\partial_L K}|P\circ F_L| \leqslant \max_{K_0}|P\circ F_L| = \max_{\tilde{F}_s}|P| < |P(y_0)|.$$

This is impossible, and so $\tilde{F}(U) = B_{s_0}$ as claimed. Thus $B_{s_0} \subset F(U)$, i.e. $F(U)$ is a neighbourhood of y_0.

LEMMA 2.5. *Let X, Y be locally compact spaces and let F be a proper mapping of X onto Y. Then for every continuous function $h\colon X \to C$ the function*

$$h_F(y) = \max\{|h(x)|\colon x \in F^{-1}(y)\}, \qquad y \in Y,$$

is upper-semicontinuous on Y.

Proof. Fix $s \in \mathbf{R}$. Then $\{y \in Y\colon h_F(y) < s\} = Y \setminus \{y \in Y\colon h_F(y) \geqslant s\} = Y \setminus F\big(|h|^{-1}([s, +\infty))\big)$. Since F is proper and h is continuous, the set $\{h_F(y) < s\}$ is open. Thus h_F is upper semicontinuous on Y.

PROPOSITION 2.6. *Let $\{A, X \xrightarrow{F} \Omega\}$ be a structural system of order n. Then for every $g \in A$ the function $\log g_F$ is plurisubharmonic in Ω.*

Proof. Fix $g \in A$. By the previous lemma the function g_F, and hence $\log g_F$, is upper semicontinuous on Ω. Let $L = \lambda(C)$ be an affine complex line in C^n with $\Omega \cap L \neq \emptyset$. Since A_L is a m.m.a. on X_L, the function $\log (g_L)_{F_L}$, where $g_L = g|_{X_L}$, is subharmonic on Ω_L ([9], Lemma 1). Obviously, the equality $g_F \circ \lambda = (g_L)_{F_L}$ holds true on Ω_L. Hence $\log g_F$ is plurisubharmonic in Ω.

If $S \subset C$ is compact, we define the νth ($\nu \geqslant 1$) diameter of S by the formula

$$D^{(\nu)}(S) = \max\Big\{\prod_{i<j} |s_i - s_j|\colon \{s_1, \ldots, s_\nu\} \subset S\Big\}.$$

Thus $D^{(1)}(S)$ is the diameter of S. Note that $\# S < k$ implies $D^{(k)}(S) = 0$.

PROPOSITION 2.7. *Let $\{A, X \xrightarrow{F} \Omega\}$ be a structural system of order n. Fix $g \in A$ and define*

$$D_g^{(\nu)}(y) = D^{(\nu)}\big(g(F^{-1}(y))\big), \qquad y \in \Omega,\ \nu \geqslant 2.$$

Then the function $\log D_g^{(\nu)}$ is plurisubharmonic in Ω.

Proof. Let us fix $g \in A$ and $\nu \geqslant 2$. We first show the upper semicontinuity of $\log D_g^{(\nu)}$. Let $\pi = \underbrace{F \times \ldots \times F}_{\nu \text{ times}}$, $X^\nu = \underbrace{X \times \ldots \times X}_{\nu \text{ times}}$, $\Omega^\nu = \underbrace{\Omega \times \ldots \times \Omega}_{\nu \text{ times}}$. Then $\pi\colon X^\nu \to \Omega^\nu$ is a proper mapping. Define

$$G(x_1, \ldots, x_\nu) = \prod_{i<j} \big(g(x_i) - g(x_j)\big), \qquad (x_1, \ldots, x_\nu) \in X^\nu.$$

Obviously, G is continuous on X^ν. It is easy to see that $D_g^{(\nu)} = G \circ \Delta$, where Δ: $\Omega \ni y \to (y, \ldots, y) \in \Omega^\nu$. According to Lemma 2.5, $D_g^{(\nu)}$, and hence $\log D_g^{(\nu)}$, is upper semicontinuous on Ω. Let $L = \lambda(\boldsymbol{C})$ be an affine complex line in $\boldsymbol{C}^n$ with $\Omega \cap L \neq \emptyset$ and let $g_L = g|_{X_L}$. By Wermer's result ([10], Theorem 2) the function $\log D_{g_L}^{(\nu)}$ is subharmonic on Ω_L. Since we have the equality $D_{g_L}^{(\nu)} = D_g^{(\nu)} \circ \lambda$, we conclude that $\log D_g^{(\nu)}$ is plurisubharmonic in Ω.

