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On the inner parts of certain analytic functions

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1. Introduction. It is a classical theorem that a bounded analytic function on the open unit disk has an essentially unique factorization into an inner part and an outer part ([3], p. 63). The inner part further factors into a Blaschke product and a singular function ([3], p. 66). More generally, if we consider bounded analytic functions on a finite open Riemann surface R, a similar factorization theorem is to be expected since the open unit disk is the universal covering space for R. But this procedure involves the use of the so-called multiple-valued functions [7]. We have recently proved in [4] the existence and the essential uniqueness of Blaschke products for R without using the above techniques. Our method was based on the H^p-space theory developed in [1]. If X is the boundary of R, let H^∞ denote the space of non-tangential boundary values on X of bounded analytic functions on R. We give in Section 2 an intrinsic factorization of functions in H^∞.

If f is a function in H^∞ and a is a complex number, $f-a$ is in H^∞ and we have

$$(*) \qquad f-a = B_a S_a F_a,$$

where B_a is a Blaschke product, S_a is a singular function and F_a is an outer function. If $\hat{f}$ is the bounded analytic function on R whose non-tangential boundary values are equal to f, it is clear that the factor B_a appears in $(*)$ if and only if a belongs to the range of $\hat{f}$. The main part of this paper, Section 3, deals with conditions on a under which the factor S_a is absent in $(*)$. We show [Theorem 3.4] that if $f(x)$ misses, for almost every point x in X, a σ-compact set V in C, then for every point a in V, except for a subset of logarithmic capacity zero, the factor S_a is missing in $(*)$. The method of proof is adopted from a similar classical result proved by Frostman in [2], p. 111–112.

Now, if f is bounded away from a point a_0 in C, there certainly exists a sequence $(a_n)_n$ converging to a_0 such that S_{a_n} is absent in $(*)$. Thus, $f-a_n/F_{a_n}$ is a Blaschke product for each n. One would expect that the outer parts F_a would vary continuously with a, i.e. if a_n tends to a_0, then $\hat{F}_{a_n}$ would converge to $\hat{F}_{a_0}$ uniformly on R. This, in particular, would generalize to the case of a finite open Riemann surface Frostman's theorem for the unit disk, stating that every inner function is a uniform limit of Blaschke products. Although we are unable to prove this, we can show that if f is any inner function, there exists a sequence $(B_n)_n$ of Blaschke products such that $(\hat{B}_n)_n$ converges to $\hat{f}$ uniformly on compact subsets of R and $(|\hat{B}_n|)_n$ converges to $|\hat{f}|$ uniformly on R [Theorem 4.1].

2. The factorization theorem. Let δ be the first Betti number of the finite open Riemann surface R, and $\{\gamma_1, \ldots, \gamma_\sigma\}$ a homology basis for the closed paths in R. Then there exist nowhere-vanishing functions ([8], Lemma 1) $Z_1, \ldots, Z_\sigma$ which are analytic on R and can be extended analytically across the boundary X of R such that

$$\frac{1}{2\pi} \int_{\gamma_j} {}^*d(\log|Z_k|) = \delta_{j,k} \quad \text{for } 1 \leqslant j, k \leqslant \sigma.$$

Fix a point z_0 in R and consider the harmonic measure m on X with respect to z_0. Then a function f in $L^\infty(dm)$ belongs to H^∞ iff there exists a bounded analytic function $\hat{f}$ on R whose non-tangential boundary values are equal to f. Following [1], we call a function f in H^∞ an inner function if there exist real numbers $a_1, \ldots, a_\sigma$ such that $|f| = |Z_1|^{a_1} \ldots |Z_\sigma|^{a_\sigma}$, a.e. dm on X. In the classical case, where R is the open unit disk and $\sigma = 0$, this becomes $|f| = 1$ a.e. $d\theta$ on the unit circle.

