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The u-invariant of fields with 16 and 32 square classes. I

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Abstract. We discuss here the conjectures of Kaplansky and of Lam concerning the u-univariant of a field of characteristic different from two. Both conjectures are shown to hold true for any field having at most 32 square classes.

Introduction. For a non-real field K the u-invariant is defined to be the maximal dimension of anisotropic quadratic forms over K. Thus, for the field of complex numbers, for finite fields and for p-adic fields u is 1, 2 and 4, respectively. It was conjectured by I. Kaplansky in 1953 that u, if finite, is always a power of two, but this has been verified only in some special cases.

Elman and Lam [3] generalized the notion of u-invariant to the case of an arbitrary field K. The general u-invariant is defined to be the maximal dimension of anisotropic torsion quadratic forms in the Witt ring $W(K)$ of K. If $I = I(K)$ is the fundamental ideal of $W(K)$, then from Arason–Pfister's Hauptsatz [1] it follows that I^n is torsion-free whenever $u < 2^n$ (cf. [3], Corollary 1.2). T. Y. Lam ([4], p. 333) asks whether the converse is true, that is, does $I^n \cap W_t = 0$ imply $u < 2^n$? (here W_t denotes the torsion ideal of $W(K)$).

In this paper we answer Lam's question in the affirmative for any field K having at most 32 square classes, and for any such field we prove that u is a power of two, thus verifying Kaplansky's conjecture for this class of fields.

The first section contains the results for the case of fields with 16 square classes. If the numbers q of square classes is $\leqslant 8$, the above statements holds true as was proved by R. Elman and T. Y. Lam.

In the second section we consider the fields with $q = 32$ and we prove that $I^n \cap W_t = 0$ implies $u < 2^n$, when $n = 3$. The cases where $n < 3$ have been settled by Elman and Lam ([3], Proposition 1.8). The remaining cases will be considered in the second part of the paper.

We use standard notation and terminology. Thus u is always the general u-invariant, $q = q(K)$ is the number of square classes, that is, the order of the group $g(K) = K^*/K^{*2}$, $s = s(K)$ is the level of the field K, i.e., the minimal number of summands in a representation of -1 as the sum of squares. For a formally real field K we put $s = \infty$. Both q and s, if finite, are powers of two.

The diagonalized quadratic form $\varphi = \sum a_i x_i^2$ is denoted by $\varphi = (a_1, \ldots, a_n)$ and the set of square classes represented by φ is denoted by $D(\varphi)$.

The form $\varphi = (1, a_1) \cdot \ldots \cdot (1, a_n)$ is called *n-fold Pfister form* and is denoted by $\langle\langle a_1, \ldots, a_n \rangle\rangle$. The value set $D(\varphi)$ of a Pfister form is always a subgroup of $g(K)$.

The cardinality of a set X is denoted $|X|$. Thus $q = |g(K)|$. The fields under consideration are assumed to have characteristic different from two.

1. Fields with 16 square classes. First we state and prove a couple of lemmas which will be of use in both sections. The main result of this section is Theorem 1 which verifies Kaplansky's conjecture for any field with $q = 16$ and answers Lam's question for the same class of fields.

LEMMA 1.1. *If $I^3 \cap W_t = 0$ and $I^2 \cap W_t$ does not contain any 6-dimensional anisotropic form, then $u \leqslant 4$.*

Proof. Let $\varphi \neq 0$ be any anisotropic form in $I^2 \cap W_t$. According to Theorem 2.8 of [2] $\varphi = \sum \langle\langle a_i, b_i \rangle\rangle$, where the Pfister form $\langle\langle a_i, b_i \rangle\rangle$ is torsion. Since $2\langle\langle a_i, b_i \rangle\rangle \in I^3 \cap W_t = 0$, we have $f = \langle\langle a_i, b_i \rangle\rangle + \langle\langle a_j, b_j \rangle\rangle = \langle\langle a_i, b_i \rangle\rangle - \langle\langle a_j, b_j \rangle\rangle = (a_i, b_i, a_i b_i, -a_j, -b_j, -a_j b_j) \in W(K)$. If $f \neq 0$, its anisotropic subform f_{an} has dimension 4. Indeed, $\dim f_{an} = 6$ is excluded by assumption and $\dim f_{an} = 2$ is impossible by the Hauptsatz of Arason–Pfister [1]. Now, $f_{an} \in I^2$ implies that $d(f_{an}) = 1$, and so f_{an} is a scalar multiple of a 2-fold Pfister form. But $I^3 \cap W_t = 0$ implies that f_{an} is universal (cf. [3], Lemma 1.3); hence f_{an} is a 2-fold Pfister form. It follows that φ itself is a 2-fold Pfister form. Hence $u \leqslant 4$ by [3] Proposition 1.8 (3). Q.E.D.

