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Fundamental solution for operators preserving a quadratic form

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Dedicated to the memory of Jacek Szarski

Abstract. Explicit formulas are given for an invariant fundamental solution for arbitrary operators which preserve the form $\sum_{i=1}^{m} x_i^2 - \sum_{i=1}^{n} y_i^2$, $m, n \geqslant 1$, m, n not simultaneously even. The solution is given in the form of a series of distributions.

Introduction. In this paper we present a method for establishing formulas for an invariant fundamental solution of the operator $P(\square)$, where $\square = \sum_{i=1}^{m} \partial^2/\partial x_i^2 - \sum_{j=1}^{n} \partial^2/\partial y_j^2$ and P an arbitrary polynomial. The method applied here is an extension of that used in [1] for homogeneous operators. In computing the fundamental solution we make use of the Lorentz invariance of the operator P without ever mentioning Fourier transformation. For the sake of the uniformity of the method we decided to restrict ourselves to the case where m, n are not simultaneously even (since then the operator $\square^q$ has an invariant homogeneous fundamental solution).

1. Notation and definitions. $\boldsymbol{R}^k$ will denote the k-dimensional Euclidean space. $\boldsymbol{N}_0$ stands for the set of non-negative integers. By $\boldsymbol{N}_0^k$ we denote the set of all multiindices $a = (a_1, \ldots, a_k)$, $a_i \in \boldsymbol{N}_0$ $(i = 1, \ldots, k)$. Let $x = (x_1, \ldots, x_m) \in \boldsymbol{R}^m$, $y = (y_1, \ldots, y_n) \in \boldsymbol{R}^n$ and

$$s(x, y) = |x|^2 - |y|^2 = \sum_{i=1}^{m} x_i^2 - \sum_{j=1}^{n} y_j^2.$$

We apply the notation commonly used in the theory of distributions and of differential operators. In particular $C_0^k(\Omega)$ stands for the set of compactly supported C^k $(0 \leqslant k \leqslant \infty)$ functions with supports in an open set $\Omega \subset \boldsymbol{R}^k$. The value of a distribution u on a test function $\varphi \in C_0^\infty(\Omega)$ will be written as $u[\varphi]$. By δ we denote the Dirac measure at zero and Y is the Heaviside function.

By S_k we denote the set $S_k = \{(x_1, \ldots, x_k): x_1^2 + \ldots + x_k^2 = 1\}$, ω_k is the Lebesgue measure on this surface and

$$|S_k| = \int_{S_k} d\omega_k .$$

If $k = 1$ we set $S_1 = \{+1, -1\}$, $\omega_1 = \delta_1 + \delta_{-1}$ and consequently $|S_1| = 2$.

We consider the operator

$$\Box_{mn} = \sum_{i=1}^{m} \partial^2/\partial x_i^2 - \sum_{j=1}^{n} \partial^2/\partial y_j^2$$

and form an operator with constant coefficients a_ν

$$P_q = \sum_{\nu=0}^{q} a_\nu (\Box_{mn})^\nu, \quad (\Box_{mn})^0 = \mathrm{Id} \quad \text{in } \mathscr{D}'(\boldsymbol{R}^{m+n}), \quad a_q = 1.$$

The operator $\Box_{mn}$ and consequently P_q preserve the form s: the formula

$$\Box_{mn} f\big(s(x, y)\big) = (Lf)\big(s(x, y)\big) \quad \text{for } f \in C^\infty(\boldsymbol{R}^1)$$

determines the operator

$$L = 2(n+m)\frac{d}{ds} + 4s\frac{d^2}{ds^2}$$

whose transpose equals

$$L^{\mathrm{tr}} = 2(4-n-m)\frac{d}{ds} + 4s\frac{d^2}{ds^2}.$$

We shall denote

$$L_q = \sum_{\nu=0}^{q} a_\nu L^\nu, \quad L^0 = \mathrm{Id} \quad \text{in } \mathscr{D}'(\boldsymbol{R}^1)$$

and observe that

$$P_q(f\circ s) = L_q f\circ s.$$

Definition (a convention). Let $g \in C^{2q}(\boldsymbol{R}^1\setminus\{0\})$. By $L^{\mathrm{tr}}g$ we shall denote the function given by

$$(L^{\mathrm{tr}}g)(s) = 2(4-n-m)\frac{dg(s)}{ds} + 4s\frac{d^2 g(s)}{ds^2} \quad \text{for } s \in \boldsymbol{R}^1\setminus\{0\}$$

and its continuous extension to the whole of $\boldsymbol{R}^1$ if such an extension exists.

2. Description of the method of finding a solution of $P_q E = \delta$. In [1] we computed explicitly fundamental solutions of the homogeneous operators invariant with respect to the form $\sum_{i=1}^{m} x_i^p - \sum_{j=1}^{n} y_i^q$ (p, q arbitrary even positive integers) and thus in particular invariant with respect to the form s. We shall apply the results of [1] to determine a fundamental solution of the non-homogeneous operator P_q. We shall make use of both the theorems proved in [1] and the constants computed there. We begin by characterizing the method employed in [1]. By means of an operation K of averaging (which played a fundamental role in [1]) the study of an invariant $(m+n)$-dimensional operator P was reduced to the study of the one-dimensional operator L. It turned out that there exists a sequence of functions $\{\gamma_{ij}\}$ such that every function $K\varphi$ with $\varphi \in C_0^\infty(\boldsymbol{R}^{m+n})$ can be represented modulo a function of an arbitrarily prescribed order of smoothness, as a linear combination of a finite number of the functions $\{\gamma_{ij}\}$. The operator L^{tr} acts inside the set $K(C_0^\infty)$ (cf. Theorem 1 in Part 2 in [1] and Theorem 1 below). Finding a fundamental solution of the invariant operator P reduces to computing the value of L^{tr} on the functions γ_{ij} and on functions which are sufficiently smooth. Below we quote Lemma 1 from [1] concerning the existence and uniqueness of the operation K and then the definition of the functions γ_{ij} together with Lemma 2 concerning the decomposition of the function $K\varphi$.