Let $(a_n)_n$ be a sequence of points in R. In [4], we defined a Blaschke product for R with respect to the sequence $(a_n)_n$ as an inner function B such that $\hat{B}$ has zeros at $(a_n)_n$ and B is minimal in the following sense: if f is any function in H^∞ such that $\hat{f}$ has zeros at $(a_n)_n$, then B divides f in H^∞. Now, if f is a function in H^∞, the zeros of $\hat{f}$ on R can be factored out with the help of an essentially unique Blaschke product and we are left with a function g in H^∞ such that $\hat{g}$ has no zeros on R. Since $Z_1, \ldots, Z_\sigma$ constitute a basis of the multiplicative group of the nowhere vanishing analytic functions on R modulo the subgroup of the exponentials, g differs from an exponential e^{-h} by a factor of $Z_1^{m_1} \ldots Z_\sigma^{m_\sigma}$, where $m_1, \ldots, m_\sigma$ are integers. The real part u of the function h in the exponent can be taken to be non-negative and has an integral representation

$$u(z) = \int_X P(x, z)\, d\mu(x)/P(x, z_0),$$

where $P(x, z)$ is the normal derivative at x of the Green function for R with singularity at z, and μ is a non-negative finite Baire measure on X determined by u. In order to represent the analytic function h, one needs an analog of the complex Poisson kernel. The difficulty lies in the fact that, for a fixed point x in X, the harmonic function $P(x, z)$ may not have a (single-valued) conjugate on R. For this reason, let

$$\lambda_j(x) = \frac{1}{2\pi} \int_{\gamma_j} {}^*dP(x, \cdot) \quad \text{for } 1 \leqslant j \leqslant \sigma,$$

and consider, for a fixed x in X, the harmonic function

$$K(x, z) = P(x, z) - \sum_{j=1}^{\sigma} \lambda_j(x) \log |Z_j(z)|.$$

Then the analytic function $H(x, z)$ whose real part is $K(x, z)$ and which is real at z_0 is a suitable analog of the complex Poisson kernel. Thus we obtain

$$h(z) = \int_X H(x, z)\, d\mu(x)/P(x, z_0).$$

Let $\mathfrak{M}$ denote the set of all finite signed Baire measures μ' on X such that

$$\int_X \lambda_j(x)\, d\mu'(x)/P(x, z_0) = 0 \quad \text{for } 1 \leqslant j \leqslant \sigma.$$

The above measure μ above belongs to $\mathfrak{M}$ and has the Lebesgue decomposition into absolutely continuous and singular parts: $\mu = \mu_a + \mu_s$. We shall now give names to the analytic functions represented by μ_a and μ_s.

DEFINITION 2.1. (i) A function f in H^∞ is called an *outer function* if

$$\log |\hat{f}(z_0)| = \int_X \log |f|\, dm.$$

(ii) An inner function S is called a *singular function* if $\hat{S}$ has no zeros on R.

It can be seen fairly easily that there exist real numbers $\beta_1, \ldots, \beta_\sigma$ such that $d\mu_s + \big(\sum_{j=1}^{\sigma} \beta_j \log |Z_j|\big) dm$ belongs to $\mathfrak{M}$ and represents a singular function and that μ_a represents an outer function. We thus get the following factorization theorem.

THEOREM 2.2. *Let f be a function in H^∞. Then*

$$f = wF, \quad w = BS,$$

where w is an inner function;

B is a Blaschke product,

$$|B| = |Z_1|^{\alpha_1} \ldots |Z_\sigma|^{\alpha_\sigma}, \quad \textit{a.e. } dm \textit{ on } X;$$

S is a singular function,

$$\hat{S}(z) = \exp - \int_X H(x, z) \Big[\Big(\sum_{j=1}^{\sigma} \beta_j \log |Z_j|\Big)\, dm + d\mu_s\Big]/P(x, z_0);$$

F is an outer function,

$$\hat{F}(z) = \exp \int_X H(x, z) \Big[\sum_{j=1}^{\sigma} (\beta_j - a_j) \log |Z_j| + \log |f| \Big] \, dm / P(x, z_0),$$

where $\mu_s \geqslant 0$,

$$\Big(\sum_{j=1}^{\sigma} \beta_j \log |Z_j| \Big) \, dm + d\mu_s \quad \textit{and} \quad \Big(\sum_{j=1}^{\sigma} (\beta_j - a_j) \log |Z_j| + \log |f| \Big) \, dm$$

belong to $\mathfrak{M}$ *and*

$$\arg \big(\hat{f} / \hat{B}\hat{S}(z_0) \big) = \arg \big(\hat{f} / \hat{B}(z_0) \big) = 0.$$

The singular measure μ_s and the real numbers a_j and β_j, $j = 1, \ldots, \sigma$, are determined by f.