LEMMA 1.2. *If $I^3 \cap W_t = 0$ and φ is an arbitrary 6-dimensional form in $I^2 \cap W_t$ then for any $a \in K^*$ there exist $x, y \in K^*$ such that*

$$(1.2.1) \qquad \varphi \cong (1, -b) \perp x(1, -c) \perp y(1, -bc), \quad \textit{where } b, c \in D(1, a).$$

Proof. For any $a \in K^*$ we have $(1, a) \cdot \varphi \in I^3 \cap W_t = 0$; hence $(1, a) \cdot \varphi$ is hyperbolic and by Corollary 2.3 of [3], $\varphi \cong z(1, -b) \perp x(1, -c) \perp y(1, d)$, where $b, c, d \in D(1, a)$.

Moreover, $(1, -z) \cdot \varphi = 0$ in $W(K)$ implies that $\varphi \cong z\varphi$, so we can assume that $z = 1$. Further, $\varphi \in I^2$ implies that $d(\varphi) = 1$ (cf. [5], Satz 13); hence $bcd = 1$ and $d = bc$ in $g(K)$, as required.

The next lemma records some instances where a 6-dimensional form in $I^2 \cap W_t$ is necessarily isotropic. The result will be used repeatedly in the sequel.

LEMMA 1.3. *Suppose K is a field with $q = 2^n$ and $I^3 \cap W_t = 0$. Let $\varphi \cong (1, -b) \perp x(1, -c) \perp y(1, -bc)$ be an arbitrary 6-dimensional form in $I^2 \cap W_t$.*

(i) *If $g(K) = D(1, -b) \cdot D(1, -c)$, then φ is isotropic.*

(ii) *If $D(1, -b) \subset D(1, -c)$ and $(1, -b) \in W_t$ or $(1, x) \in W_t$, then φ is isotropic.*

(iii) *If $|D(1, -b)| \geqslant 2^{n-1}$ and $(1, -c) \in W_t$, then φ is isotropic.*

Proof. Assume (i). Then $-x = kl$, where $k \in D(1, -b)$, $l \in D(1, -c)$ and the form $(1, -b) \perp x(1, -c) \cong (1, -b) \perp (-k)(1, -c)$ is isotropic; hence φ itself is isotropic.

Assume (ii). Then $D\big((1, -b) \perp x(1, -c)\big) \supset D\big((1, -b) \perp x(1, -b)\big) = D(\langle\langle x, -b\rangle\rangle) = g(K)$, since $I^3 \cap W_t = 0$ implies that any torsion 2-fold Pfister form is universal, and $\langle\langle x, -b\rangle\rangle \in W_t$ since $(1, -b) \in W_t$ or $(1, x) \in W_t$. Thus φ contains a 4-dimensional universal subform and is isotropic.

Now consider (iii). If $D(1, -b)$ does not contain $D(1, -c)$, then $D(1, -b) \cdot D(1, -c) = g(K)$ and, by (i), we are finished. If $D(1, -c) \subset D(1, -b)$, then by (ii), $x\varphi$ is isotropic. This proves the lemma.

The next three lemmas give some sufficient conditions for $u \leqslant 4$ to hold for a field K.

LEMMA 1.4. *Let K be a field such that $I^3 \cap W_t = 0$ and suppose that there exists $a \in K^*$ such that $|D(1, a)| = 2$. Then $u \leqslant 4$.*

Proof. Suppose there exists a 6-dimensional anisotropic form $\varphi \in I^2 \cap W_t$. Then we have the decomposition (1.2.1) for φ. Since $b, c, bc \in D(1, a)$ and $|D(1, a)| = 2$, one of the three elements must be a square and then, by (1.2.1), φ is isotropic. Hence $I^2 \cap W_t$ does not contain anisotropic 6-dimensional forms, and by Lemma 1.1, $u \leqslant 4$.