To conclude the preliminaries we present a multidimensional analogue of a well known result by Hartogs (see, for example, [1], Theorem II.17, p. 174 and [7], Lemma 3, p. 59). The proof given in [1] can be easily adopted to our situation.

PROPOSITION 2.8 *Let $B \subset \boldsymbol{C}^n$ be an open ball and let f: $B \to \boldsymbol{C}$ be a bounded function such that* $\log|f-a|$ *is plurisubharmonic in B for every $a \in \boldsymbol{C}$. Then either f or $\bar{f}$ is holomorphic in B.*

III. The main result. Before the formulation of the theorem on analytic structure we remind two definitions:

DEFINITION 3.1. We say that a subset E of a domain $\Omega \subset \boldsymbol{C}^n$ is *pluripolar* if there exists a function u, plurisubharmonic in Ω, $u \not\equiv -\infty$, such that $u = -\infty$ on E.

DEFINITION 3.2 ([6], Definition 3). We say that a triple (X, F, Y) is an *analytic cover* with the critical set S if

(i) X is a locally compact Hausdorff space, Y is a complex manifold and F is a continuous proper mapping of X onto Y with finite fibres;

(ii) S is a proper analytic subset of Y such that the set $F^{-1}(S)$ is negligible in X (i.e. $F^{-1}(S)$ is nowhere dense and for every $a \in F^{-1}(S)$ and every connected neighbourhood U of a there exists a neighbourhood $U' \subset U$ of a such that $U' \setminus F^{-1}(S)$ is connected) and the mapping F: $X \setminus F^{-1}(S) \to Y \setminus S$ is locally homeomorphic.

Our main result is the following

THEOREM 3.3. *Let $\{A, X \xrightarrow{F} \Omega\}$ be a structural system of order n. Suppose that there exists a non-pluripolar subset E of Ω such that $\# F^{-1}(y) < \infty$ for every $y \in E$. Let $\Omega_\nu = \{y \in \Omega\colon \# F^{-1}(y) = \nu\}$ for $\nu = 1, 2, 3, \ldots$ Then*

(1) *there exists $k \in \boldsymbol{N}$ such that $\Omega = \Omega_1 \cup \ldots \cup \Omega_k$ and $\Omega_k \neq \emptyset$;*

(2) *the set $S = \Omega_1 \cup \ldots \cup \Omega_{k-1}$ is a closed analytic subset of Ω with* $\dim S \leqslant n-1$;

(3) *the triple (X, F, Ω) is an analytic cover with the critical set S;*

(4) *there exists an analytic space structure of pure dimension n on X such that $A \subset \mathcal{O}(X)$.*

Proof. (1) Since a countable union of pluripolar sets is pluripolar, it is enough to use Proposition 2.7 and the fact that A separates points of X.

(2) Let $y_0 \in \Omega_k$ and $F^{-1}(y_0) = \{x_1, \ldots, x_k\}$. Let $U_1, \ldots, U_k$ be disjoint neighbourhoods of $x_1, \ldots, x_k$ in X. Since the mapping F is open (by (1) and Proposition 2.4), the set $V = F(U_1) \cap \ldots \cap F(U_k)$ is a neighbourhood of y_0. Obviously, $V \subset \Omega_k$. Hence the set Ω_k is open. Therefore $S = \Omega_1 \cup \ldots \cup \Omega_{k-1} = \Omega \setminus \Omega_k$ is closed. Moreover, the mapping $F\colon F^{-1}(\Omega_k) \to \Omega_k$ is a local homeomorphism. Thus we get a uniquely determined structure of n-dimensional complex manifold on $F^{-1}(\Omega_k)$ making F a holomorphic. mapping.