3. Absence of the singular factor. Let f be a function in H^∞ and a a complex number. Let $f - a = B_a S_a F_a$, as in Theorem 2.2. We would like to find conditions on a which imply the absence of the factor S_a. Clearly, if $f - a$ were an outer function, S_a would be equal to 1. Let W be the range of f in C, and ∂W its boundary. If a were not in $W \cup \partial W$, $-a$ would be an invertible function in H^∞ and, since we can use Jensen's inequality both ways, this would imply that $f - a$ is outer. If a belongs to W, certainly $f - a$ would not be outer. Thus, it is interesting to find points a in ∂W for which $f - a$ is outer.

DEFINITION 3.1. Let E be a subset of C and a a point in C. Then E is said to *omit a half-plane near a* if there exist a disk D with centre at a and a diameter of D such that $E \cap D$ lies on only one side of the diameter.

PROPOSITION 3.2. *Let f be a function in H^∞, W the range of $\hat{f}$ and a a point in ∂W. If W omits a half-plane near a, then $f - a$ is outer, and hence S_a is absent.*

Proof. We can assume without loss of generality that $a = 0$ and that there exists a $\delta > 0$ such that $W \cap \{z \in C, \ |z| < \delta, \ \operatorname{Re} z < 0\} = \emptyset$, because a translation and a suitable rotation would give the required result.

Now, for each ε such that $0 < \varepsilon < \delta/2$, $f + \varepsilon$ is invertible in H^∞ and is, therefore, an outer function, i.e.

$$\log |\hat{f}(z_0) + \varepsilon| = \int_X \log |f + \varepsilon| \, dm.$$

As ε tends to zero, $\hat{f}(z_0) + \varepsilon$ tends to $\hat{f}(z_0) \neq 0$, and since

$$\min(\log \delta/2, \log |f|) \leqslant \log |f + \varepsilon| \leqslant M,$$

for some constant M, $\int_X \log |f + \varepsilon| \, dm$ tends to $\int_X \log |f| \, dm$, and this shows that f is an outer function.

We now investigate the absence of the factor S_a when $f-a$ is not necessarily an outer function. Certainly, $f-a/F_a$ is an inner function and we would like to know when, in fact, it is a Blaschke product. In order to generalize the classical criterion for this to the case of a finite open Riemann surface R, we introduce the following subregions of R. Let $G(\cdot, z_0)$ be the Green function for R with singularity at z_0, and

$$R_n = \left\{z \in R, G(z, z_0) > \frac{1}{n}\right\}, \qquad X_n = \left\{z \in R, G(z, z_0) = \frac{1}{n}\right\}.$$

For all sufficiently large n, R_n is a finite open Riemann surface with boundary X_n, $\bar{R}_n \subset R_{n+1}$ and $R = \bigcup R_n$. Let m_n be the harmonic measure on X_n with respect to z_0.

LEMMA 3.3. *Let f be an inner function, $|f| = |Z_1|^{\delta_1} \ldots |Z_\sigma|^{\delta_\sigma}$, a.e. dm on X. Then f is a Blaschke product if and only if*

$$\lim_{n\to\infty} \int_{X_n} \log\{|\hat{f}|\, |Z_1|^{-\delta_1} \ldots |Z_\sigma|^{-\delta_\sigma}\}\, dm_n = 0.$$

Proof. Let $f = BS$, as in Theorem 2.2. Note that here $a_j - \beta_j = \delta_j$, for $1 \leqslant j \leqslant \sigma$.

If a is a point in R and h is a Blaschke factor for R with respect to a with $h = |Z_1|^{\varepsilon_1} \ldots |Z_\sigma|^{\varepsilon_\sigma}$ a.e. dm on X, it has been shown in Section 2 of [4] that for z in R,

$$\log|\hat{h}(z)| = \sum_{j=1}^{\sigma} \varepsilon_j \log|Z_j(z)| - G(z, a).$$

Now let $(a_k)_k$ be the sequence of the zeros of $\hat{B}$ on R, so that $\sum_{k=1}^{\infty} G(z, a_k) < \infty$. It is clear from the construction of a Blaschke product as given in Theorem 2.5 of [4] and the essential uniqueness of a Blaschke product as proved in Theorem 3.1 of [4] that

$$\log|\hat{B}(z)| = \sum_{j=1}^{\sigma} a_j \log|Z_j(z)| - \sum_{k=1}^{\infty} G(z, a_k).$$