LEMMA 1.5. *Let K be a non-real field with $s(K) = 1$ and assume $I^3 = 0$. If there exists $a \in K^*$ such that $|D(1, a)| = 4$, then $u \leqslant 4$.*

Proof. If $a = 1$, then $4 = |D(1, 1)| = q$, and $u \leqslant 4$. So assume that a is a non-square in K. Then there exists $b \in K^*$ such that $D(1, a) = \{1, a, b, ab\}$ and from $s = 1$ we obtain easily $D(1, a) \subset D(1, b)$. According to Lemma 1.2, any anisotropic 6-dimensional form in I^2 ($I^2 = I^2 \cap W_t$) has the decomposition

$$\varphi \cong (1, a) \perp x(1, b) \perp y(1, ab),$$

and by Lemma 1.3(ii) φ is isotropic, a contradiction. Hence $u \leqslant 4$, by Lemma 1.1.

LEMMA 1.6. *Let K be a non-real field with $s(K) = 2$ and assume $I^3 = 0$. If there exists $a \in D(1, 1)$ such that $|D(1, a)| = 4$, then $u \leqslant 4$.*

Proof. We have $D(1, a) \subset D(1, 1)$. Indeed, if $s = 2$, then $a \in D(1, 1)$ implies $-a \in D(1, 1)$ whence $-1 \in D(1, 1)$. Hence if a is a non-square, then $D(1, a) = \{\pm 1, \pm a\}$ is contained in $D(1, 1)$. Now, by Lemma 1.2, if φ is a 6-dimensional anisotropic form in I^2, then

$$\varphi \cong (1, a) \perp x(1, -a) \perp y(1, 1),$$

and Lemma 1.3(ii) implies that φ is isotropic. Hence $u \leqslant 4$, by Lemma 1.1.

Now we can prove Kaplansky's conjecture for all fields with $q \leqslant 16$.

THEOREM 1. *Let K be any field with square class number $q \leqslant 16$. Then*

(i) *$u = u(K)$ is a power of two,*

(ii) *$I^k \cap W_t = 0$ implies $u < 2^k$ for any $k \geqslant 1$.*

Proof. (i) By the results of Elman and Lam ([3], Theorem 2.7; Theorem 2.7′; Corollary A.6) we have either $u \leqslant 8$ or $u = 16$. It is well known that $u \neq 3, 5, 7$. Hence, to prove (i) we must show that $u \neq 6$. First observe that $u < 8$ implies $I^3 \cap W_t = 0$ (by Arason–Pfister's Hauptsatz) and so we can apply our Lemmas 1.1 through 1.6. If $u < 8$ and there exists a binary form representing exactly 2 square classes, then $u \leqslant 4$, by Lemma 1.4. So we may assume that $|D(\beta)| \geqslant 4$, for any anisotropic binary form β over K. If the field is non-real and $s(K) = 1$, then Lemma 1.5 applies in the case where $|D(\beta)| = 4$ for a binary form β, and if $|D(\beta)| \geqslant 8$ for every binary form β, then Lemma 1.3(iii) applies, and in both cases we obtain $u \leqslant 4$. If $s(K) = 2$ and there exists a form $(1, a)$, $a \in D(1, 1)$, representing exactly 4 square classes, then by Lemma 1.6 we obtain $u \leqslant 4$, and if every such binary anisotropic form represents 8 square classes, then, by Lemma 1.3(iii), again we get $u \leqslant 4$.

Thus we may assume that $s > 2$ and $|D(\beta)| \geqslant 4$ for every anisotropic binary form β. If $u = 6$, then by Lemmas 1.1 and 1.2, there exists a 6-dimensional anisotropic form $\varphi \in I^2 \cap W_t$, and

$$\varphi \cong (1, -a_1) \perp x(1, -a_2) \perp y(1, -a_1 a_2), \quad \text{where } a_1, a_2 \in D(1, 1).$$

Since, by Lemma 1.3 we can assume $|D(\beta_i)| < 8 = q/2$, the forms $\beta_1 = (1, -a_1)$, $\beta_2 = (1, -a_2)$, $\beta_3 = (1, -a_1 a_2)$ represent exactly 4 square classes. So we have $D(\beta_1) = \{1, -1, a_1, -a_1\}$, $D(\beta_2) = \{1, -1, a_2, -a_2\}$, $D(\beta_3) = \{1, -1, a_1 a_2, -a_1 a_2\}$, whence $D(\beta_1)\cdot D(\beta_2) = D(\beta_1)\cdot D(\beta_3) = D(\beta_2)\cdot D(\beta_3)$. Observe that x and y do not belong to the set $D(\beta_1)\cdot D(\beta_2)$, since otherwise φ would be isotropic. Hence

$$g(K) = D(\beta_1)\cdot D(\beta_2)\cdot\{1, x\} = D(\beta_2)\cdot D(\beta_3)\cdot\{1, x\}.$$

