ANNALES
POLONICI MATHEMATICI
XXIX (1974)

On some non-local boundary value problems with distributional data

by H. MARCINKOWSKA (Wrocław)

Abstract. The paper is a continuation of the author's earlier work [5]. It is well known that the homogeneous elliptic boundary value problem (with differential boundary conditions) yields a family Λ_s ($s = -1, -2, \ldots$) of homeomorphisms between suitably constructed spaces, which are subspaces of a Sobolev space or are adjoined to them. We are going to prove that the mapping Λ_s defines another boundary value problem. The differential equation is the same as in the basic problem, but the boundary conditions are the closures of integro-differential operators and the boundary data are distributions, non-vanishing in general. Similar results are obtained for homeomorphisms connected with a non-homogeneous elliptic boundary value problem.

Let us consider a properly elliptic differential operator L of order $2m$, defined in a bounded domain $\Omega \subset R_n$, covered by a normal set $B = \{B_j\}_{j=1}^m$ of boundary differential operators. All the coefficients of L, B_j and the boundary $\partial\Omega$ are assumed to be infinitely differentiable. We shall use the following notations: $D_j = \partial/\partial x_j$; $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$; L^+ — the differential operator formally adjoined to L; $B^+ = \{B_j'\}_{j=1}^m$ — a normal set of boundary differential operators adjoined to B with respect to L; m_j = order of B_j, m_j' = order of B_j' ($0 \leqslant m_j, m_j' < 2m$); l_j = order of C_j (see (25)) ($l_j + m_j' = 2m-1$); l_j' = order of C_j' (see (25)) ($l_j' + m_j = 2m-1$); $H_k(\Omega, B) = \{u \in H_k(\Omega): B_j u|_{\partial\Omega} = 0 \text{ iff } m_j < k\}$ ($k = 0, 1, 2, \ldots$); $\overset{0}{H}_k(\Omega)$ = the closure of $C_0^\infty(\Omega)$ in $H_k(\Omega)$; $N = \{u \in H_{2m}(\Omega): Lu = 0, B_j u|_{\partial\Omega} = 0$ for $j = 1, \ldots, m\}$; $N^+ = \{u \in H_{2m}(\Omega): L^+ u = 0, B_j' u|_{\partial\Omega} = 0$ for $j = 1, \ldots, m\}$; $\ominus$ — orthogonal substraction in $L_2(\Omega)$; $(,)$ — the scalar product in $L_2(\Omega)$ or its extension onto $H_k(\Omega) \times H_{-k}(\Omega)$; $\langle,\rangle$ — the scalar product in $L_2(\Omega)$ or its extension onto $H_s(\partial\Omega) \times H_{-s}(\partial\Omega)$; X^* = the dual space of X; $H_{-k}(\Omega) = \big(H_k(\Omega)\big)^*$.

It is well known [2], [4], [7] that the homogeneous boundary value problem $Lu = f$, $B_j u|_{\partial\Omega} = 0$ ($j = 1, \ldots, m$) with $f \in L_2(\Omega) \ominus N^+$ is correctly posed in the space $H_{2m}(\Omega, B) \ominus N$. This theorem may be formulated as follows: the mapping

$$\Lambda: H_{2m}(\Omega, B) \ominus N \ni u \to Lu \in L_2(\Omega) \ominus N^+$$

is a linear homeomorphism. It was proved in [1], [2] that the closure Λ_s of Λ with $s = -1, -2, \ldots$ is a homeomorphism

$$\left.\begin{array}{ll}(-2m \leqslant s < 0) & H_{2m+s}(\Omega, B)\ominus N \\ (s < -2m) & (H_{-2m-s}(\Omega)\ominus N)^*\end{array}\right\} \to (H_{-s}(\Omega, B^+)\ominus N^+)^*.$$

For $s = 0, 1, 2, \ldots$ we denote by Λ_s the restriction of Λ to $H_{2m+s}(\Omega, B)$. The author has proved in [5] that the mapping Λ_s defines a boundary value problem (non-homogeneous in general) in which the boundary data are distributions on $\partial\Omega$. The aim of the present paper is a more detailed investigation of this question. In particular, we are going to show that the boundary operators connected with Λ_s (which have been introduced in [5]) are the closures of the integro-differential operators with weakly singular kernels. In the last section we obtain similar results concerning the homeomorphisms connected with the non-homogeneous boundary value problem. For the convenience of the reader we shall recall here some results of [5].

1. Linear functionals over Sobolev spaces. The results of this section are based on the following theorem, which is the special case of a more general theorem due to L. N. Slobodetskiĭ [8].

THEOREM A. *For every $j = 0, 1, \ldots, l-1$ (l positive integer) the mapping*

$$H_l(\Omega) \ni u \to \left.\frac{\partial^j u}{\partial \nu^j}\right|_{\partial\Omega} \in H_{l-j-1/2}(\partial\Omega)$$

(where $\nu = (\nu_1, \ldots, \nu_n)$ is the normal unit vector field on $\partial\Omega$) is continuous (the boundary value is understood as the trace on $\partial\Omega$).

Conversely, for any given functions $\varphi_j \in H_{l-j-1/2}(\partial\Omega)$ ($j = 0, 1, \ldots, l-1$) one can construct an $u \in H_l(\Omega)$ which satisfies the conditions

$$\left.\frac{\partial^j u}{\partial \nu^j}\right|_{\partial\Omega} = \varphi_j \quad (j = 0, 1, \ldots, l-1)$$

and which depends continuously on φ_j.

