ANNALES
POLONICI MATHEMATICI
XLVIII (1988)

Directional qualitative cluster sets

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Abstract. For arbitrary function f defined on the half plane H above the real line R, the interesection properties between directional and sectorial qualitative cluster sets $C_q(f, x, \theta)$ and $C_q(f, x, S)$, and between directional cluster set $C(f, x, \theta)$ and directional qualitative cluster set $C_q(f, x, \theta)$ are studied. The results proved here state that: If $f\colon H \to W$, W is a compact, normal and second countable topological space, then (i) except a countable set of points on R, for every sector S in H, $C_q(f, x, S)$ intersects $C_q(f, x, \theta)$ for a residual set of θ in $(0, \pi)$, (ii) except a first category set on R, for fixed θ in $(0, \pi)$, $C(f, x, \theta)$ intersects $C_q(f, x, \varphi)$ for each $\varphi(0, \pi)$, and (iii) except a countable set on R, for every φ in $(0, \pi)$, $C_q(f, x, \varphi)$ intersects $C(f, x, \theta)$ for a residual set of θ in $(0, \pi)$.

1. Let R be the real line and $H = R \times (0, \alpha)$. Let $L_\theta(x)$ be a ray in H emanating from x in R and in the direction θ, $0 < \theta < \pi$. Also, let $S_{\alpha\beta}$ denote a sector in H having vertex at the origin, defined by

$$S_{\alpha\beta} = \{Z\colon\ Z \in H\colon\ 0 < \alpha < \arg(z) < \beta < \pi\}.$$

$S_{\alpha\beta}(x)$ is the translate of $S_{\alpha\beta}$, obtained by taking the origin at x. If there is no confusion, simply S and $S(x)$ will stand to denote $S_{\alpha\beta}$ and $S_{\alpha\beta}(x)$. For $x \in R$ and $r > 0$, set

$$K(x, r) = \{Z\colon\ Z \in H\colon\ |Z - x| < r\},$$
$$S(x, r) = S(x) \cap K(x, r),$$

and

$$L_\theta(x, r) = L_\theta(x) \cap K(x, r).$$

Further, $\bar{E}$ will denote the closure of the set E.

Throughout the paper, f.c. and s.c. will mean first category and second category, respectively.

If $f\colon H \to W$, W is a topological space, the *qualitative cluster set* $C_q(f, x)$ of f at x is the set of all w in W for which $f^{-1}(u) \cap K(x, r)$ is s.c. for all $r > 0$ and for every open set u in W containing w. The definitions of sectorial qualitative cluster set $C_q(f, x, S)$ and directional qualitative cluster set $C_q(f, x, \theta)$ are similar with $K(x, r)$ replaced by $S(x, r)$ and $L_\theta(x, r)$, respectively. The cluster set $C(f, x, S)$ (resp. $C(f, x, \theta)$) of f at x in S (in the direction θ) is the set of all w in W for which $x \in \overline{f^{-1}(u) \in S(x)}$ $(x \in \overline{f^{-1}(u) \cap L_\theta(x)})$ for every open set u in W containing w.

2. Wilczyński [6] proved that if f: $H \to R$ has the Baire property and $\theta \in (0, \pi)$ is a fixed direction, then $C_q(f, x) = C_q(f, x, \theta)$ at all x but a f.c. set on R. Supplementing this result, Evans and Humke [1] proved that $C_q(f, x) = C_q(f, x, \theta)$ for a residual set of directions θ in $(0, \pi)$ at all x but a f.c. set on R. In [3] it is also proved that if $\{S\}$ is the collection of all sectors in H then

$$\bigcup \{C_q(f, x, S): S \in \{S\}\} = C_q(f, x, \theta)$$

for a residual set of directions $\theta \in (0, \pi)$ at all x but a σ-porous set [7] on R. In [2], intersecting properties of qualitative cluster sets are studied. A result of this paper states that if f: $H \to W$ is arbitrary, W is a compact, normal and second countable topological space, then except a countable set of points x in R, $C_q(f, x, S_1) \cap C_q(f, x, S_2) \neq \emptyset$ for each pair of sectors S_1 and S_2 in H. Here further properties of qualitative cluster sets are studied.