LEMMA 1. *There exists a unique linear operation K, $C_0^\infty(\boldsymbol{R}^{m+n}) \ni \varphi \mapsto K\varphi \in C^\infty(\boldsymbol{R}^1 \setminus \{0\})$, $\operatorname{supp} K\varphi$ being bounded, satisfying*

1° $K\colon C_0^\infty(\boldsymbol{R}^{m+n}\setminus\{0\}) \to C_0^\infty(\boldsymbol{R}^1)$ *is a surjection,*

2° *for every function $f \in C^0(\boldsymbol{R}^1)$ and $\varphi \in C_0^0(\boldsymbol{R}^{m+n})$*

$$(f\circ s)[\varphi] = f[K\varphi],$$

3° *if $\varphi_\nu \in C_0^\infty(\boldsymbol{R}^{m+n})$, $\nu = 1, 2, \ldots$, have commonly bounded supports, then so have the functions $K\varphi_\nu$ $(\nu = 1, 2, \ldots)$.*

Before giving the definition of the functions γ_{ij} we introduce the notation

$$a_0 = \tfrac{1}{2}(m-2), \quad b_0 = \tfrac{1}{2}(n-2), \quad \mu = a_0+b_0+1 = \tfrac{1}{2}(m+n-2),$$

$$a_i = a_0+\tfrac{1}{2}i, \quad b_j = b_0+\tfrac{1}{2}j, \quad \mu_{ij} = a_i+b_j+1 \quad (i, j = 1, 2, \ldots).$$

Let

$$\begin{aligned} \gamma_{ij}(s) &= A(a_i, b_j)\,Y(s)\,s^{\mu_{ij}} && \text{if } b_j \in \boldsymbol{N}_0, \\ &= B(a_i, b_j)\,Y(-s)\,|s|^{\mu_{ij}} && \text{if } a_i \in \boldsymbol{N}_0,\ b_j \notin \boldsymbol{N}_0, \\ &= \big(A(a_i, b_j)\ln|s| + C(a_i, b_j)\,Y(s)\big)s^{\mu_{ij}} && \text{if } a_i, b_j \notin \boldsymbol{N}_0, \end{aligned}$$

where the constants $A(a_i, b_j)$, $B(a_i, b_j)$, $C(a_i, b_j)$ are defined in Lemma 3, Part I of [1]; in all cases $A(a_i, b_j)$, $B(a_i, b_j)$ are different from zero.

We denote by χ a function in $C_0^\infty(\boldsymbol{R}^1)$ equal to one in a neighbourhood of zero.

LEMMA 2(¹). *Let $\varphi \in C_0^\infty(\boldsymbol{R}^{m+n})$. For every number $N \in \boldsymbol{N}_0$ there exists*

(¹) See [1], Part 1, Corollary 1.

a number $k \in N_0$ and a function $h_k(\varphi;\cdot)$ such that

$$(K\varphi)(s) = \chi(s) \sum_{\substack{0 \leqslant i+j \leqslant k \\ \mu_{ij} \leqslant N}} \Lambda_{ij}(\varphi)\gamma_{ij}(s) + h_k(\varphi;\, s) \quad \textit{for } s \in \boldsymbol{R}^1,$$

$$\Lambda_{ij}(\varphi) = \tfrac{1}{4} \sum_{|\alpha|=i} \sum_{|\beta|=j} \frac{1}{\alpha!\beta!} (D_x^\alpha D_y^\beta \varphi)(0) C_{\alpha\beta},$$

where the constants $C_{\alpha\beta} = \int_{S_m}\int_{S_n} \xi^\alpha \eta^\beta d\omega_m d\omega_n$ are different from zero if and only if each of the numbers $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ is either even or zero. In particular $\Lambda_{ij}(\varphi) = 0$ if one of the numbers i, j is odd and $\Lambda_{00} = B_0\varphi(0)$, where

$$B_0 = \tfrac{1}{4}|S_m||S_n|.$$

The function $h_k(\varphi;\cdot) \in C_0^N$ and $h_k(\varphi_\nu;\cdot) \underset{\nu\to\infty}{\to} 0$ in $\mathscr{D}^N(\boldsymbol{R}^1)$ if $\varphi_\nu \underset{\nu\to\infty}{\to} 0$ in $\mathscr{D}(\boldsymbol{R}^{n+m})$.

THEOREM 1. *Let P, T be operators related by the invariance relation*

$$P(f\circ s) = Tf\circ s \quad \textit{for } f \in C^{2q}(\boldsymbol{R}^1).$$

Then for every function $\varphi \in C_0^\infty(\boldsymbol{R}^{m+n})$ we have

$$K(P^{\mathrm{tr}}\varphi)(s) = T^{\mathrm{tr}}(K\varphi)(s) \quad \textit{for } s \neq 0. \tag{1}$$

Proof. Applying the definition of K we have for every function $f \in C_0^\infty(\boldsymbol{R}^1\setminus\{0\})$ the relations

$$\int_{\boldsymbol{R}^1} f(s)K(P^{\mathrm{tr}}\varphi)(s)\,ds = \int_{\boldsymbol{R}^{m+n}} (f\circ s)P^{\mathrm{tr}}\varphi(x,y)\,d(x,y) = \int_{\boldsymbol{R}^{m+n}} P(f\circ s)\varphi(x,y)\,d(x,y),$$

(2)

$$\int_{\boldsymbol{R}^1} f(s)T^{\mathrm{tr}}(K\varphi)(s) = \int_{\boldsymbol{R}^1} (Tf)(s)(K\varphi)(s)\,ds = \int_{\boldsymbol{R}^{m+n}} (Tf\circ s)\varphi(x,y)\,d(x,y).$$

Using the invariance relation we deduce from (2) our assertion.

THEOREM 2. *Suppose $0 \neq E \in \mathscr{D}'(\boldsymbol{R}^1)$ and the formal definition*

$$u[\varphi] = E[K\varphi] \quad \textit{for } \varphi \in C_0^\infty(\boldsymbol{R}^{m+n}) \tag{3}$$

defines a distribution on $\boldsymbol{R}^{m+n}$. If

$$L_q E[K\varphi] = A\varphi(0) \quad \textit{for } \varphi \in C_0^\infty(\boldsymbol{R}^{m+n}), \tag{4}$$

then u satisfies the identity $P_q u = A\delta$ and therefore is a non-zero solution of the equation $P_q u = 0$ if $A = 0$ and $\frac{1}{A}u$ is a fundamental solution of P_q otherwise.