The proof may be found in [6], [8]. The above theorem may be reformulated in the following way:

THEOREM A_1. *The linear mapping*

$$\boldsymbol{n}\colon H_k(\Omega) \ni u \to \left\{\left.\frac{\partial^j u}{\partial \nu^j}\right|_{\partial\Omega}\right\}_{j=0}^{k-1} \in \prod_{j=0}^{k-1} H_{k-j-1/2}(\partial\Omega)$$

is an epimorphism and

$$\ker \boldsymbol{n} = \overset{0}{H}_k(\Omega). \tag{1}$$

Proof. We have only to show (1). It follows from the continuity of $\boldsymbol{n}$ that $\ker \boldsymbol{n} \supset \overset{0}{H}_k(\Omega)$, thus it remains to show the converse inclusion. Every $u \in H_k(\Omega)$ may be decomposed into the sum

$$u = \overset{0}{u} + u_{\perp}$$

with $\overset{0}{u} \in \overset{0}{H}_k(\Omega)$ and $u_{\perp}$ orthogonal to $\overset{0}{H}_k(\Omega)$. If $u \in \ker \boldsymbol{n}$, so does $u_{\perp}$. But

$$(u_{\perp}, v)_k = 0 \quad (v \in \overset{0}{H}_k(\Omega)) \tag{2}$$

and for $v \in \overset{0}{H}_k(\Omega) \cap H_{2k}(\Omega)$ we can integrate by parts the left-hand side of (2), obtaining

$$(u_{\perp}, \Delta_k v) = 0, \tag{3}$$

where $\Delta_k = \sum_{|\alpha| \leqslant k} (-1)^{|\alpha|} D^{2\alpha}$. As is well known, to every $f \in L_2(\Omega)$ there may be found an $v \in \overset{0}{H}_k(\Omega) \cap H_{2k}(\Omega)$ such that

$$\Delta_k v = f.$$

So (3) yields $u_{\perp} = 0$ and therefore $u = \overset{0}{u} \in \overset{0}{H}_k(\Omega)$, which completes the proof.

COROLLARY 1. *Let $M_k(\Omega)$ be the orthogonal completion of $\overset{0}{H}_k(\Omega)$ in $H_k(\Omega)$. Then $\boldsymbol{n}$ maps isomorphically $M_k(\Omega)$ onto the product $\prod_{j=0}^{k-1} H_{k-j-1/2}(\partial\Omega)$.*

According to the orthogonal decomposition

$$H_k(\Omega) = \overset{0}{H}_k(\Omega) \oplus M_k(\Omega),$$

every linear functional f over $H_k(\Omega)$ may be uniquely represented in the form

$$f = f_{\Omega} + f_{\partial\Omega},$$

where f_{Ω} (the inner part of f) vanishes on $M_k(\Omega)$ and $f_{\partial\Omega}$ (the boundary part of f) vanishes on $\overset{0}{H}_k(\Omega)$. It follows from the Riesz theorem that f_{Ω} is a linear functional over $\overset{0}{H}_k(\Omega)$, which may be identified with a distribution in Ω of the form $\sum_{|\alpha| \leqslant k} D^{\alpha} f_{\alpha}$ with $f_{\alpha} \in L_2(\Omega)$. The boundary part $f_{\partial\Omega}$ belongs to $M_k^*(\Omega)$, so Corollary 1 yields

THEOREM B [5]. *For every $f \in H_k^*(\Omega)$ its boundary part admits the representation*

$$(\varphi, f_{\partial\Omega}) = \sum_{j=0}^{k-1} \left\langle \frac{\partial^j \varphi}{\partial \nu^j}, f_j \right\rangle \quad (\varphi \in H_k(\Omega)), \tag{4}$$

where the mapping

$$M_k^*(\Omega) \ni f_{\partial\Omega} \to \{f_j\}_{j=0}^{k-1} \in \prod_{j=0}^{k-1} H_{-k+j+1/2}(\partial\Omega)$$

is a linear homeomorphism.

Every function $f \in L_2(\Omega)$ defines a linear functional over $H_k(\Omega)$ by the formula

$$H_k(\Omega) \ni \varphi \to (\varphi, f). \tag{5}$$

Denoting this functional by this same letter f we shall study it more detailly. It follows from the definition of the interior part of a functional that

$$(\varphi, f) = (\varphi, f_\Omega)$$

for $\varphi \in C_0^\infty(\Omega)$; so the distributions f_Ω and f are equal. Let us now consider the boundary part. Applying the Riesz theorem to the space $H_k(\Omega)$ we get

$$(\varphi, f) = (\varphi, h)_k \quad (\varphi \in H_k(\Omega)) \tag{6}$$

with an $h \in H_k(\Omega)$. It follows from identity (6) that h is a solution of the generalized Neumann problem. Thus $h \in H_{2k}(\Omega)$ and

$$h = G_k f, \tag{7}$$

where G_k denotes the corresponding resolving operator (it is well known that it has an integral form on the set $C_0^\infty(\Omega)$, see e.g. [2]).

If we decompose $h = h_0 + h_1$ with $h_0 \in \overset{0}{H}_k(\Omega)$ and $h_1 \in M_k(\Omega)$, then we get

$$(\varphi, f) = (\varphi, h_0)_k \quad (\varphi \in \overset{0}{H}_k(\Omega)),$$

so h_0 is a solution of the generalized Dirichlet problem. Denoting by A_k the resolving operator (it is an integral operator in the same sense as G_k) we obtain

$$h_0 = A_k f,$$

where $h_0 \in \overset{0}{H}_k(\Omega) \cap H_{2k}(\Omega)$. So

$$(\varphi, f_{\partial\Omega}) = (\varphi, h_1)_k \quad (\varphi \in H_k(\Omega)), \tag{8}$$

where h_1 belongs to $M_k(\Omega) \cap H_{2k}(\Omega)$ and has the form

$$h_1 = (G_k - A_k) f. \tag{9}$$

For arbitrary $\varphi \in H_k(\Omega)$, $g \in H_{2k}(\Omega)$ we have the well-known Green's formula

$$(10) \qquad (\varphi, g)_k = (\varphi, \Delta_k g) + \sum_{j=0}^{k-1} \left\langle \frac{\partial^j \varphi}{\partial \nu^j}, T_j g \right\rangle,$$

where T_j is a differential operator of order $2k-j-1$. It follows from (10) that $\Delta_k g = 0$ if $g \in M_k(\Omega)$, so according to (8), (9) we obtain the following representation of the boundary part of f:

$$(11) \qquad (\varphi, f_{\partial\Omega}) = \sum_{j=0}^{k-1} \left\langle \frac{\partial^j \varphi}{\partial \nu^j}, T_j(G_k - A_k)f \right\rangle \quad (\varphi \in H_k(\Omega)).$$

Comparison with identity (4) yields

$$(12) \qquad f_j = T_j(G_k - A_k)f$$

for $f \in L_2(\Omega)$.