3. In this section, some sets are defined which will be used in the sequel. For $E \subset H$ and $x \in R$ set

$$E(x) = \{\theta\colon 0 < \theta < \pi\colon L_\theta(x, r) \cap E \text{ is residual in } L_\theta(x, r) \text{ for some } r > 0\},$$

$$\hat{E}(x) = \{\theta\colon 0 < \theta < \pi\colon x \notin \overline{L_\theta(x) \cap E}\},$$

and

$$E[x] = \{S\colon S \subset H\colon S(x, r) \cap E \text{ is f.c. for some } r > 0\}.$$

For positive integer n and rationals α, β in $(0, \pi)$, let

$$E_n(x) = \{\theta\colon 0 < \theta < \pi\colon L_\theta(x, 1/n) \cap E \text{ is residual in } L_\theta(x, 1/n)\},$$

$$\hat{E}_n(x) = \{\theta\colon 0 < \theta < \pi\colon L_\theta(x, 1/n) \cap E = \emptyset\},$$

$$E_n[x] = \{S\colon S \subset H\colon S(x, 1/n) \cap E \text{ is f.c.}\},$$

$$E_n[x, \alpha, \beta] = \{S\colon S \subset H \setminus \bar{S}_{\alpha\beta},\ S(x, 1/n) \cap E \text{ is f.c.}\},$$

and, for any set $\Theta \subset (0, \pi)$, let

$$\Theta_{\alpha\beta} = \Theta \cap (\alpha, \beta).$$

For a set $E \subset H$ we also define

$$(E) = \{x\colon x \in R;\ E(x) \text{ is s.c. and } E[x] \neq \emptyset\},$$

$$(E)_\theta^1 = \{x\colon x \in R;\ x \notin \overline{L_\theta(x) \cap E} \text{ and } E(x) \neq \emptyset\},$$

$$(E)^0 = \{x\colon x \in R;\ \hat{E}(x) \text{ is s.c. and } E(x) \neq \emptyset\}.$$

LEMMA 1. *If $E \subset H$ is arbitrary, then the set (E) is countable.*

Proof. For positive integers n and p and rationals α, β, $0<\alpha<\beta<\pi$: let

$$T_{n\alpha\alpha\beta} = \{x\colon\ x\in R;\ E_{n\alpha\beta}(x) \text{ is s.c. and } E_p[x, \alpha, \beta] \neq \emptyset\},$$

where $E_{n\alpha\beta}(x) = E_n(x) \cap (\alpha, \beta)$.

If $x\in(E)$ then $E[x] \neq \emptyset$ and $E(x)$ is s.c. Thus one can choose positive integers p, n and two rationals γ, δ; $0<\gamma<\delta<\pi$ such that $S_{\gamma\delta}(x, 1/p)\cap E$ is f.c. and $E_n(x)$ is s.c. Then by a result in [5], p. 56, either $(0, \gamma)$ or (δ, π) contains a point $\theta'\in E_n(x)$ such that every neighbourhood of θ' intersects $E_n(x)$ in a s.c. set. Thus there exist rationals α, β; $0<\alpha<\beta<\pi$ such that $(\alpha, \beta)\cap E_n(x)$ is a s.c. set and $[\alpha, \beta]\cap[\gamma, \delta] = \emptyset$. Hence $E_{n\alpha\beta}(x)$ is a s.c. set and $E_p[x, \alpha\beta] \neq \emptyset$. These imply that $x\in T_{np\alpha\beta}$, and hence (E) is contained in a countable union of sets $T_{np\alpha\beta}$.

If possible, let $T = T_{np\alpha\beta}$ be uncountable for some n, p and α and β. Then there is x_0 in T which is a two-sided limit point of T. Let $S(x_0)\in E_p[x_0, \alpha, \beta]$. Then $S(x_0, 1/p)\cap E$ is a f.c. set. Let $r>0$. Then there is $\eta\in(x_0-r, x_0+r)$ such that $S_{\alpha\beta}(\eta, 1/n)$ intersects $S(x_0, 1/p)$ in a quadrilateral Q (say), and $E_{n\alpha\beta}(\eta)$ is a s.c. set. Since for $\theta\in E_{n\alpha\beta}(\eta)$, $L_\theta(\eta, 1/n)\cap E$ is residual in $L_\theta(\eta, 1/n)$ and $E_{n\alpha\beta}(\eta)$ is a s.c. set, therefore $Q\cap E$ is a second category set in H [5], p. 56. Thus $S(x_0, 1/p)\cap E$ is a second category set, a contradiction. Thus $T_{np\alpha\beta}$ is countable for all positive integers n, p and rationals α, β; $0<\alpha<\beta<\pi$. Hence (E) is countable.