Also,

$$\log|\hat{S}(z)| = -\int_X P(x, z)\left[\left(\sum_{j=1}^{\sigma} \beta_j \log|Z_j|\right) dm + d\mu_s\right]/P(x, z_0).$$

Hence,

$$\log|\hat{f}(z)| = \sum_{j=1}^{\sigma} (a_j - \beta_j)\log|Z_j(z)| - \sum_{k=1}^{\infty} G(z, a_k) - \int_X P(x, z)\, d\mu_s / P(x, z_0).$$

Since $\alpha_j - \beta_j = \delta_j$, this gives

$$\int_{X_n} \log\{|\hat{f}|\,|Z_1|^{-\delta_1}\ldots|Z|^{-\delta_\sigma}\}\,dm_n$$

$$= -\int_{X_n}\Big[\sum_{k=1}^{\infty} G(z, a_k)\Big]\,dm_n - \int_{X_n}\Big[\int_X P(x, z)\,d\mu_s(x)/P(x, z_0)\Big]\,dm_n(z).$$

If we interchange the order of integration in the second term, we readily find that it is equal to $\int_X d\mu_s$. Since f is a Blaschke product if and only if $S \equiv 1$, i.e. $\mu_s \equiv 0$, to conclude the proof of Lemma 3.3, it is enough to show that

$$\lim_{n\to\infty} \int_{X_n}\Big[\sum_{k=1}^{\infty} G(z, a_k)\Big]\,dm_n = 0.$$

Let $u(z) = \sum_{k=1}^{\infty} G(z, a_k)$ and, for each natural number N, $u_N(z) = \sum_{k=N}^{\infty} G(z, a_k)$. Now, $u - u_N$ is continuous on a neighbourhood of X and is zero on X. Hence, for all sufficiently large n and each N,

$$\int_{X_n} u(z)\,dm_n = \int_{X_n} u_N(z)\,dm_n.$$

Fix N for the time being. Only a finite number of a_k's, say $k = N, \ldots, N_1$, lie in $R_n \cup X_n$. If a_k is in R_n and if $G_n(\cdot, z_0)$ is the Green function for R_n with singularity at z_0, we have

$$\begin{aligned}\int_{X_n} G(z, a_k)\,dm_n &= \int_{X_n} [G(z, a_k) - G_n(z, a_k)]\,dm_n\\ &= G(z_0, a_k) - G_n(z_0, a_k)\\ &\leqslant G(z_0, a_k).\end{aligned}$$

If a_k is in X_n, we still get

$$\int_{X_n} G(z, a_k)\,dm_n \leqslant G(z_0, a_k),$$

by approximating a_k by a sequence of points in R_n.

For points a_k outside $R_n \cup X_n$, we have

$$\int_{X_n}\Big[\sum_{k=N_1+1}^{\infty} G(z, a_k)\Big]\,dm_n = \sum_{k=N_1+1}^{\infty} G(z_0, a_k).$$

Thus,

$$\int_{X_n} u_N(z)\,dm_n \leqslant u_N(z_0) \quad \text{for each } N.$$

But $u_N(z_0)$ tends to zero as N tends to infinity. This completes the proof.

We shall now apply the criterion in Lemma 3.3 to find sets of points a for which the singular factor S_a is missing. We have already noted that the Blaschke factor B_a is missing if and only if a lies off the range of $\hat{f}$. Curiously enough, the condition we give for the absence of S_a requires that in some sense a lie off "the range of f".

THEOREM 3.4. *Let f be a function in H^∞. If V is any σ-compact subset of C such that*

$$m\big(\{x \in X, f(x) \in V\}\big) = 0,$$

then for every point a in V, except for a subset of logarithmic capacity zero, the singular factor S_a in $f-a = B_a S_a F_a$ is absent.

Proof. Since the logarithmic capacity of the union of a countable number of sets of logarithmic capacity zero is zero, we can assume without loss of generality that V is compact.