Now we show that every function $g \in A$ is holomorphic on the manifold $F^{-1}(\Omega_k)$. Fix $g \in A$, a point $y_0 \in \Omega_k$ and $x_0 \in F^{-1}(y_0)$. Let us also fix an open ball $B \subset \Omega_k$ centered at y_0. Choose an open neighbourhood U of x_0 in the space X such that the mapping $F_0 = F|_U\colon U \to B$ is homeomorphic It may easily be shown that $\{A|_U, U \xrightarrow{F_0} B\}$ is a structural system of. order n. By Proposition 2.6 the function $\log(g-a)_{F_0} = \log|g \circ F_0^{-1} - a|$ is plurisubharmonic in B for every $a \in C$. Applying Proposition 2.8 we see that either $g \circ F_0^{-1}$ or $\overline{g \circ F_0^{-1}}$ is holomorphic in B.

Choose $\nu \in \{1, 2, \ldots, n\}$ for which the function f_ν is non-constant in U. Then

$$(*) \qquad (f_\nu g)\big(F_0^{-1}(y)\big) = y_\nu g\big(F_0^{-1}(y)\big), \qquad y = (y_1, \ldots, y_n) \in B.$$

Applying the same argument to the function $f_\nu g \in A$, we conclude that either $(f_\nu g) \circ F_0^{-1}$ or $\overline{(f_\nu g) \circ F_0^{-1}}$ is holomorphic in B.

Using the standard argument (see [4] for details) we show that the closed set $S = \Omega_1 \cup \ldots \cup \Omega_{k-1}$ is analytic in Ω with $\dim S \leqslant n-1$.

(3) It is easy to see that the set $F^{-1}(S)$ is negligible in X. Indeed, by the openness of F, we have

$$X = F^{-1}(\Omega) = F^{-1}(\overline{\Omega \setminus S}) = \overline{F^{-1}(\Omega \setminus S)} = \overline{X \setminus F^{-1}(S)},$$

i.e. the set $X \setminus F^{-1}(S)$ is dense in X. Since $F^{-1}(S)$ is closed, it is nowhere dense in X. Obviously, for every $x \in F^{-1}(S)$ and for every connected neighbourhood U of x the set $U \setminus F^{-1}(S)$ is connected. We have already shown that the mapping $F\colon X \setminus F^{-1}(S) \to \Omega \setminus S$ is locally homeomorphic. Thus we conclude that the triple (X, F, Ω) is an analytic cover with the critical set S.

(4) By the above and by [6], Theorem 32, there exists an analytic space structure of pure dimension n on X in which F is holomorphic. Let $g \in A$. Then g is continuous on X and holomorphic in the set $F^{-1}(\Omega_k) = X \setminus F^{-1}(S)$, where the set $F^{-1}(S)$ is analytic, nowhere dense in X.

By Riemann's extension theorem ([6], Theorem 13) g is holomorphic on X. Therefore $A \subset \mathcal{O}(X)$.

Using the notation of Example 2.3 we have

COROLLARY 3.4 ([2], Theorem 2.13). *Let $n \in N$ and let $\partial_{n-1}A \subset T$. Let $\hat{F} \in \hat{A}^n$ and let Ω be a connected component of $C^n \setminus \hat{F}(T)$ such that $\hat{F}(M) \cap \cap \Omega \neq \emptyset$. Suppose that there exists a non-pluripolar set $E \subset \Omega$ such that $\# \hat{F}^{-1}(z) < \infty$ for every $z \in E$. Put $X = \hat{F}^{-1}(\Omega)$, $F = \hat{F}|_X$ and $A_0 = \hat{A}|_X$. Then statements* (1)–(4) *of Theorem* 3.3 *are true.*

Let us note that in original Basener's version of the above theorem ([4], Theorem 2) it is assumed that the set E is of positive $(2n)$-dimensional Lebesgue measure.

Finally, let us compare the notion of a maximum modulus algebra and a structural system. Theorem 3.3 implies the following relation:

COROLLARY 3.5. *If $\{A, X \xrightarrow{F} \Omega\}$ is a structural system of order n, then A is a m.m.a.*

In general, a m.m.a. on X does not generate a structural system of order $n \geqslant 1$.

EXAMPLE 3.6. *Let $A = \{f \in C(C^2): \forall b \in C, f(\cdot, b) \in \mathcal{O}(C)\}$. Then A is a m.m.a. on C^2, whereas $\{A, C^2 \xrightarrow{\text{id}} C^2\}$ is not a structural system of order* 2.

The author is very indebted to Piotr Jakóbczak for his helpful remarks.

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Reçu par la Rédaction le 12.02.1981