Proof. By (3), (4), (1) and Theorem 1 for every function $\varphi \in C_0^\infty(\boldsymbol{R}^{m+n})$ we have the relations:

$$P_q u[\varphi] = u[(P_q)^{\mathrm{tr}}\varphi] = E[K(P_q)^{\mathrm{tr}}\varphi] = E[(L_q)^{\mathrm{tr}}K\varphi] = L_q E[K\varphi] = A\varphi(0).$$

3. Determining non-zero solutions of the equation $L_q E = 0$. Looking for a non-zero solution of the equation

$$(5) \qquad L_q E = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}^1)$$

we distinguish two cases: I where μ is an integer and NI where μ is not. In every of these cases we select two subcases: $[\mu] < q$ and $[\mu] \geqslant q$. So we obtain four cases which we symbolise in the form: $I_{\mu<q}$, $I_{\mu\geqslant q}$, $NI_{[\mu]<q}$, $NI_{[\mu]\geqslant q}$. Later in Section 4 we will have to consider separately the case where n is even and the case where n is odd. We shall denote this writting e or o as a superscript in the previous symbols. For instance the case $I_{\mu<q}$ splits into two subcases $I^{\mathrm{e}}_{\mu<q}$ and $I^{\mathrm{o}}_{\mu<q}$.

For convenience let us note the following distributional formulas

$$(6) \qquad \begin{aligned} L\delta^{(p)} &= 4(\mu-1-p)\delta^{(p+1)} \quad \text{for } p = 0, 1, 2, \ldots, \\ LY &= 4\mu\delta, \\ L\left(\frac{s^p}{p!}\,Y\right) &= 4(\mu+p)\frac{s^{p+1}}{(p-1)!}\,Y \quad \text{for } p = 1, 2, \ldots \end{aligned}$$

Case $I_{\mu<q}$. We seek a solution of (5) in the form

$$(7) \qquad E = \sum_{j=0}^{\infty} c_j \frac{s^{t+j}}{(t+j)!}\,Y, \quad \text{where } t = q-\mu-1 \geqslant 0,\ c_0 = 1.$$

Let us substitute formally series (7) to (5) and arrange it with respect to the orders of the distributions δ and the powers of $s^\lambda Y$. By equating to zero the expressions thus obtained we arrive(²) at the following system of equations for the unknown $c_1, c_2, \ldots$

$$(8) \qquad \sum_{j=0}^{\min(k,q)} a_{q-j} c_{k-j} L^{q-j}\left(\frac{s^{t+k-j}}{(t+k-j)!}\,Y\right) = 0, \quad k = 1, 2, \ldots$$

Applying (6) we observe that system (8) is satisfied by the solution of the system

$$\sum_{j=0}^{\min(k,q)} a_{q-j} c_{k-j} 4^{q-j}(k+q-j-1)! = 0, \quad k = 1, 2, \ldots$$

(²) Observe that $L^q\left(\frac{s^t}{t!}\,Y\right) = 0$.

which after a change of indices can be written as

$$(9)\qquad \sum_{i=\max(k-q,0)}^{k} a_{q-k+i} c_i 4^{q-k+i}(q+i-1)! = 0.$$

Let us form the infinite matrix $A = (A_{ki})_{\substack{k=1,2,\dots\\ i=0,1,2,\dots}}$

$$(10)\qquad A_{ki} = \begin{cases} a_{q-k+i}4^{q-k+i}(q+i-1)! & \text{if } 0 \leqslant q-k+i \leqslant q,\\ 0 & \text{otherwise.}\end{cases}$$

System (9) can be written as

$$(11)\qquad Ac = 0, \quad \text{i.e.,} \quad \sum_{i=0}^{\infty} A_{ki}c_i = 0, \quad k = 1, 2, \dots, c_0 = 1.$$

The possibility of determining c_k from the kth equation of system (11) (knowing $c_0, c_1, \dots, c_{k-1}$) results from the fact that $A_{kk} = a_q 4^q(q+k-1)! \neq 0$ (since by assumption $a_q = 1$).

Next we shall prove that series (7) with coefficients c_j satisfying system (11) is absolutely uniformly convergent on every finite interval. To this end we shall estimate the growth of c_j. First we denote

$$M = \max_{0\leqslant k\leqslant q} |a_k|, \qquad B = \max_{0\leqslant j\leqslant q} |c_j|.$$

Then $M \geqslant 1$, $B \geqslant 1$. We shall establish by induction that

$$(12)\qquad |c_{q+j}| \leqslant \frac{M^j B}{8^j}, \qquad j = 0, 1, 2, \dots$$

For $j = 0$: $|c_q| \leqslant B$ so (12) holds. Assuming that (11) is true for a fixed j, we compute c_{q+j+1} from the $q+j+1$-th equation in (11):

$$c_{q+j+1} = -\frac{1}{4^q(2q+j)!}\sum_{i=j+1}^{q+j} a_{i-j-1}4^{i-j-1}(q+i-1)!\,c_i.$$

Therefore

$$|c_{q+j+1}| \leqslant \frac{M}{4}\,\frac{q}{(2q+j)}\,\frac{M^j B}{8^j} \leqslant \frac{M^{j+1}B}{8^{j+1}}.$$

It follows from (12) that starting from the qth term series (7) can be dominated by

$$(13)\qquad \sum_{j=0}^{\infty} |c_{j+q}| \frac{s^{t+j+q}}{(t+j+q)!} \leqslant \frac{Bs^{t+q}}{(t+q)!}\sum_{j=0}^{\infty}\frac{M^j s^j}{j!} = \frac{Bs^{t+q}}{(t+q)!}e^{Ms}.$$

Case $I_{\mu\geqslant q}$. We seek a solution to (5) in the form of the series

$$(14)\qquad E = \sum_{k=0}^{\mu-q} c_k \delta^{(\mu-k-q)} + \sum_{j=0}^{\infty} c_{\mu-q+1+j}\frac{s^j}{j!}\,Y, \qquad c_0 = 1.$$

Proceeding as in the previous case we obtain by (6) the same infinite system of equations (11) for the unknowns $c_1, c_2, \ldots$ with the coefficients A_{ki} given by (10). If $q \geqslant 2$, then by neglecting the first $q-1$ terms in the series $\sum_{j=0}^{\infty} c_{\mu-q+j+1} \frac{s^j}{j!} Y$ we estimate

$$\sum_{j=q-1}^{\infty} |c_{\mu-q+j+1}| \frac{|s|^j}{j!} = \sum_{k=\mu-q}^{\infty} |c_{k+q}| \frac{|s|^{k+2q-\mu-1}}{(k+2q-\mu-1)!}$$

and prove its convergence as in (13).