Let $f-a = w_a F_a$ as in Theorem 2.2, where $|w_a| = |Z_1|^{\delta_1(a)} \ldots |Z_\sigma|^{\delta_\sigma(a)}$, a.e. dm on X. According to Lemma 3.3, the inner function $w_a = f-a/F_a$ is a Blaschke product if and only if

$$\lim_{n\to\infty} L(n, a) = 0,$$

where

$$L(n, a) = \int_{X_n} K(z, a)\, dm_n,$$

where

$$K(z, a) = \log\{|\hat{F}_a(z)/\hat{f}(z)-a|\ |Z_1(z)|^{\delta_1(a)} \ldots |Z_\sigma(z)|^{\delta_\sigma(a)}\}.$$

Let $l(a) = \overline{\lim_{n\to\infty}} L(n, a)$, and

$$E = \{a \in V, l(a) > 0\}.$$

We wish to show that the logarithmic capacity of E is zero. If it were not, let μ be its equilibrium distribution. We shall show that $\int_E l(a)\, d\mu(a) = 0$ and get a contradiction.

Now, by Jensen's inequality,

$$\begin{aligned} K(z, a) &= \log\{|\hat{F}_a(z)/\hat{f}(z)-a|\ |Z_1(z)|^{\delta_1(a)} \ldots |Z_\sigma(z)|^{\delta_\sigma(a)}\} \\ &= \log\left\{\exp \frac{1}{2\pi} \int_X P(x, z)\Big[\log|f-a| - \sum_{j=1}^{\sigma} \delta_j(a) \log|Z_j|\Big]\, ds_x\right\} - \\ &\qquad\qquad - \log|\hat{f}(z)-a| + \sum_{j=1}^{\sigma} \delta_j(a) \log|Z_j(z)| \\ &= \frac{1}{2\pi} \int_X P(x, z) \log|f(x)-a|\, ds_x - \log|\hat{f}(z)-a| \geqslant 0. \end{aligned}$$

Hence it is enough to show that

$$\lim_{n\to\infty} \int_E L(n, a)\,d\mu(a) = 0.$$

But, $\int_E L(n, a)\,d\mu(a) = \int_{X_n} v(z)\,dm_n(z)$, where

$$v(z) = \int_{a\in E} K(z, a)\,d\mu(a), \qquad z \text{ in } R.$$

Since μ is the equilibrium distribution of E, there exists a constant M_1 uch tha t, for every complex number b,

$$-\int_E \log|b-a|\,d\mu(a) < M_1.$$

Also, since $E \subset V$ and V is compact, there exists a constant M_2 such that, for a in E, $|a| < M_2$. If M is the bound for f, we thus get

$$0 \leqslant v(z) = \int_E \left[\frac{1}{2\pi}\int_X P(x, z)\log|f(x)-a|\,ds_x\right] d\mu(a) - \\ - \int_E \log|\hat{f}(z)-a|\,d\mu(a) \\ \leqslant \log(M+M_2)+M_1.$$

Now, for almost all (dm) x in X, if $(z_k)_k$ is a sequence in R tending non-tangentially to x, $v(z_k)$ tends to zero. This can be seen as follows. For almost all x and $(z_k)_k$ tending non-tangentially to x, the harmonic function

$$u(z) = \int_E \left[\frac{1}{2\pi}\int_X P(x, z)\log|f(x)-a|\,ds_x\right] d\mu(a) \\ = \frac{1}{2\pi}\int_X P(x, z)\left[\int_E \log|f(x)-a|\,d\mu(a)\right] ds_x$$

tends to $\int_E \log|f(x)-a|\,d\mu(a)$. Similarly, $\hat{f}(z_k)$ tends to $f(x)$, and since $\int_E \log|b-a|\,d\mu(a)$ is harmonic outside cl. $E \subset V$ and $f(x)$ lies off V for almost all x,

$$\int_E \log|\hat{f}(z_k)-a|\,d\mu(a)$$

also tends to $\int_E \log|f(x)-a|\,d\mu(a)$.

Since v is non-negative and bounded above, it only remains to consider boundary strips around the boundary components of X (cf., section 4. b of [6]) to conclude that

$$\lim_{n\to\infty} \int_{X_n} v(z)\,dm_n = 0.$$

COROLLARY 3.5. *Let S be a singular function which is not trivial, i.e. not of the form $cZ_1^{m_1} \dots Z_\sigma^{m_\sigma}$, where c is a complex constant of absolute value 1 and $m_1, \dots, m_\sigma$ are integers. Then $\hat{S}$ omits only a set of logarithmic capacity zero near 0. If F is any outer function, $\hat{S}\hat{F}$ cannot omit a half-plane near 0.*