LEMMA 2. *If $E\subset H$ is arbitrary and if $\theta\in(0, \pi)$ is a fixed direction, then the set $(E)^1$ is a first category set.*

Proof. For positive integers m and n, let

$$P_{mn} = \{x\colon\ x\in R;\ L_\theta(x, 1/m)\cap E = \emptyset \text{ and } E_n(x) \neq \emptyset\}.$$

Then $(E)^1$ is contained in a countable union of the sets P_{mn}.

If possible, let P_{mn} be a s.c. set for some m and n. Let $x_0\in P_{mn}$ be such that $(x_0-r, x_0)\cap P_{mn}$ and $(x_0, x_0+r)\cap P_{mn}$ are second category sets for each $r>0$. Let $\theta_0\in E_n(x_0)$. Then clearly $\theta\neq\theta_0$. Let $\eta\in(x_0-r, x_0+r)\cap P_{mn}$ be such that $L_\theta(\eta, 1/m)\cap L_{\theta_0}(x_0, 1/n) \neq \emptyset$. Let $|x_0-\eta| = r_0$. Set

$$Z = \{z\colon\ z\in L_\theta(x, 1/m)\cap L_{\theta_0}(x_0, 1/n) \text{ for } x\in(x_0-r, x_0+r)\cap P_{mn}\}.$$

Then $Z\subset H\backslash E$ and Z is a second category set in $L_{\theta_0}(x_0, 1/n)$. This contradicts the fact that $\theta_0\in E_n(x_0)$. Hence P_{mn} is a first category set for all m and n. This proves that $(E)^1$ is a first category set.

LEMMA 3. *If $E\subset H$ is arbitrary, then the set $(E)^0$ is countable.*

Proof. For rationals α, β, $0<\alpha<\beta<\pi$ and positive integers m and n let

$$M_{mn\alpha\beta} = \{x\colon\ x\in R;\ \hat{E}_{m\alpha\beta}(x) \text{ is s.c. and } E_n(x)\backslash[\alpha, \beta] \neq \emptyset\}.$$

If $\hat{E}(x)$ is s.c. and $E(x) \neq \emptyset$, then, for some m and n, $\hat{E}_m(x)$ is s.c. set and $E_n(x) \neq \emptyset$, i.e., there is some $\theta \in \hat{E}_m(x)$ such that $(\theta - \Delta\theta, \theta + \Delta\theta) \cap E_m(x)$ is s.c. for each $\Delta\theta > 0$ and $E_n(x) \neq \emptyset$. If $\Phi \in E_m(x)$, then $\theta \neq \Phi$. So there are rationals α, β in $(0, \pi)$ such that $\hat{E}_{m\alpha\beta}(x)$ is s.c. and $E_n(x) \setminus [\alpha, \beta] \neq \emptyset$. Thus $x \in (E)^0$ implies $x \in M_{mn\alpha\beta}$ for some m, n and α, β. Hence $(E)^0$ is contained in a countable union of sets $M_{mn\alpha\beta}$.

Suppose $M = M_{mn\alpha\beta}$ is uncountable. Let $x_0 \in M$ be such that, for each $r > 0$, $(x_0 - r, x_0) \cap M$ and $(x_0, x_0 + r) \cap M$ are non-void. Let $\theta_0 \in E_n(x_0) \setminus [\alpha, \beta]$. Then there is $\eta \in (x_0 - r, x_0 + r)$ such that $\eta \in M$ also, and $S_{\alpha\beta}(\eta, 1/m) \cap L_{\theta_0}(x_0, 1/n)$ is a segment J (say) with end points at $L_\alpha(\eta, 1/m) \cap L_{\theta_0}(x_0, 1/n)$ and $L_\beta(\eta, 1/m) \cap L_{\theta_0}(x_0, 1/n)$. Since for $\theta \in \hat{E}_{m\alpha\beta}(\eta)$, $L_\theta(\eta, 1/m) \cap E = \emptyset$ and since $\hat{E}_{m\alpha\beta}(\eta)$ is a second category set in (α, β), the set $J \cap (H \setminus E)$ is a second category set in $L_{\theta_0}(x_0, 1/n)$. This contradicts the fact that $\theta_0 \in E(x_0)$. Thus $M = M_{mn\alpha\beta}$ is countable for all positive integers m, n and rationals a, β, $0 < \alpha < \beta < \pi$; and hence the set $(E)^0$ is countable.