Case $NI_{[\mu]<q}$. Let $c_0 = 1$, $t = q-\mu-1$. We construct first a solution to (5) outside the origin in the form(³)

$$E^*(s) = \sum_{j=0}^{\infty} c_j \frac{s^{j+t}}{(j+t)!} \qquad \text{for } s > 0, \tag{15}$$
$$= \sum_{j=0}^{\infty} c_j (-1)^j \frac{|s|^{j+t}}{(j+t)!} \qquad \text{for } s < 0.$$

The formulae

$$L\left(\frac{s^h}{h!}\right) = 4(\mu+h) \frac{s^{h-1}}{(h-1)!} \qquad \text{for } s > 0,$$
$$h = t, t+1, \ldots, \tag{16}$$
$$L\left(\frac{|s|^h}{h!}\right) = -4(\mu+h) \frac{|s|^{h-1}}{(h-1)!} \qquad \text{for } s < 0,$$

lead as before to the same system (11) for the coefficients $c_1, c_2, \ldots$ with A_{ki} given by (10). So by (12) series (15) is convergent. Thus E^* given by (15) is a solution to $L_q E(s) = 0$ for $s \neq 0$. Define

$$E(s) = \sum_{j=0}^{\infty} c_j Y(s) \frac{s^t s^j}{(t+j)!}, \tag{17}$$

$$\tilde{E}(s) = \sum_{j=0}^{\infty} c_j Y(-s) \frac{|s|^t s^j}{(t+j)!}. \tag{18}$$

We shall prove that $L_q E = 0$; the proof that $L_q \tilde{E} = 0$ is analogous. We know that $L_q E[a] = 0$ for $a \in C_0^\infty(\boldsymbol{R}^1 \setminus \{0\})$, so

$$L_q E = \sum_{k=0}^{M} \lambda_k \delta^{(k)},$$

(³) In this case μ and $t = q-\mu-1$ are not integers. We shall use the notation: $h! = h(h-1)\ldots(h-[h])$ if $h \notin N$, $h > 0$ and $h! = 1$ if $-1 < h < 0$.

where M is some integer and λ_k some complex numbers. We shall prove that all λ_k are zero. For thus purpose we write

$$L_q E = \sum_{k=0}^{\infty} v_k,$$

where

$$v_0[a] = \frac{1}{t!} \int_0^{\infty} |s|^t (L^{\mathrm{tr}})^q a(s)\, ds,$$

$$v_k[a] = \sum_{j=0}^{\min(k,q)} a_{q-j} c_{k-j} \frac{1}{(t+k-j)!} \int_0^{\infty} |s|^t s^{k-j} (L^{\mathrm{tr}})^{q-j} a(s)\, ds,$$

$k = 1, 2, \ldots$ for $a \in C_0^{\infty}(\boldsymbol{R}^1)$.

We observe that v_k is homogeneous of order $k-\mu-1$ which is not an integer number. Thus we have

$$\sum_{k=0}^{\infty} v_k = \sum_{k=0}^{M} \lambda_k \delta^{(k)}$$

and since $\delta^{(k)}$ is homogeneous of order $-1-k$ we prove easily that all $\lambda_k = 0$. Therefore

$$L_q E = \sum_{k=0}^{\infty} v_k = 0 \quad \text{in } \mathscr{D}'(\boldsymbol{R}^1)$$

and since every v_k has a different order of homogeneity we conclude that

$$v_k[a] = 0 \quad \text{for } a \in C_0^{\infty}(\boldsymbol{R}^1),\ k = 0, 1, 2 \ldots \tag{19}$$

Case $NI_{[\mu] \geqslant q}$. In this case $\mu \geqslant \frac{3}{2}$, $t = q-\mu-1 \leqslant -\frac{3}{2}$, hence $[t] \leqslant -2$. As in the case $NI_{[\mu]<q}$ we seek first a solution to (5) outside the origin, but now in the form

$$\begin{aligned} E^*(s) &= \sum_{\nu=0}^{-2-[t]} (-t-1-\nu)!(-1)^{\nu} c_{\nu} s^{t+\nu} + \sum_{\nu=-[t]-1}^{\infty} c_{\nu} \frac{s^{t+\nu}}{(t+\nu)!} \\ &\qquad \text{for } s > 0,\ c_0 = 1, \\ E^*(s) &= \sum_{\nu=0}^{-2-[t]} (-t-1-\nu)!\, c_{\nu} |s|^{t+\nu} + \sum_{\nu=-[t]-1}^{\infty} c_{\nu} (-1)^{\nu} \frac{|s|^{t+\nu}}{(t+\nu)!} \\ &\qquad \text{for } s < 0,\ c_0 = 1. \end{aligned} \tag{20}$$

Formulae (16) lead as before to the same system (11) for the coefficients $c_1, c_2, \ldots$ with A_{ki} given by (10) and we conclude that E^* given by (20) is a solution to $L_q E^* = 0$ outside the origin.

Denote $\tilde{m} = [\mu]-q+1$,

$$b_\nu = \begin{cases} (-t-1-\nu)!(-1)^\nu c_\nu & \text{for } \nu = 0,1,\dots,-2-[t], \\ \dfrac{c_\nu}{(t+\nu)!} & \text{for } \nu = -[t]-1,\ [-t],\dots \end{cases} \tag{21}$$

and define

$$\varkappa(s) = \sum_{\nu=0}^{\infty} b_\nu s^\nu, \tag{22}$$

$$E[a] = \int_0^\infty s^{-1/2} D^{\tilde{m}}\big(a(s)\varkappa(s)\big)\,ds \quad \text{for } a \in C_0^{\tilde{m}}(\boldsymbol{R}^1), \tag{23}$$

$$\tilde{E}[a] = \int_{-\infty}^0 |s|^{-1/2} D^{\tilde{m}}\big(a(s)\varkappa(s)\big)\,ds. \tag{24}$$