Proof. Since S is an inner function, there exists a $\delta > 0$ such that $|S| \geqslant \delta$ a.e. dm on X. Then the inner part w_a of $S-a$ is a Blaschke product for each a satisfying $|a| < \delta$ and not belonging to a fixed set of logarithmic capacity zero. Moreover, for any a such that $|a| < \delta/3$, w_a cannot be trivial, since $S-a$ cannot be an outer function. Indeed in that case, for z in R,

$$\log\{|\hat{S}(z)| + |a|\} \geqslant \log|\hat{S}(z) - a| = \frac{1}{2\pi} \int_X \log|S-a| P(x, z)\, ds_x \geqslant \log(\delta - |a|),$$

and hence, $|\hat{S}(z)| \geqslant \delta/3$ for z in R. This would imply that S is trivial. Thus, $\hat{S}$ takes on every value a in $|a| < \delta/3$ for which w_a is a Blaschke product.

The second statement in the corollary follows immediately from Proposition 3.2.

4. Frostman's theorem. Let us consider the classical case where R is the open unit disk, X the unit circle, $z_0 = 0$ and $dm = d\theta$. Suppose that f is an inner function, i.e. f is in H^∞ and $|f| = 1$ a.e. $d\theta$ on the unit circle. Then it follows that if $|a| \geqslant 1$, then $f-a$ is outer and if $|a| < 1$ and a does not belong to a fixed set of logarithmic capacity zero, then $f-a/F_a$ is a Blaschke product. In this case, F_a has a neat expression, viz. $F_a = c\,(1-\bar{a}f)$, where c is a complex constant of absolute value one. Frostman proved this result in this form in [2], p. 111. A weaker result was proved by Newman [5]. Since, as a tends to 0, $\hat{f}(z) - a/1-\bar{a}\hat{f}(z)$ tends to $\hat{f}(z)$ uniformly on the open unit disk, this shows that Blaschke products are norm-dense in the set of all inner functions. Since no such neat expression is available for F_a in the general case, we are unable to generalize Frostman's theorem. But we can still prove a slightly weaker result.

THEOREM 4.1. *Let f be an inner function, $|f| = |Z_1|^{\delta_1} \dots |Z_\sigma|^{\delta_\sigma}$, a.e. dm on X. Then there exists a sequence $(B_n)_n$ of Blaschke products such that $(\hat{B}_n)_n$ converges to $\hat{f}$ uniformly on compact subsets of R and $(|\hat{B}_n|)_n$ converges to $|\hat{f}|$ uniformly on R.*

Proof. There exists an $\varepsilon > 0$ such that $|f| \geqslant \varepsilon$ a.e. dm on X. Hence, by Theorem 3.4, there exists a sequence $(a_n)_n$ tending to 0 such that $w_{a_n} = f - a_n/F_{a_n}$ is a Blaschke product.

Now, for z in R,

$$\hat{F}_{a_n}(z) = \exp \int_X H(x, z) \Big[\log|f - a_n| - \sum_{j=1}^{\sigma} \delta_j(a_n) \log|Z_j|\Big]\, ds_x/2\pi.$$

A discerning look at Theorem 2.2 shows that, for $1 \leqslant j \leqslant \sigma$,

$$2\pi\delta_j(a_n) = \int_X \lambda_j(x)\log|f(x)-a_n|\,ds_x,$$

whereas

$$2\pi\delta_j = \int_X \lambda_j(x)\Big(\sum_{k=1}^{\sigma}\delta_k\log|Z_k(x)|\Big)\,ds_x = \int_X \lambda_j(x)\log|f(x)|\,ds_x.$$

This shows that $\delta_j(a_n)$ tends to δ_j, $1 \leqslant j \leqslant \sigma$, as a_n tends to 0. Thus, since f is bounded away from 0,

$$g_n = \log|f-a_n| - \sum_{j=1}^{\sigma}\delta_j(a_n)\log|Z_j|$$

tends to 0 uniformly on X. Since the kernel $H(x, z)$ is uniformly bounded for x in X and z in a compact subset of R and since

$$\operatorname{Re}\int_X H(x,z)g_n(x)\,ds_x = \int_X P(x,z)g_n(x)\,ds_x,$$

where $P(x,z) \geqslant 0$ for x in X and z in R, it follows that $(\hat{F}_{a_n})_n$ converges to 1 uniformly on compact subsets of R and $(|\hat{F}_{a_n}|)_n$ converges to 1 uniformly on R. If we let $B_n = f-a_n/F_{a_n}$, this proves the theorem.

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