Now, $y = b_2 b_3 x$, where $b_i \in D(\beta_i)$. Then

$$x\beta_2 \perp y\beta_3 = -xb_2\beta_2 \perp xb_2\beta_3,$$

and we conclude that $x\beta_2 \perp y\beta_3$ is isotropic. Thus also φ is isotropic, a contradiction.

Summarizing, we have proved that in any case $u \neq 6$, and so u is a power of two.

(ii) In the above part of the proof we have checked (ii) for $k = 3$. If $k = 1, 2$, the result is true for any field K, as proved by Elman and Lam ([3], Proposition 1.8). If $k \geqslant 5$, then $u \leqslant q < 2^k$, so that (ii) is satisfied. It remains to check (ii) for $k = 4$. Suppose $u \geqslant 2^k = 16$; then from $q \leqslant 16$ and from Kneser's theorem we obtain $u = 16 = q$. Now, $u = q$ can happen only in a non-real field K with $s = 1$ or 2 (cf. [3], 2,7; 2,7′; A.6). If $s = 1$, the u-dimensional anisotropic form has $q = u = 16$ different entries in the diagonalization and so it is a 4-fold Pfister form. If $s = 2$, and $g(K) = \{1, -1\}\cdot D$, then it is easily seen that the u-dimensional anisotropic form is $2\cdot\varphi$, where φ has the diagonalization consisting of all elements of D, since any other u-dimensional form is necessarily isotropic.

Thus $2\cdot\varphi$ is again a 4-fold Pfister form. Hence, $u \geqslant 2^4$ implies $I^4 \cap W_t \neq 0$, and the theorem is proved.

2. Fields with 32 square classes. In this section we consider fields K with 32 square classes and such that $I^3 \cap W_t = 0$. The main result is contained in the following theorem.

THEOREM 2. *For any field K with $q = 32$ and $I^3 \cap W_t = 0$, the u-invariant satisfies $u \leqslant 4$.*

The theorem combined with Corollary 1.2 in [3] and with the results of the first section, gives immediately the following

COROLLARY. *For any field K with $q \leqslant 32$, we have $u \neq 6$.*

To prove the theorem, it suffices to show that $I^2 \cap W_t$ does not contain 6-dimensional anisotropic forms (cf. Lemma 1.1). If such a form φ does exist, we take its β-decomposition (cf. [3], p. 289)

$$\varphi = \beta_1 \perp \beta_2 \perp \beta_3, \quad \text{where } \beta_i = x_i\langle 1, -a_i\rangle,\ a_i \in D(1, 1),\ i = 1, 2, 3.$$

Now, we can assume that $|D(\beta_i)| \leqslant 8$, for $i = 1, 2, 3$ (by Lemma 1.3) and also we can assume that $|D(\beta_i)| \geqslant 4$, $i = 1, 2, 3$ (by Lemma 1.4).

In the cases where $s \leqslant 2$, we may even assume that $|D(\beta_i)| \geqslant 8$, $i = 1, 2, 3$, by Lemmas 1.5 and 1.6. Thus it remains to consider the following three cases:

1° $|D(\beta_1)| = |D(\beta_2)| = 8$;

2° $|D(\beta_1)| = 8$ and $|D(\beta_2)| = |D(\beta_3)| = 4$ and $s > 2$;

3° $|D(\beta_i)| = 4$, $i = 1, 2, 3$, and $s > 2$.

We prove that in the first two cases the form φ is necessarily isotropic. This is done in Lemma 2.2 and Lemma 2.3, respectively. Lemma 2.1 shows that the third case cannot occur for any field with $q \geqslant 32$ and $I^3 \cap W_t = 0$. Thus in every case any 6-dimensional form in $I^2 \cap W_t$ is isotropic, and so $u \leqslant 4$, by Lemma 1.1.

We now complete the details of the above discussion.