2. The case of a subspace defined by a normal set of boundary operators. Let e_r be the unit vector of the x_r-axis. At every fixed point $x \in \partial\Omega$ we have the decomposition

$$(13) \qquad e_r = \nu_r(x)\nu(x) + \tau_r(x),$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the normal unit vector field and τ_r is a tangent vector field (which may vanish in some points). Therefore the derivation at the point x may be decomposed in the following way:

$$(14) \qquad D_r = \nu_r(x) \frac{\partial}{\partial \nu} + \frac{\partial}{\partial \tau_r}.$$

As $\partial\Omega$ is a regular surface (it would be sufficient to assume that it is of class C^2), every point $\overset{0}{x} \in \partial\Omega$ has an n-dimmensional neighbourhood Ξ with the following property: through every point $y \in \Xi$ there goes a uniquely defined normal to $\partial\Omega$. Shifting the tangent plane along this normal we can extend in a natural way the fields ν and τ_r onto Ξ. So the decompositions (13) and (14) hold in the whole of Ξ. For to calculate the derivatives in the point $\overset{0}{x}$ of a function u, which is defined in a domain containing $\overline{\Omega}$, it is sufficient to consider only $u|_{\Xi}$. The above remarks enable us to decompose the derivation of higher order as follows:

$$(15) \qquad D^a = \left(\nu_1 \frac{\partial}{\partial \nu} + \frac{\partial}{\partial \tau_1}\right)^{a_1} \cdots \left(\nu_n \frac{\partial}{\partial \nu} + \frac{\partial}{\partial \tau_n}\right)^{a_n}.$$

An elementary calculation shows that the commutator $\left[\frac{\partial}{\partial \nu}, \frac{\partial}{\partial \tau_r}\right]$ is a differential operator containing the tangential derivatives only.

Using this fact and (15) we can bring the boundary operator

$$B_t u = \sum_{|\alpha|\leqslant m_t} b_{t,\alpha}(x) D^\alpha u$$

to the form

$$B_t u = \tilde{b}_t \frac{\partial^{m_t} u}{\partial \nu^{m_t}} + \sum_{r=0}^{m_t-1} \tilde{b}_{t,r} \frac{\partial^r u}{\partial \nu^r}, \tag{16}$$

where

$$\tilde{b}_t = \sum_{|\alpha|=m_t} b_{t,\alpha}(x)\nu^\alpha \neq 0$$

(because B_t, $t = 1, \ldots, m$, form a normal set). We shall denote in the sequel by Θ_k (with $k = 0, 1, 2, \ldots$) the class of all boundary differential operators of order at most k, involving only the tangential derivations. By Θ_{-k} we shall mean the class consisting of the identically vanishing operator only. Then it is easy to see that $\tilde{b}_{t,r} \in \Theta_{t-r}$. Thus the space $H_k(\Omega, B)$ may be characterized as the set of all functions $\varphi \in H_k(\Omega)$ which satisfy on the boundary $\partial\Omega$ the following relations:

$$\frac{\partial^r \varphi}{\partial \nu^r} = \sum_{j\in P} d_{r,j} \frac{\partial^j \varphi}{\partial \nu^j}, \tag{17}$$

where $r \in \{0, 1, \ldots, k-1\} \cap \{m_j\}_{j=1}^m$, $P = \{0, 1, \ldots, k-1\}/\{m_j\}_{j=1}^m$ and $d_{r,j} \in \Theta_{r-j}$. As in the case of the whole space $H_k(\Omega)$, the orthogonal decomposition

$$H_k(\Omega, B) = \overset{0}{H}_k(\Omega) \oplus M_k(\Omega, B)$$

yields the decomposition of a functional f over $H_k(\Omega, B)$

$$f = f_\Omega + f_{\partial\Omega},$$

where the inner part f_Ω vanishes on $M_k(\Omega, B)$ and the boundary part $f_{\partial\Omega}$ vanishes on $\overset{0}{H}_k(\Omega)$. Similarly as in Section 1, the inner part belongs to $\overset{0}{H}_{-k}(\Omega)$ and may be identified with a distribution. Let us now consider the boundary part. It follows from Theorem A_1 that the mapping

$$n|_{H_k(\Omega,B)}\colon H_k(\Omega, B) \to \prod_{j\in P} H_{k-j-1/2}(\partial\Omega)$$

is an epimorphism with the kernel $\overset{0}{H}_k(\Omega)$. This yields

THEOREM C [5]. *For every $f \in H_k^*(\Omega, B)$ its boundary part admits the representation*

$$(\varphi, f_{\partial\Omega}) = \sum_{j\in P} \left\langle \frac{\partial^j \varphi}{\partial \nu^j}, f_j \right\rangle \quad (\varphi \in H_k(\Omega, B)), \tag{18}$$

where the mapping

$$M_k^*(\Omega, B) \ni f_{\partial\Omega} \to \{f_j\}_{j\in P} \in \prod_{j\in P} H_{-k+j+1/2}(\partial\Omega)$$

is a linear homeomorphism.

In order to consider more detailly the case of a functional defined by a square integrable function f, we shall need the following lemma:

LEMMA 1. *For every $u, v \in H_2(\Omega)$ and for every tangent vector field τ of class C^1 over $\partial\Omega$ we have the formula of integration by parts*

$$\int_{\partial\Omega} \frac{\partial u}{\partial \tau} v\, d\sigma = \int_{\partial\Omega} u \frac{\partial^+ v}{\partial \tau}\, d\sigma, \tag{19}$$

where

$$\frac{\partial^+ v}{\partial \tau} = -\frac{\partial v}{\partial \tau} + pv$$

and p is an infinitely differentiable function on $\partial\Omega$.

Proof. The surface $\partial\Omega$ may be locally described by means of the equation

$$x_n = h(x') \qquad (x' = (x_1, \ldots, x_{n-1}) \in R_{n-1})$$

with an h of class C^∞ (possibly after a suitable change of the numeration of the coordinates). Let us put $s_{jk} = \delta_{jk}$ $(j, k = 1, \ldots, n-1)$ and $s_{jn} = D_j h$. Then the vectors $\sigma_j = (s_{j1}, \ldots, s_{jn})$ $(j = 1, \ldots, n-1)$ give a natural basis in the tangent plane and considering $u|_{\partial\Omega}$ as a function of local parameters x' we have

$$\frac{\partial u}{\partial \tau} = \sum_{j=1}^{n-1} t^j D_j u|_{\partial\Omega} \tag{20}$$

if $\tau = \sum_{j=1}^{n-1} t^j \sigma_j$. Using the natural Riemannian structure on $\partial\Omega$ with $g_{\alpha\beta} = \sum_{r=1}^{n} s_{\alpha r} s_{\beta r}$ we can write (20) in an invariant form

$$\frac{\partial u}{\partial \tau} = t^\alpha \nabla_\alpha u, \tag{21}$$

where ∇_α denotes the covariant derivation. Our lemma now follows immediately from the well-known formula (see e.g. [9])

$$\int_{\partial\Omega} \nabla_\alpha \gamma^\alpha\, d\sigma = 0,$$

where γ^α is a contravariant vector field of class C^1 over $\partial\Omega$. We have only to put

$$\gamma^a = uvt^a$$

with $u, v \in C^1(\bar{\Omega})$ and note that according to (21)

$$\nabla_a(uvt^a) = \frac{\partial u}{\partial \tau} v + u \frac{\partial v}{\partial \tau} + uv\nabla_a t^a.$$

This yields (19) for smooth u, v and therefore for all $u, v \in H_2(\Omega)$ if we understand the boundary value in the sence of the trace.