Note that $L_q E[a] = 0$ for $a \in C_0^\infty(\boldsymbol{R}^1\setminus\{0\})$ and write $L_q E$ in the form $L_q E = \sum_{k=0}^{\infty} v_k$, where

$$v_0[a] = (\mu-q)!\int_0^\infty s^{-1/2} D^{\tilde{m}}\big((L^{\mathrm{tr}})^q a(s)\big)\,ds,$$

$$v_k[a] = \sum_{j=0}^{\min(k,q)} a_{q-j} b_{k-j} \int_0^\infty s^{-1/2} D^{\tilde{m}}\big(s^{k-j}(L^{\mathrm{tr}})^{q-j} a(s)\big)\,ds,$$

$$k = 1,2,\dots$$

We observe that v_k is homogeneous of order $k-\mu-1$. This leads as before to the conclusion (19). Similarly

$$L_q\tilde{E} = \sum_{k=0}^{\infty} \tilde{v}_k \tag{25}$$

with

$$\tilde{v}_k[a] = \sum_{j=0}^{\min(k,q)} a_{q-j} b_{k-j} \int_{-\infty}^0 |s|^{-1/2} D^{\tilde{m}}\big(s^{k-j}(L^{\mathrm{tr}})^{q-j} a(s)\big)\,ds,$$

$$k = 1,2,\dots.$$

homogeneous of order $k-\mu-1$. We conclude that

$$\tilde{v}_k[a] = 0 \quad \text{for } a \in C_0^\infty(\boldsymbol{R}^1),\ k = 0,1,2,\dots \tag{26}$$

We shall summarize the results of this part in the following

THEOREM 3. *The distribution E given by*

1° (7) *in case* $I_{\mu<q}$,

2° (14) *in case* $I_{\mu\geqslant q}$,

and the distributions E and $\tilde{E}$ given by

3° (17) *and* (18) *in case* $NI_{[\mu]<q}$,

4° (23) *and* (24) *with* $x(\cdot)$ *defined by* (22) *and* (21) *in case* $NI_{[\mu]\geq q}$, *are non-zero solutions of equation* (5). *In all the cases the coefficients* c_j *are computed from system* (11) *with the matrix* A_{ki} *defined by* (10).

4. Determining fundamental solutions of the operator P_q and certain non-zero solutions of the homogeneous equation $P_q u = 0$. We shall prove that the distributions E, $\tilde{E}$ constructed in Theorem 3 satisfy the formula

$$L_q S[K\varphi] = A\varphi(0) \quad \text{for } \varphi \in C_0^\infty(\mathbf{R}^{m+n}) \tag{27}$$

with adequate constants A. Then by Theorem 2 the distribution (3) satisfies $P_q u = A\delta$ with the same constant A. Again we consider separately cases I and NI. We begin with

Case $NI_{[\mu]\geq q}$. Define

$$E_q[a] = \int_0^\infty s^{-1/2} D^{\tilde{m}} a(s)\, ds \qquad \text{for } a \in C_0^{\tilde{m}}(\mathbf{R}^1),$$

$$\tilde{E}_q[a] = \int_{-\infty}^0 |s|^{-1/2} D^{\tilde{m}} a(s)\, ds \qquad \text{for } a \in C_0^{\tilde{m}}(\mathbf{R}^1).$$

Instead of proving formula (27) we shall show that

$$L_q E[K\varphi] = (\mu - q)!\, L^q E_q[K\varphi] \quad \text{for } \varphi \in C_0^\infty(\mathbf{R}^{m+n}) \tag{28}$$

and then apply the results of [1] for the homogeneous operator L^q and the distributions E_q, $\tilde{E}_q$ to establish (27).

We introduce a one-dimensional operator

$$T_k^{\mathrm{tr}} = \sum_{j=0}^{\min(k,q)} a_{q-j} b_{k-j} s^{k-j} (L^{\mathrm{tr}})^{q-j} \quad \text{for } k = 0, 1, 2, \ldots \tag{29}$$

which is homogeneous of order $\lambda = q - k$, and an $m+n$-dimensional operator

$$P_k^{\mathrm{tr}} = \sum_{j=0}^{\min(k,q)} a_{q-j} b_{k-j} (x^2 - y^2)^{k-j} (\Box_{mn})^{q-j}.$$

Applying condition 2° of the definition of the operator K from Lemma 1 it is not difficult to show that

$$K(P_k^{\mathrm{tr}}\varphi)(s) = T_k^{\mathrm{tr}}(K\varphi)(s) \quad \text{for } s \neq 0.$$

Hence by (2) (with P_k, T_k instead of P, T) we get

$$P_k(f \circ s) = T_k f \circ s.$$

In terms of the operator T_k^{tr} the distributions v_k, $\tilde{v}_k$ assume the form

$$\begin{aligned} v_k[a] &= \int_0^\infty s^{-1/2} D^{\tilde{m}}(T_k^{\mathrm{tr}} a)(s)\,ds = E_q[T_k^{\mathrm{tr}} a], \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 0, 1, 2, \ldots \\ \tilde{v}_k[a] &= \int_{-\infty}^0 |s|^{-1/2} D^{\tilde{m}}(T_k^{\mathrm{tr}} a)(s)\,ds = \tilde{E}_q[T_k^{\mathrm{tr}} a]. \end{aligned}$$

To compute $L_q E[K\varphi]$, $L_q \tilde{E}[K\varphi]$ it is enough to find $v_k[K\varphi]$, $\tilde{v}_k[K\varphi]$ $(k = 0, 1, 2, \ldots)$. To this end we apply as we did in [1](4) the decomposition of $K\varphi$ given in Lemma 2 and use integration by parts to show that in view of (19) and (26)

$$(30) \qquad v_k[g], \tilde{v}_k[g] = 0 \quad \text{for } g \in \tilde{C}_0^{2q+m}(\boldsymbol{R}^1).$$

Next we compute v_k on the functions γ_{ij} entering into the asymptotic expansion of the function $K\varphi$.

In particular for the functions

$$(31) \qquad \sigma_1(s) = \chi(s) Y(s) s^r, \qquad \sigma_2(s) = \chi(s) Y(-s) s^r \qquad (s \in \boldsymbol{R}^1)$$

we obtain $v_k[\sigma_2] = 0$,

$$(32) \qquad v_k[\sigma_1] = \sum_{p=0}^{2q} c_p \int_0^\infty s^{-1/2} D^{\tilde{m}}(\chi^{(p)} s^{r+p-\lambda})\,ds = c_1^* \int_0^\infty s^{r-\lambda-\tilde{m}-1/2} \chi(s)\,ds$$

if $r - \lambda - \tilde{m} - \frac{1}{2} = r + k - \mu - 1 > -1$, i.e. if $r > \mu - k$.