Lemma 4. *If $f\colon H \to W$ is arbitrary, where W is a compact topological space, and if U is an open set containing $C_q(f, x, \theta)$, then $L_\theta(x, r) \cap f^{-1}(U)$ is residual in $L_\theta(x, r)$ for some $r > 0$.*

The proof follows from Lemma 3 in [2].

Lemma 5. *If $f\colon H \to W$ is arbitrary, W is a compact topological space, and if V_1 and V_2 are closed sets such that $C_q(f, x, S) \cap V_1 = \emptyset$ and $C(f, x, \theta) \cap V_2 = \emptyset$, then $S(x, r) \cap f^{-1}(V_1)$ is f.c. set for some $r > 0$ and $x \notin \overline{L_\theta(x) \cap f^{-1}(V_2)}$.*

Proof of the first part follows from Lemma 4 in [2] and the proof of the second part is similar.

4. Throughout this section, W is taken to be a compact, normal and second countable topological space.

Theorem 1. *If $f\colon H \to W$ is arbitrary, then except a countable set of points x on R, for every sector $S \subset H$,*

$$C_q(f, x, \theta) \cap C_q(f, x, S) \neq \emptyset$$

for a residual set of directions $\theta \in (0, \pi)$.

Proof. Let I be the exceptional set of Theorem 1 in R. Let $\mathscr{G}$ be the collection of all sets G which can be expressed as a finite union of members of B, where $B = \{V_n\}$ is a countable basis for the topology of W. Then $\mathscr{G}$ is a countable collection. For $G \in \mathscr{G}$ let $f^{-1}(\bar{G}) = G^*$. Let $x_0 \in I$. Then there are $S(x_0) \subset H$ and a second category set $O(x_0)$ in $(0, \pi)$ such that $C_q(f, x_0, S)$ and $C_q(f, x_0, \theta)$ are disjoint closed sets for every $\theta \in O(x_0)$. Since W is compact and normal, there is $G \in \mathscr{G}$ such that $C_q(f, x_0, S) \cap \bar{G} = \emptyset$ and $C_q(f, x_0, \theta) \subset G$ for a second category set of directions $\theta \in O(x_0)$. Then by Lemma 5, $S(x_0, r) \cap G^*$ is a f.c. set for at least one $r > 0$, and by Lemma 4, $L_\theta(x_0, r) \cap G^*$ is residual for some $r > 0$, for a s.c. set of $\theta \in O(x_0)$. Thus the set $G^*[x_0] \neq \emptyset$ and $G^*(x_0)$ is a s.c. set in $(0, \pi)$. These imply $x_0 \in (G^*)$ and hence

$$I \subset \bigcup \{(G^*)\colon\ G \in \mathscr{G}\}.$$

By the result in Lemma 1, (G^*) is countable for each $G \in \mathscr{G}$, and hence I is countable. This completes the proof.

THEOREM 1′. *If $f\colon H \to W$ is arbitrary, then, except a countable set on R, at every x in R there is a residual set $O(x) \subset (0, \pi)$ such that for every $\theta \in O(x)$ and every $S \subset H$*

$$C_q(f, x, \theta) \cap C_q(f, x, S) \neq \emptyset.$$

This theorem can be proved by considering the collection $\{S_{\alpha\beta}\}$ of all sectors with rationals α, β in $(0, \pi)$ and applying Theorem 1. (As the proof of Theorem 2 in [4].)

COROLLARY 1. *Let $f\colon H \to W$ be arbitrary. Then, except possibly a countable set on R, the degeneracy of $C_q(f, x, S)$ for any S in H implies that $C_q(f, x, \theta)$ have common value for a residual set of directions $\theta \in (0, \pi)$.*

(Degeneracy of $C_q(f, x, S)$ means that $C_q(f, x, S)$ is singleton.)