Remark. Obviously our proof remains valid if we replace $\partial\Omega$ by an arbitrary regular surface Σ of class C^2, assuming only one of the functions $u|_\Sigma$, $v|_\Sigma$ to have a compact support.

The above proved lemma yields now

COROLLARY 2. *Let b be a tangential differential operator of order k on $\partial\Omega$ with smooth coefficients. There exists a tangential differential operator b^+ of the same order and such that*

$$\langle bu, v\rangle = \langle u, b^+ v\rangle$$

for $u, v \in H_{k+1}(\Omega)$.

Using equalities (17) together with Corollary 2 we now obtain the following form of formula (10) valid for $\varphi \in H_k(\Omega, B)$ and $g \in H_{2k}(\Omega)$:

$$(\varphi, g)_k = (\varphi, \Delta_k g) + \sum_{j\in P} \left\langle \frac{\partial^j \varphi}{\partial \nu^j}, R_j g \right\rangle. \tag{22}$$

Here

$$R_j = T_j + \sum_{s=1}^{m} d^+_{m_s, j} T_{m_s}$$

is a boundary differential operator of order $2k-j-1$.

Now consider a function $f \in L_2(\Omega)$ and the linear functional (5) on $H_k(\Omega, B)$ defined by f. We obtain from (8), (9) and (22) the following representation of its boundary part:

$$(\varphi, f_{\partial\Omega}) = \sum_{j\in P} \left\langle \frac{\partial^j \varphi}{\partial \nu^j}, R_j(G_k - A_k)f \right\rangle \quad (\varphi \in H_k(\Omega, B)). \tag{24}$$

Comparison with (8) yields in this case

$$f_j = R_j(G_k - A_k)f.$$

3. Boundary value problem connected with the homeomorphism Λ_s. We shall apply in this section the well-known generalized Green's formula (for the proof see e.g. [4], [6]):

$$(25)\qquad (Lu, v)+\sum_{j=1}^{m}\langle B_j u, C_j' v\rangle = (u, L^+ v)+\sum_{j=1}^{m}\langle C_j u, B_j' v\rangle .$$

It is well known that the operators C_j may be arbitrarily chosen to complete the system $\{B_j\}_{j=1}^{m}$ to a Dirichlet system of order $2m$. If this choice is fixed, then $\{B_j', C_j'\}_{j=1}^{m}$ is a uniquely determined Dirichlet system of order $2m$.

Let us first consider the case $-2m \leqslant s < 0$. Denoting by Λ_s^+ the homeomorphism corresponding to the adjoined boundary value problem we get from (25)

$$(26)\qquad (\Lambda_s u, v) = (u, \Lambda_{-s-2m}^{+} v)$$

for $u \in H_{2m+s}(\Omega, B)$, $v \in H_{-s}(\Omega, B^+)$, thus the equality

$$(27)\qquad \Lambda_s u = f$$

is equivalent to

$$(28)\qquad (\Lambda_{-s-2m}^{+} v, u) = (v, f) \qquad (v \in H_{-s}(\Omega, B^+)).$$

Splitting the functional $\Lambda_s u$ into the interior and the boundary part we see that (27) holds iff u is a solution of the following boundary value problem:

$$(29)\qquad Lu = f_\Omega ,$$

$$(30)\qquad B_j u|_{\partial\Omega} = 0 \quad \text{iff } m_j < 2m+s,$$

$$(31)\qquad Q_j u = f_j \quad (j \in I).$$

Here $I = \{0, 1, \ldots, -s-1\}\setminus\{m_j'\}_{j=1}^{m}$ and $Q_j u$ are defined by $\Lambda_s u$ according to Theorem C, so we have

$$(32)\qquad \big(\varphi, (\Lambda_s u)_{\partial\Omega}\big) = \sum_{j \in I}\left\langle \frac{\partial^j \varphi}{\partial \nu^j}, Q_j u\right\rangle \quad (\varphi \in H_{-s}(\Omega, B^+)).$$

It was remarked in [5] that the boundary value problem (29)–(31) is well posed in $H_{2m+s}(\Omega)\ominus N$ for arbitrarily given $f_\Omega \in \overset{0}{H}_s(\Omega)$ and $f_j \in H_{s+j+1/2}(\partial\Omega)$ satisfying the orthogonality condition

$$(\varphi, f_\Omega)+\sum_{j \in I}\left\langle \frac{\partial^j \varphi}{\partial \nu^j}, f_j\right\rangle = 0 \quad (\varphi \in N^+).$$

Using the fact that $m_j'+l_j = 2m-1$ one can easily calculate that there are m boundary conditions (30), (31). For $u \in H_{2m}(\Omega, B)$ we have

$\Lambda_s u = Lu \in L_2(\Omega)$ and it follows from the investigations of Section 2 that

$$(33)\qquad Q_j u = R_j(G_{-s} - \Lambda_{-s})Lu \quad (u \in H_{2m}(\Omega, B);\ j \in I)$$

in this case. Thus the closure of a homogeneous elliptic boundary value problem yields another boundary value problem with the same differential equation but with other boundary conditions.

According to (31), (33) the boundary operators are the closures of integro-differential expressions and the boundary data are distributions on $\partial\Omega$ non vanishing in general.