The distributions $v_k[\sigma_1]$ being independent of the choice of the function χ we conclude as in [1] that $c_1^* = 0$. Analogous considerations in the case of the function

$$(33) \qquad \sigma_3 = \ln|s| s^r \chi(s)$$

lead to the expression

$$(34) \qquad v_k[\sigma_3] = c_2^* \int_0^\infty s^{r+k-\mu-1} \chi(s)\,ds + c_3^* \int_0^\infty s^{r+k-\mu-1} \ln s\, \chi(s)\,ds$$

for $r > \mu - k$ which again gives $c_2^* = c_3^* = 0$. Similarly we prove that

$$(35) \qquad \begin{aligned} \tilde{v}_k[\sigma_1] &= 0, \quad k = 0, 1, 2, \ldots, \\ \tilde{v}_k[\sigma_2] &= 0 \quad \text{if } r > \mu - k, \\ \tilde{v}_k[\sigma_3] &= 0 \quad \text{if } r > \mu - k. \end{aligned}$$

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(4) It is useful to note here that the proof presented in [1] (formulae (π_1)–(π_4) and (12)–(20) in Part 1 can be simplified by showing (π_3)–(π_4) for all $t > \mu$ without considering separately the sum (12).

From Lemma 2 and (30)–(34) we obtain (28). In fact

$$L_qE[K\varphi] = \sum_{k=0}^{\infty} v_k[K\varphi] = v_0[K\varphi] = E_q[T_0^{\mathrm{tr}}K\varphi]$$
$$= (\mu-q)!E_q[(L^q)^{\mathrm{tr}}K\varphi] = (\mu-q)!L^qE_q[K\varphi],$$
$$L_q\tilde{E}[K\varphi] = \sum_{k=0}^{\infty} \tilde{v}_k[K\varphi] = \tilde{v}_0[K\varphi] = \tilde{E}_q[T_0^{\mathrm{tr}}K\varphi]$$
$$= (\mu-q)!\tilde{E}_q[(L^q)^{\mathrm{tr}}K\varphi] = (\mu-q)!L^q\tilde{E}_q[K\varphi].$$

To prove (27) we apply 1.1° in Part 2 in [1] taking $L = (\mu-q)!L^q$, $\lambda = q$, $M = 2q$, $E = E_q$. Then we get in

Case $NI^{e}_{[\mu]\geqslant q}$:

(36) $$(\mu-q)!L^qE_q[K\varphi] = A\varphi(0)$$

with the constant A defined in [1] by formulae (21) in Part 2,

(37) $$L^q\tilde{E}_q[K\varphi] = 0.$$

Case $NI^{o}_{[\mu]\geqslant q}$:

$$L^qE_q[K\varphi] = 0,$$
$$(\mu-q)!L^q\tilde{E}_q[K\varphi] = \tilde{A}\varphi(0)$$

with the constant A defined in [1] by formulae (29) in Part 2.

Case $NI_{[\mu]<q}$. In this case we follow the method presented above for $[\mu] > q-1$. Let

$$E_q = Y(s)s^t, \quad \tilde{E}_q = Y(-s)s^t.$$

To show (27) we prove that

$$L_qE[K\varphi] = \frac{1}{(q-\mu-1)!}L^qE_q[K\varphi], \qquad \varphi \in C_0^\infty(\boldsymbol{R}^{m+n}),$$
$$L_q\tilde{E}[K\varphi] = \frac{1}{(q-\mu-1)!}L^q\tilde{E}_q[K\varphi],$$

and then apply 1.1° in Part 2 in [1] putting

$$L = \frac{1}{(q-\mu-1)!}L^q, \quad \lambda = q,\ M = 2q.$$

We consider the subcases

Case $NI^{e}_{[\mu]<q}$. By formula (24) in Part 2 in [1] we then get

$$\frac{1}{(q-\mu-1)!}L^qE_q[K\varphi] = A\varphi(0), \quad L^q\tilde{E}_q[K\varphi] = 0, \quad \varphi \in C_0^\infty(\boldsymbol{R}^{m+n})$$

the constant A being defined in [1] by formulae (27).

Case $NI^{\circ}_{[\mu]<q}$. We apply assertions 1° and 4° in Theorem 3 in [1] and obtain

$$\frac{1}{(q-\mu-1)!}L^q\tilde{E}_q[K\varphi] = \tilde{A}\varphi(0), \quad L^qE_q[K\varphi] = 0, \quad \varphi\in C_0^\infty(\boldsymbol{R}^{m+n}),$$

where the constant $\tilde{A}$ is defined in [1] by (29) with

$$c^*(\mu) = \sum_{p=1}^{M} c_p^*(\mu)(-1)^p(p-1)!, \quad c_p^* \text{ given by (31) in [1].}$$

Case $I_{\mu<q}$. We consider distribution (7) with c_j satisfying (11). As in the case NI we write

$$L_qE = \sum_{k=0}^{\infty} v_k, \tag{38}$$

where this time

$$v_k[a] = \sum_{j=0}^{\min(k,q)} a_{q-j}c_{k-j}\int_0^\infty \frac{s^{t+k-j}}{(t+k-j)!}(L^{\mathrm{tr}})^{q-j}a(s)\,ds,$$

$k = 0,1,2,\ldots$ for $a\in C_0^{2q}(\boldsymbol{R}^1)$, and consider functions (25), (27). It is not difficult to show that there exist constants c_{1kr}, c_{2kr}, c_{3kr} such that

$$v_k[\sigma_1] = c_{1kr}\int_0^\infty s^{r-q+t+k}\chi(s)\,ds,$$

$$v_k[\sigma_3] = c_{2kr}\int_0^\infty s^{r-q+t+k}\chi(s)\,ds + c_{3kr}\int_0^\infty s^{r-q+t+k}\ln s\,\chi(s)\,ds$$

if $r-q+t+k = r-\mu-1+k\geqslant 0$. Obviously

$$v_k[\sigma_2] = 0.$$

Since $v_k[a] = 0$ for $a\in C_0^{2q}(\boldsymbol{R}^1)$ $(k = 0,1,2,\ldots)$ it follows that

$$c_{1kr} = c_{2kr} = c_{3kr} = 0, \quad \text{if } r-\mu-1+k\geqslant 0. \tag{39}$$