THEOREM 2. *If $f\colon H \to W$ is arbitrary and if $\theta \in (0, \pi)$ is a fixed direction, then except a first category set of points x on R*

$$C(f, x, \theta) \cap C_q(f, x, \Phi) \neq \emptyset$$

for every $\Phi \in (0, \pi)$.

Proof. Let K be the exceptional set in Theorem 2. Let $\mathscr{G}$ and G^* be the same as in Theorem 1. Let $x_0 \in K$. Then there is $\theta_0 \in (0, \pi)$ such that $C(f, x, \theta)$ and $C_q(f, x_0, \theta_0)$ are disjoint. Let $G \in \mathscr{G}$ be such that $C(f, x_0, \theta) \cap \bar{G} = \emptyset$ and $C_q(f, x_0, \theta_0) \subset G$. Then by Lemma 4 and Lemma 5, $L_{\theta_0}(x_0, r) \cap G^*$ is residual in $L_{\theta_0}(x_0, r)$ for some $r > 0$, i.e., $G^*(x_0) \neq \emptyset$ and $x_0 \notin \overline{L_\theta(x_0) \cap G^*}$. Thus $x_0 \in (G^*)^1$, and consequently it is proved that

$$K \subset \bigcup \{(G^*)^1\colon\ G \in \mathscr{G}\}.$$

By Lemma 2, $(G^*)^1$ is a first category set and hence K is a first category set. This completes the proof.

COROLLARY 2. *If $f\colon H \to W$ is arbitrary and if $\theta \in (0, \pi)$ is a fixed* *direction, then except a first category set of points x in R, the degeneracy of $C(f, x, \theta)$ implies that*

$$\bigcap_{0<\Phi<\pi} C_q(f, x, \Phi) \neq \emptyset.$$

Now we give an example to ensure that the exceptional set of Theorem 2 need not be of measure zero.

Example. Let $P \subset R$ be any set of the first category but of positive measure. Set

$$F = \bigcup \{L_{\pi/2}(x):\ x \in P\}.$$

Let f be the characteristic function of F. Then clearly for $x \in P$, $C_q(f, x, \theta) = \{0\}$ for every $\theta \in (0, \pi)\setminus\{\pi/2\}$ and $C(f, x, \pi/2) = \{1\}$. Hence at every $x \in P$, $C_q(f, x, \theta)$ and $C(f, x, \pi/2)$ are disjoint for every $\theta \in (0, \pi)\setminus\{\pi/2\}$

Theorem 3. *If f: $H \to W$ is arbitrary, then except a countable set of points x on R for each $\Phi \in (0, \pi)$*

$$C(f, x, \theta) \cap C_q(f, x, \Phi) \neq \emptyset$$

for a residual set of direction $\theta \in (0, \pi)$.

Proof. If T be the exceptional set of Theorem 3, then by using Lemma 4 and Lemma 5 as in Theorem 1 and Theorem 2, it can be shown that

$$T \subset \bigcup \{(G^*)^0:\ G \in \mathscr{G}\}.$$

By Lemma 3, $(G^*)^0$ is countable and hence T is countable. This completes the proof.

Corollary 3. *If f: $H \to W$ is arbitrary, then except a countable set of points x on R for each sector $S \subset H$ and each $\theta \in (0, \pi)$*

$$C(f, x, S) \cap C_q(f, x, \theta) \neq \emptyset.$$

Proof. For each $S \subset H$ there are α' and β' in $(0, \pi)$ such that $S = S_{\alpha'\beta'}$. And for each $\theta \in (\alpha', \beta')$, $C(f, x, \theta) \subset C(f, x, S)$. Thus the proof is complete by Theorem 3.

Corollary 4. *If f: $H \to W$ is arbitrary, then, except a countable set of points x in R, the degeneracy of $C_q(f, x, \theta)$ for any $\theta \in (0, \pi)$ implies that the sets $C(f, x, \theta)$ have a common value for a residual set of directions $\theta \in (0, \pi)$.*

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Reçu par la Rédaction le 1985.04.01