Let us pass to the case $s < -2m$. Formula (26) now has the form

$$(34)\qquad (\Lambda_s u, v) = (u, L^+ v) \quad (u \in H_{2m+s}(\Omega, B),\ v \in H_{-s}(\Omega, B^+)).$$

Splitting the functional $\Lambda_s u$ into its interior and boundary part we can define the operators Q_j by formula (32) just in this same manner as in the preceding case. We shall study them more detailly. Note that u is also a functional as well, so it can be split

$$u = u_\Omega + u_{\partial\Omega};$$

this yields

$$(35)\qquad Q_j u = Q_j u_\Omega + Q_j u_{\partial\Omega} \quad (j \in I)$$

and

$$(36)\qquad \Lambda_s u = \Lambda_s u_\Omega + \Lambda_s u_{\partial\Omega}.$$

Replacing in (34) u by $u_{\partial\Omega}$ we obtain

$$(37)\qquad (L^+\varphi, u_{\partial\Omega}) = (\varphi, \Lambda_s u_{\partial\Omega}) \quad (\varphi \in H_{-s}(\Omega, B^+))$$

and it is easy to see from the last identity that the functional $\Lambda_s u_{\partial\Omega}$ has a vanishing interior part. The study of its boundary part give us some information about the operators Q_j acting on $u_{\partial\Omega}$.

The comparison of (37) and (32) with u replaced by $u_{\partial\Omega}$ yields, according to Theorem B

$$(38)\qquad \sum_{j=0}^{-s-2m-1} \left\langle \frac{\partial^j}{\partial\nu^j} L^+\varphi, u_j \right\rangle = \sum_{t \in K} \left\langle \frac{\partial^t \varphi}{\partial\nu^t}, Q_t u_{\partial\Omega} \right\rangle + \\ + \sum_{t=2m}^{-s-1} \left\langle \frac{\partial^t \varphi}{\partial\nu^t}, Q_t u_{\partial\Omega} \right\rangle \quad (\varphi \in H_{-s}(\Omega, B^+)),$$

where $K = \{0, 1, \ldots, 2m-1\} \setminus \{m_j'\}_{j=1}$.

Writing

$$L\varphi = \sum_{|\alpha|\leqslant 2m} a_\alpha(x) D^\alpha \varphi$$

and using the decomposition (15) we can bring the differential expression $\frac{\partial^j}{\partial\nu^j} L^+\varphi|_{\partial\Omega}$ to the form

$$\frac{\partial^j}{\partial\nu^j} L^+\varphi = \sum_{t=0}^{2m+j} b_{t,j} \frac{\partial^t \varphi}{\partial\nu^t}, \tag{39}$$

where $b_{t,j} \in \Theta_{2m+j-t}$ and

$$b_{2m+j,j} = \sum_{|\alpha|=2m} \bar{a}_\alpha \nu^\alpha \tag{39'}$$

(so $b_{2m+j,j}$ does not vanish, according to the ellipticity of L).

Using (39), Corollary 2 and relations (7) with P replaced by K we can transform (38) as follows:

$$\sum_{t\in K} \left\langle \frac{\partial^t\varphi}{\partial\nu^t}, \sum_{j=0}^{-s-2m-1} p_{t,j} u_j \right\rangle + \sum_{t=2m}^{-s-1} \left\langle \frac{\partial^t\varphi}{\partial\nu^t}, \sum_{j=t-2m}^{-s-2m-1} b_{t,j}^+ u_j \right\rangle \tag{40}$$

$$= \sum_{t\in K} \left\langle \frac{\partial^t\varphi}{\partial\nu^t}, Q_t u_{\partial\Omega} \right\rangle + \sum_{t=2m}^{-s-1} \left\langle \frac{\partial^t\varphi}{\partial\nu^t}, Q_t u_{\partial\Omega} \right\rangle \quad (\varphi \in H_{-s}(\Omega, B^+)),$$

where

$$p_{t,j} = b_{t,j}^+ + \sum_{s=1}^{m} d_{m_s',t}^+ b_{m_s',j}^+,$$

so $p_{t,j} \in \Theta_{2m+j-t}$. According to Theorem A the derivative $\partial^t\varphi/\partial\nu^t$ may be an arbitrary function from $H_{-s-t-1/2}(\partial\Omega)$, therefore (40) is equivalent to the following relations:

$$Q_t u_{\partial\Omega} = \sum_{j=0}^{-s-2m-1} p_{t,j} u_j \quad (t \in K) \tag{41}$$

and

$$Q_t u_{\partial\Omega} = \sum_{j=t-2m}^{-s-2m-1} b_{t,j}^+ u_j \quad (t = 2m, \ldots, -s-1). \tag{42}$$

We have thus proved that the application of the operator Q_t to the boundary part of u may be described by means of suitably defined tangential derivations applied to the "components" of $u_{\partial\Omega}$ defined by Theorem C.

Note that according to (39′)

$$b^{+}_{t,t-2m} = \sum_{|\alpha|=2m} a_\alpha \nu^\alpha \neq 0$$

and therefore the system of equations (42) may be solved with respect to u_j. We obtain in this way

$$(43) \qquad u_j = \sum_{r=j+2m}^{-s-1} q_{j,r} Q_r u_{\partial\Omega} \qquad (j = 0, \ldots, -s-2m-1),$$

where $q_{j,r} \in \Theta_{r-j-2m}$ and substituting (43) into (41) we get

$$(44) \qquad Q_t u_{\partial\Omega} = \sum_{r=2m}^{-s-1} \tau_{t,r} Q_r u_{\partial\Omega} \qquad (t \in K)$$

with $\tau_{t,r} = \sum_{j=0}^{r-2m} p_{t,j} q_{j,r} \in \Theta_{r-t}$. Using the fact that according to (35)

$$Q_j u_{\partial\Omega} = Q_j u - Q_j u_\Omega,$$

we obtain from (44) the following statement:

THEOREM 1. *If $\Lambda_s u = f$ (with $u \in H_{2m+s}(\Omega)$, $s < -2m$), then the interior part u_Ω is a solution of the boundary value problem*

$$(45) \qquad Lu_\Omega = f_\Omega,$$

$$(46) \qquad Q_t u_\Omega = h_t \qquad (t \in K)$$

with

$$h_t = \sum_{r=2m}^{-s-1} \tau_{t,r} f_r - f_t$$

(so $h_t \in H_{s+t+1/2}(\partial\Omega)$).

Note that for $u \in H_{2m}(\Omega, B)$ we have $f = Lu \in L_2(\Omega)$ and $u_\Omega = u$.