Hence by (38) and Lemma 2 we conclude that

$$L_qE[K\varphi] = v_0[K\varphi] \quad \text{for } \varphi\in C_0^\infty(\boldsymbol{R}^{m+n}).$$

To prove relation (4) we observe first that

$$L^{\mathrm{tr}}(s^\mu) = 0, \quad (L^{\mathrm{tr}})^{\mu+1}(s^\mu\ln s) = 0$$

which easily implies the existence of the constants c_1, c_2, c_3 such that

$$v_0[\chi(s)s^\mu\ln|s|] = c_1\int_0^\infty \chi'(s)\,ds + c_2\int_0^\infty \ln s\,\chi'(s)\,ds$$

$$= -c_1 + c_2\int_0^\infty \ln s\,\chi'(s)\,ds,$$

$$v_0[\chi(s)s^\mu Y(s)] = v_0[\chi(s)s^\mu] = 0,$$

v_0 being zero on smooth functions we conclude that $c_2 = 0$. Hence

$$v_0[K\varphi] = A\varphi(0),$$

where

$$A = -\tfrac{1}{4}|S_m||S_n|A(a_0, b_0)c_1 \quad \text{if } a \notin \boldsymbol{N}_0,\ b_0 \notin \boldsymbol{N}_0,$$
$$A = 0 \quad \text{if } a_0, b_0 \in \boldsymbol{N}_0.$$

Case $I_{\mu>q}$. The distribution E given by (14) is now a candidate for a solution of (27). A standard procedure requires that we write (38) with v_k homogeneous but this time we put

$$v_k = v_{k1} + v_{k2},$$

$$(40)\quad v_{k1} = \sum_{j=\max(0,k-(\mu-q))}^{\min(k,q)} c_{k-j} a_{q-j} L^{q-j} \delta^{(-k+j+\mu-q)} \quad \text{if } k \leqslant \mu,$$

$$v_{k1} = 0 \quad \text{if } k > \mu,$$

$$v_{k2} = \sum_{j=0}^{k-\mu+q-1} c_{k-j} a_{q-j} L^{q-j} \frac{s^{k-j-\mu+q-1}}{(k-j-\mu+q-1)!} Y \quad \text{if } k \geqslant -q+1+\mu,$$

$$(41)\quad v_{k2} = 0 \quad \text{if } k < \mu - q + 1.$$

Next we prove that(5)

$$(42)\quad v_{k1}[s^r\chi(s)Y(s)] = v_{k1}[s^r\chi(s)\ln|s|] = 0 \quad \text{if } r > \mu - k \quad (k = 0, 1, 2, \ldots)$$

and we observe that $v_{k1}[s^r\chi(s)Y(s)]$, $v_{k1}[s^r\chi(s)\ln|s|]$ are independent of χ and by (41) also v_{k2} $(k = 0, 1, 2, \ldots)$ possess this property.

It follows from the form of v_{k2} and from the considerations conducted in case $I_{\mu<q}$ that

$$v_{k2}[K\varphi] = 0 \quad \text{for } k = 1, 2, \ldots$$

Hence from (38), (41) and (42) we deduce

$$L_q E[K\varphi] = v_{01}[K\varphi] + v_{02}[K\varphi] = L^q \delta^{(\mu-q)}[K\varphi]$$
$$= \delta^{(\mu-q)}[(L^{\mathrm{tr}})^q(K\varphi)]$$
$$= (-1)^{\mu-q} D^{m-q}(L^{\mathrm{tr}})^q(K\varphi)(s)|_{s=0}.$$

---

(5) In the proof of the second of these equalities we employ the following fact: Let L be a one-dimensional differential operator, homogeneous of order λ. Then there exist constants $c_1(k)$, $c_2(k)$, $c_3(k)$ such that

$$L^{\mathrm{tr}}(s^k \ln|s|) = \begin{cases} c_1(k)s^{k-\lambda}\ln|s| + c_2(k)s^{k-\lambda} & \text{if } k-\lambda \geqslant 0, \\ c_3(k)s^{k-\lambda} & \text{if } k-\lambda < 0. \end{cases}$$

If $a_0, b_0 \notin N_0$, then it follows from the proof of assertion 4° of Theorem 4 in Part 2 in [1] that([6])

$$D^{\mu-q}(L^{\mathrm{tr}})^q(K\varphi)(s)|_{s=0} = \varphi(0)\tilde{A}(\mu-q)!.$$

To obtain the value of the coefficient $\tilde{A}$ one can apply Part 2 in [1] with $\tilde{\lambda} = q$, $M = 2q$, $L^{\mathrm{tr}} = (L^q)^{\mathrm{tr}}$. Then $\tilde{A}$ is given by assertion 4° in Theorem 4 [1].

5. Computation of the constants A and $\tilde{A}$. We now pass to an explicit computation of the constants A and $\tilde{A}$. The method will be essentially the same in all cases considered previously. We shall outline it in the case NI^e and give the final result in the remaining cases. We define distributions

$$\begin{aligned} E_q[a] &= \int_0^\infty \frac{1}{\sqrt{s}} D^{[\mu]-q+1} a(s)\,ds \quad \text{if } [\mu] \geqslant q, \\ &= \int_0^\infty s^{q-\mu-1} a(s)\,ds \quad \text{if } [\mu] < q. \end{aligned} \qquad q = 1, 2, \ldots$$

In view of (28) we need only compute the constants A_q in

$$L^q E_q[K\varphi] = A_q \varphi(0).$$

We shall prove that

$$\begin{aligned} LE_q &= 4(q-1)E_{q-1} && \text{if } 2 \leqslant q \leqslant [\mu]+1, \\ LE_q &= 4(q-1)(q-\mu-1)E_{q-1} && \text{if } q > [\mu]+1 \end{aligned} \tag{43}$$

as distributions on $\boldsymbol{R}^1$. In fact (43) is obvious outside zero and extends to zero by a homogeneity argument. The homogeneity argument applied to the distribution

$$u_q[\varphi] = E_q[K\varphi], \qquad \varphi \in C_0^\infty(\boldsymbol{R}^{m+n})$$

shows that (43) is also valid for functions $K\varphi$. By (43) we get the identity

$$\begin{aligned} A_q &= 4(q-1)A_{q-1}, && 2 \leqslant q \leqslant [\mu]+1, \\ A_q &= 4(q-1)(q-\mu-1)A_{q-1}, && q > [\mu]+1. \end{aligned} \tag{44}$$

The value of A_1 is well known (it can also be computed from [1]) and equals

$$A_1 = \frac{16\mu}{(\mu-1)!}\,\frac{(2\pi)^{\mu+1}}{\Gamma(\mu+1)}. \tag{45}$$

([6]) Note that the operator L^q is homogeneous of order $\lambda = q$.