According to what we have proved in Section 2, the boundary value problem now gets the following form:

$$Lu = f,$$

$$Q_t u = h_t \qquad (t \in K),$$

where

$$Q_t u = \sum_{r=2m}^{-s-1} \tau_{t,r} R_r (G_{-s} - A_{-s}) Lu - R_t (G_{-s} - A_{-s}) Lu$$

– so Q_t has an integro-differential form in such a case.

4. The case of non-homogeneous boundary conditions. It was proved in [3] that the non-homogeneous elliptic boundary value problem

$$Lu = f,$$

$$B_j u|_{\partial\Omega} = g_j \qquad (j = 1, \ldots, m)$$

yields a family of homeomorphisms between suitably constructed product spaces. Assuming we have fixed the operators C_j occurring in Green's formula (25) let us introduce the following notations:

$$K_t(\Omega) = \begin{cases} \{U = (u, C_1u, \ldots, C_m u):\ u \in H_t(\Omega)\} \text{ for } t \geqslant 2m \\ \text{the closure of } K_{2m}(\Omega) \text{ in the space} \\ H_t(\Omega) \times \prod_{j=0}^{m} H_{t-l_j-1/2}(\partial\Omega) \text{ for } t < 2m, \end{cases}$$

$K_t^+(\Omega)$ — the analogue of $K_t(\Omega)$ with C_j replaced by C_j'; $\ominus$, $(\ ,\ )$ — the orthogonal substraction and the scalar product in $L_2(\Omega) \times \underbrace{L_2(\partial\Omega) \times \ldots \times L_2(\partial\Omega)}_{m}$ (or its closure if necessary).

According to Theorem 1 of [3] the closure $\mathscr{L}_s$ $(s = -1, -2, \ldots)$ of the mapping

$$\mathscr{L}:\ (u, C_1u, \ldots, C_m u) \to (Lu, B_1u, \ldots, B_m u) \qquad (u \in H_{2m}(\Omega))$$

is a homeomorphism

$$\mathscr{L}_s:\ K_{2m+s}(\Omega) \ominus \mathscr{N} \to (K_{-s}^+(\Omega) \ominus \mathscr{N}^+)^*,$$

where $\mathscr{N} = \ker \mathscr{L}$ and $\mathscr{N}^+ = \ker \mathscr{L}^+$ with

$$\mathscr{L}^+:\ (v, C_1'v, \ldots, C_m'v) \to (L^+v, B_1'v, \ldots, B_m'v) \qquad (v \in H_{2m}(\Omega)).$$

For $s = 0, 1, \ldots$ we denote by $\mathscr{L}_s$ the restriction of $\mathscr{L}$ to $K_{2m+s}(\Omega)$.

Denoting by $\mathscr{L}_t^+$ the closure of the operator $\mathscr{L}^+$ considered as the mapping

$$\mathscr{L}_t^+:\ K_{2m+t}^+(\Omega) \to \big(K_{-t}(\Omega)\big)^*$$

we can write Green's formula in the following form (see [3]):

$$(\mathscr{L}_{-2m-s}^+ \Phi, U) = (\Phi, \mathscr{L}_s U) \tag{46'}$$

valid for arbitrary $\Phi \in K_{-s}^+(\Omega)$, $U \in K_{2m+s}(\Omega)$.

We can assume without loss of generality that the numbers l_j and l_j' $(j = 1, \ldots, m)$ form two increasing sequences. For fixed t let p be the greatest number such that

$$l_j < t \quad \text{for } j = 1, \ldots, p \tag{47}$$

(if (47) is not satisfied for any j we put $p = 0$). Let us denote

$$\mathscr{H}_t(\Omega) = \begin{cases} \{(u, C_1u, \ldots, C_p u):\ u \in H_t(\Omega)\} & \text{if } 1 \leqslant p \leqslant m, \\ H_t(\Omega) & \text{if } p = 0. \end{cases}$$

We denote by $\mathscr{H}_t^+(\Omega)$ the analogue of $\mathscr{H}_t(\Omega)$ with C_j, p replaced by C_j', q respectively. It was proved in [5] that

$$K_t(\Omega) = \mathscr{H}_t(\Omega) \times \prod_{j=p+1}^{m} H_{t-l_j-1/2}(\partial\Omega) \tag{48}$$

and similarly

$$K_t^+(\Omega) = \mathscr{H}_t^+(\Omega) \times \prod_{j=q+1}^{m} H_{t-l_j'-1/2}(\partial\Omega). \tag{49}$$

Obviously the mapping

$$\mathscr{H}_t(\Omega) \ni (u, C_1 u, \ldots, C_p u) \to u \in H_t(\Omega) \qquad (1 \leqslant p \leqslant m)$$

is a linear homeomorphism, so the orthogonal decomposition

$$H_t(\Omega) = \overset{0}{H}_t(\Omega) \oplus M_t(\Omega)$$

yields the topological decomposition

$$\mathscr{H}_t(\Omega) = \mathscr{H}_t^0(\Omega) + \mathscr{M}_t(\Omega), \tag{50}$$

where $\mathscr{H}_t^0(\Omega)$ consists of all vectors of the form $(u, 0, \ldots, 0)$ with $u \in \overset{0}{H}_t(\Omega)$ and

$$\mathscr{M}_t(\Omega) = \{(u, C_1 u, \ldots, C_p u):\ u \in M_t(\Omega)\}.$$

Therefore every linear functional F over $K_t^+(\Omega)$ can be uniquely decomposed into the sum

$$F = F_\Omega + F_{\partial\Omega},$$

where F_Ω (the interior part) vanishes on $\mathscr{M}_t^+(\Omega) \times \prod_{j=q+1}^{m} H_{t-l_j'-1/2}(\partial\Omega)$ and $F_{\partial\Omega}$ (the boundary part) vanishes on $\mathscr{H}_t^0(\Omega) \times \underbrace{(0, \ldots, 0)}_{m-q}$. Obviously F_Ω is in $\overset{0}{H}{}_t^*(\Omega)$, so it may be identified with a distribution on Ω.

According to what has been precedingly proved $F_{\partial\Omega}$ acts on $\Phi = (\varphi, C_1'\varphi, \ldots, C_q'\varphi, \varphi_{q+1}, \ldots, \varphi_m) \in K_t^+(\Omega)$ as follows:

$$(\Phi, F_{\partial\Omega}) = \sum_{k=0}^{t-1} \left\langle \frac{\partial^k \varphi}{\partial \nu^k}, f_k \right\rangle + \sum_{j=q+1}^{m} \langle \varphi_j, g_j \rangle \tag{51}$$

and the mapping

$$F_{\partial\Omega} \to (f_0, \ldots, f_{t-1}, g_{q+1}, \ldots, g_m)$$

is a linear homeomorphism of $\big(\mathscr{M}_t^+(\Omega) \times \prod_{j=q+1}^{m} H_{t-l_j'-1/2}(\partial\Omega)\big)^*$ onto the product $\prod_{k=0}^{t-1} H_{-t+k+1/2}(\partial\Omega) \times \prod_{j=q+1}^{m} H_{-t+l_j'+1/2}(\partial\Omega)$.

Every vector $(v, v_1, \ldots, v_m)$ with $v \in L_2(\Omega)$, $v_j \in L_2(\partial\Omega)$ $(j = 1, \ldots, m)$ defines a linear functional V over $K_t^+(\Omega)$ by the formula

$$(\Phi, V) = (\varphi, v) + \sum_{j=1}^{q} \langle C_j'\varphi, v_j \rangle + \sum_{j=q+1}^{m} \langle \varphi_j, v_j \rangle. \tag{52}$$

If one brings C'_j to the form

$$(53) \qquad C'_j\varphi = \sum_{r=0}^{t-1} c_{j,r}\frac{\partial^r\varphi}{\partial\nu^r},$$

where $c_{j,r}\in\Theta_{l'_j-r}$, then making use of Corollary 2 and (11), (12) we obtain

$$(54) \qquad f_k = T_k(G_t-A_t)v+\sum_{j=1}^{q}c^+_{j,k}v_j \quad (k = 0, 1, \ldots, t-1).$$

For $F = \mathscr{L}_s U$ with $U\in K_{2m+s}(\Omega)$ one obtains from (51) the boundary operators $S_k U \overset{\mathrm{df}}{=} f_k$ introduced in [5]. In particular, when $U = (u, C_1u, \ldots \ldots, C_m u)$ with $u\in H_{2m}(\Omega)$, then $F = (Lu, B_1u, \ldots, B_m u)$. Then the application of identity (54) with $v = Lu$, $v_j = B_j u$ yields

$$(55) \qquad S_k U = T_k(G_t-A_t)Lu+\sum_{j=1}^{p}c^+_{j,k}B_ju.$$

Identity (55) shows that the operators S_k have on a dense subset an integro-differential form. We shall study them more detailly.

Let us first suppose $-2m\leqslant s<0$. It follows from (48) that every vector $U\in K_{2m+s}(\Omega)$ may be uniquely written as the sum

$$(55') \qquad U = U_\Omega+U_{\partial\Omega}$$

with the inner part

$$U_\Omega = (u, C_1u, \ldots, C_pu, \underbrace{0, \ldots, 0}_{m-p})$$

and the boundary part

$$U_{\partial\Omega} = (\underbrace{0, \ldots, 0}_{p+1}, u_{p+1}, \ldots, u_m),$$

where $u\in H_{2r+s}(\Omega)$ and $u_j\in H_{2m+s-l_j-1/2}(\partial\Omega)$ for $j = p+1, \ldots, m$.

Puting $U_{\partial\Omega}$ in the place of U in formula (46′) and using (51) with $F = \mathscr{L}_s U_{\partial\Omega}$ we obtain

$$(55'') \qquad \sum_{j=p+1}^{m}\langle B'_j\varphi, u_j\rangle = \sum_{j=0}^{-s-1}\left\langle\frac{\partial^j\varphi}{\partial\nu^j}, S_jU_{\partial\Omega}\right\rangle \quad (\varphi\in H_{-s}(\Omega))$$

(note that $m'_j<-s$ for $j>p$ and $B_ju = g_j = 0$ for $j>q$).

The differential expressions $B'_j\varphi$ may be written in the form analogous to (53) with $e_{j,r}$ instead of $c_{j,r}$ and $t = -s$. As the tangential derivation may be shifted to the second member of $\langle,\rangle$ (Corollary 2) and the normal derivatives of φ may be arbitrarily chosen, we obtain from (55)

$$S_r U_{\partial\Omega} = \sum_{j=p+1}^{m} e_{j,r}^{+} u_j \quad (r = 0, 1, \ldots, -s-1). \tag{56}$$

So the operators S_r are acting on the boundary part of U by means of tangential derivations (obviously in the distributional sense).

Since $e_{j,r}^{+}$ with $r = m_j'$ is a non-vanishing function, we can solve $(m-p)$-equations (56) with respect to u_j obtaining the relations

$$u_j = \sum_{k=p+1}^{m} \sigma_{j,m_k'} S_{m_k'} U_{\partial\Omega} \quad (j = p+1, \ldots, m) \tag{57}$$

with $\sigma_{j,m_k'}$ vanishing for $j < k$. Making use of (57) we can eliminate the distributions u_j in the remaining equations (56). So we obtain

$$S_r U_{\partial\Omega} = \sum_{k=p+1}^{m} \tau_{k,r} S_{m_k'} U_{\partial\Omega} \tag{58}$$

for $r \in \{0, 1, \ldots, -s-1\} \setminus \{m_j'\}_{j=1}^{m}$, where $\tau_{k,r} \in \Theta_{m_k'-r}$.

Since

$$S_r U_{\partial\Omega} = S_r U - S_r U_{\Omega}, \tag{58'}$$

equalities (58) may be written in the form

$$S_r U_{\Omega} - \sum_{k=p+1}^{m} \tau_{k,r} S_{m_k'} U_{\Omega} = S_r U - \sum_{k=p+1}^{m} \tau_{k,r} S_{m_k'} U. \tag{59}$$

The left-hand side of (59) is a boundary operator acting on u; we shall denote it by $\mathscr{S}_r u$. Comparing the right-hand side with (55) we can see, that $\mathscr{S}_r u$ has an integro-differential form for $u \in H_{2m}(\Omega)$. If we put $F = \mathscr{L}_s U$ and $-s = t$ in (51), then it follows from the definition of the operator $\mathscr{L}_s$ that

$$g_j = B_j u \quad (j = q+1, \ldots, m).$$

Thus we have the following

THEOREM 2. *Suppose $-2m \leqslant s < 0$. If $\mathscr{L}_s U = F$, then u is a solution of the following boundary value problem:*

$$Lu = F_{\Omega}, \tag{60}$$

$$B_j u|_{\partial\Omega} = g_j \quad (j = q+1, \ldots, m), \tag{61}$$

$$\mathscr{S}_r u = h_r \quad (r \in \{0, 1, \ldots, -s-1\} \setminus \{m_j'\}_{j=1}^{m}), \tag{62}$$

where $h_r = f_r - \sum_{k=p+1}^{m} \tau_{k,r} f_{m_k'}$ and the distributions f_j, g_j are given by (51).