Thus by the recurrence formula (44) we get[7]

$$A_q = 4^{q-1}(q-1)!(q-\mu-1)!\frac{16\mu}{(\mu-1)!}\cdot\frac{(2\pi)^{\mu+1}}{\Gamma(\mu+1)}. \tag{46}$$

Case *NI°*. Define

$$\tilde{E}_q[a] = \int_{-\infty}^{0} \frac{1}{\sqrt{|s|}} D^{[\mu]-q+1} a(s)\,ds \quad \text{if } [\mu] \geqslant q, \qquad q = 1, 2, \ldots,$$

$$= \int_{-\infty}^{0} |s|^{q-\mu-1} a(s)\,ds \quad \text{if } [\mu] < q,$$

then

$$L^q E_q[K\varphi] = \tilde{A}_q \varphi(0), \quad \varphi \in C_0^\infty(\boldsymbol{R}^{m+n}),$$

where

$$\tilde{A}_q = (-4)^{q-1}(q-1)!(q-\mu-1)!\frac{16\mu}{(\mu-1)!}\cdot\frac{(2\pi)^{\mu+1}}{\Gamma(\mu+1)}. \tag{47}$$

Case *I*. We define distributions

$$E_q[a] = \delta^{(\mu-q)}, \quad \mu \geqslant q,$$

$$= \int_0^\infty s^{q-\mu-1} a(s)\,ds, \quad \mu < q.$$

Then $LE_q = A_q^* E_{q-1}$, where

$$A_q^* = 4(q-1) \quad \text{if } q \leqslant \mu+1,$$

$$= 4(q-1)(q-\mu-1) \quad \text{if } q > \mu+1.$$

Since

$$A_1 = \frac{16(2\pi)^\mu}{\Gamma(\mu+1)}$$

we get

$$A_q = 4^{q-1}(q-1)!(q-\mu-1)!\,\frac{16(2\pi)^\mu}{\Gamma(\mu+1)}. \tag{48}$$

6. Remarks concerning the case $m = 1$. It follows from the preceding considerations that if n is even then if $\frac{1}{2}(n-2) \geqslant q$ we have

$$L_q E[K\varphi] = A_q \varphi(0), \quad \varphi \in C_0^\infty(\boldsymbol{R}^{1+n}), \tag{49}$$

[7] We put $(-t)! = 1$ if $t > 0$.

where E is given by (23) and A_q by (46) and if $\frac{1}{2}(n-2) < q$ we have

$$L_q E[K\varphi] = \frac{1}{(q-\mu-1)!} A_q \varphi(0), \quad \varphi \in C_0^\infty(\mathbf{R}^{1+n}) \tag{50}$$

with E given by (17), A_q given by (46).

If n is odd, then if $\frac{1}{2}(n-1) < q$ we have

$$L_q E[K\varphi] = \frac{1}{(q-\mu-1)!} A_q \varphi(0), \tag{51}$$

where E is given by (7) and A_q by (48), and if $\frac{1}{2}(n-1) \geqslant q$ we have

$$L_q E[K\varphi] = A_q \varphi(0), \tag{52}$$

where E is given by (14) and A_q by (46). We observe that the supports of these fundamental solutions are contained in the set $S = \{(x_1, y) \in \mathbf{R}^{1+n}: x_1^2 - \sum_{i=1}^{n} y_i^2 \geqslant 0\}$. It is possible, however, to improve them so that the supports be contained in

$$S_+ = S \cap \{(x_1, y) \in \mathbf{R}^{1+n}: x_1 \geqslant 0\} \quad (S_- = S \cap \{(x_1, y) \in \mathbf{R}^{1+n}: x_1 \leqslant 0\}).$$

The reason lies in the fact that the sets $\{(x_1, y): x_1^2 - \sum_{i=1}^{n} y_i^2 = s_0\}$ split for $s_0 \geqslant 0$ into two components: the one contained in S_+, the other in S_-. Consequently the operation K defined in [1] splits into two operations $K_+ = K(Y(x_1)\varphi(x_1, y))$ and $K_- = K(Y(-x_1)\varphi(x_1, y))$(8). Since both K_+ and K_- have identical properties as K, formulas (49)–(52) are valid for K replaced by K_+ or K_- with the constants divided by 2.

To underline the importance of having two fundamental solutions with supports in S_+ and S_- we shall prove

Remark. Let u_+, u_- be two fundamental solutions of an operator P on $\mathbf{R}^{1+n}$ with supports in S_+, S_- respectively. Then for every distribution $f \in \mathscr{D}'(\mathbf{R}^{1+n})$ there exists a distribution $g \in \mathscr{D}'(\mathbf{R}^{1+n})$ such that

$$Pg = f. \tag{53}$$

Proof. Let $\chi(x_1)$ be a function in $C_0^\infty(\mathbf{R}^1)$ such that $\chi(x_1) = 1$ for $x_1 \geqslant 0$, $\chi(x_1) = 0$ for $x_1 < -1$. Put

$$f_+ = \chi(x_1) f, \quad f_- = (1-\chi(x_1)) f$$

(8) Since the value of $(K\varphi)(s_0)$ for $s_0 \geqslant 0$ depends only on the values of φ on the set $\{(x_1, y): x_1^2 - \sum_{i=1}^{n} y_i^2 = s_0\}$.

and observe that there exist the convolutions

$$g_+ = f_+ * u_+ \quad \text{and} \quad g_- = f_- * u_-.$$

Hence the distribution $g = g_+ + g_-$ satisfies (53).

Reference

[1] Z. Szmydt, B. Ziemian, *Special solutions of the equations $Pu = 0$, $Pu = \delta$ for invariant linear differential operators with polynomial coefficients*, J. Differ. Eq. 38 (1980), p. 226–256.

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