The case $s < -2m$ may be treated in a quite similar manner. For arbitrary $U = (u, u_1, \ldots, u_m) \in K_{2m+s}(\Omega)$ the first component u is a linear functional over $H_{-2m-s}(\Omega)$ and equality (55′) holds with

$$U_\Omega = (u_\Omega, \underbrace{0, \ldots, 0}_{m})$$

and

$$U_{\partial\Omega} = (u_{\partial\Omega}, u_1, \ldots, u_m).$$

Formulae (46′) and (51) yield for $\varphi \in M_{-s}(\Omega)$

$$(L^+\varphi, u_{\partial\Omega}) + \sum_{j=1}^{m} \langle B_j'\varphi, u_j\rangle = \sum_{j=0}^{-s-1} \left\langle \frac{\partial^j \varphi}{\partial\nu^j}, S_j U_{\partial\Omega}\right\rangle. \tag{63}$$

According to Theorem B we have

$$(L^+\varphi, u_{\partial\Omega}) = \sum_{j=0}^{-2m-s-1} \left\langle \frac{\partial^j}{\partial\nu^j} L^+\varphi, v_j\right\rangle \quad (\varphi \in M_{-s}(\Omega)) \tag{64}$$

with v_j uniquely defined by u. Puting (39) in (64) and making use of the fact, that $\dfrac{\partial^j\varphi}{\partial\nu^j}$ may be an arbitrary function from $H_{-s-j-1/2}(\partial\Omega)$, we obtain from (63), (64)

$$S_t U_{\partial\Omega} = \sum_{j=0}^{-2m-s-1} b_{t,j}^+ v_j + \sum_{k=1}^{m} e_{k,t}^+ u_k \quad (t = 0, 1, \ldots, -s-1). \tag{65}$$

Since $b_{t,t-2m}^+$ and $e_{k,m_k'}^+$ are non-vanishing functions, one can solve $-s-m$ equations of (65) with respect to v_j, u_k obtaining in this was the relations

$$v_j = \sum_{r=j+2m}^{-s-1} \eta_{j,r} S_r U_{\partial\Omega} \quad (j = 0, 1, \ldots, -2m-s-1) \tag{66}$$

and

$$u_k = \sum_{j=1}^{k} \gamma_{j,k} S_{m_j'} U_{\partial\Omega} + \sum_{r=2m}^{-s-1} \delta_{r,k} S_r U_{\partial\Omega} \quad (k = 1, \ldots, m), \tag{67}$$

where $\eta_{j,r} \in \Theta_{r-j-2m}$, $\gamma_{j,k} \in \Theta_{m_j'-m_k'}$, and $\delta_{r,k} \in \Theta_{r-m_k'}$.

Substituting (66), (67) into the remaining equations (65) we get

$$\sum_{j=1}^{m} a_{j,t} S_{m_j'} U_{\partial\Omega} + \sum_{r=2m}^{-s-1} \beta_{r,t} S_r U_{\partial\Omega} = S_t U_{\partial\Omega} \tag{68}$$

$$(t \in \{0, 1, \ldots, -2m-1\} \setminus \{m_j'\}_{j=1}^m),$$

where $a_{j,t} \in \Theta_{m_j'-t}$ and $\beta_{r,t} \in \Theta_{r-t}$. Using (58′) we can write (68) in the form

$$(69) \qquad \mathscr{S}_t u_\Omega = \sum_{j=1}^{m} a_{j,t} S_{m_j'} U + \sum_{r=2m}^{-s-1} \beta_{r,t} S_r U - S_t U,$$

where $\mathscr{S}_t u_\Omega$ denotes the right-hand side of (69) with $S_k U$ replaced by $S_k U_\Omega$. Equality (69) yields now:

THEOREM 3. *Suppose $s < -2m$. If $\mathscr{L}_s U = F$, then u_Ω is a solution of the boundary value problem*

$$(70) \qquad Lu_\Omega = F_\Omega,$$

$$(71) \qquad \mathscr{S}_t u_\Omega = h_t \quad (t \in \{0, 1, \ldots, 2m-1\} \setminus \{m_j'\}_{j=1}^m),$$

where

$$h_t = \sum_{j=1}^{m} a_{j,t} f_{m_j'} + \sum_{r=2m}^{-s-1} \beta_{r,t} f_r - f_t$$

and the distributions f_t are given by (51).

Note that for $u \in H_{2m}(\Omega)$ we have $u_\Omega = u$ and the left-hand side of (71) has an integro-differential form.

References

[1] Ю. М. Березанский, С. Г. Крейн, Я. А. Ройтберг, *Теорема о гомеоморфизмах и локальное повичшение гладкости вплоть до границы решений эллиптических уравнений*, ДАН 148 (1963), p. 745–748.

[2] Ю. М. Березанский, *Разложение по собственным функциям самосопряженных операторов*, Киев 1965.

[3] Ю. М. Березанский, Я. А. Ройтберг, *Теорема о гомеоморфизмах и функция Грина для общих эллиптических граничных задач*, УМЖ 19 (1967), p. 3–31.

[4] J. L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, vol. 1, Paris 1968.

[5] Г. Марцинковска, *Линейные функционалы над соболевскими пространствами и граничные задачи, порожденные теоремами о гомеоморфизмах*, УМЖ 21 (1969), p. 610–626.

[6] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Prague 1967.

[7] M. Schechter, *General boundary value problems for elliptic partial differential equations*, Comm. Pure Appl. Math. 12 (1959), p. 457–486.

[8] Л. Н. Слободецкий, *Обобщенные пространства С. Л. Соболева и их приложения к краевым задачам для дифференциальных уравнений в частных производных*, Уч. зап. ЛГУ 197 (1958), p. 54–112.

[9] G. De Rham, *Variétés differentiables*, Paris 1955.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF MATHEMATICS

Reçu par la Rédaction le 17. 2. 1973