LEMMA 2.1. *Let K be a field with $|g(K)| = 2^n \geqslant 32$ and $I^3 \cap W_t = 0$ and assume that $s(K) > 2$ (we allow $s = \infty$). If a_1, a_2 are any elements of $D(1, 1)$ and $a_3 = a_1 a_2$, then there exists $i \in \{1, 2, 3\}$ such that the binary form $(1, -a_i)$ represents at least 8 elements of $g(K)$.*

Proof. Suppose the assertion does not hold. Put $\beta_i = (1, -a_i)$, $i = 1, 2, 3$. Then $a_i \in D(1, 1)$ implies that $-1 \in D(\beta_i)$, and $s > 2$ implies that $a_i \neq -1$. So we obtain $D(\beta_i) = \{\pm 1, \pm a_i\}$, $i = 1, 2, 3$, that is, all the three forms β_i represent exactly four square classes. From $I^3 \cap W_t = 0$ we conclude that any torsion 2-fold Pfister form over K is universal; hence for any $i, j \in \{1, 2, 3\}$, $i \neq j$, we have $D\langle\langle -a_i, -a_j \rangle\rangle = g(K)$. On the other hand,

$$D\langle\langle -a_i, -a_j \rangle\rangle = D(\beta_i \perp -a_j \beta_i) = \bigcup \{D(x, y) \colon x \in D(\beta_i), y \in -a_j D(\beta_i)\}.$$

Here each of x and y takes on 4 different values and so we obtain the union of 16 sets $D(x, y)$. We observe that $D(\beta_i) \subset D(1, a_i) \cup -D(1, a_i) := \pm D(1, a_i)$, $i = 1, 2, 3$, and using this we can represent $g(K)$ as the union of 8 sets $D(x, y)$ in the following way:

$$g(K) = \pm D(1, a_j) \cup \pm a_i D(1, a_j) \cup \pm D(1, a_k) \cup \pm a_i D(1, a_k),$$

where $a_k = a_i a_j$. Now, $T_{ij} = \pm D(1, a_j) \cup \pm a_i D(1, a_j)$ is a subgroup of $g(K)$, and so is T_{ik}. Thus $g(K) = T_{ij} \cup T_{ik}$ implies that one of the two subgroups contains the other and that the larger one is equal to $g(K)$. Suppose $T_{ik} \subset T_{ij} = g(K)$. Since T_{ij} consists of four cosets of $D(1, a_j)$, we obtain

$$|D(1, a_j)| \geqslant 2^{n-2}.$$

Now, if we change the role of i and j in the above argument, we obtain $g(K) = T_{ji} \cup T_{jk}$, and as above we conclude that one of the forms $(1, a_i)$ and $(1, a_k)$ represents at least 2^{n-2} square classes. Thus we have proved that two of the forms $(1, a_i)$, $i = 1, 2, 3$, represent at least 2^{n-2} square classes. Without loss of generality we can assume that

$$|D(1, a_1)| \geqslant 2^{n-2} \quad \text{and} \quad |D(1, a_2)| \geqslant 2^{n-2}.$$

It is known that for any a, b in K^*,

$$D(1, -ab) \supset D(1, a) \cap D(1, b) \quad \text{(cf. [6], p. 50).} \tag{2.1.1}$$

So we obtain

$$D(1, -a_3) \supset D(1, a_1) \cap D(1, a_2).$$

We also have $|D(1, a_1) \cap D(1, a_2)| = |D(1, a_1)| \cdot |D(1, a_2)| / |D(1, a_1) \cdot D(1, a_2)| \geqslant 2^{n-2} \cdot 2^{n-2} / 2^n = 2^{n-4} \geqslant 2$, since we have assumed $n \geqslant 5$.

Hence $\beta_3 = (1, -a_3)$ represents at least 2 elements of $D(1, a_1) \cap D(1, a_2)$. But β_3 represents 4 square classes $\{\pm 1, \pm a_3\}$ and -1 and a_3

certainly do not belong to the intersection (since $s > 2$ and $|D(\beta_3)| = 4$), so we have $-a_3 = -a_1 a_2 \in D(1, a_1) \cap D(1, a_2)$. This implies that $-a_2 \in D(1, a_1)$ and $-a_1 \in D(1, a_2)$, and so $g(K) = T_{12} = \pm D(1, a_2) \cup \cup \pm a_1 D(1, a_2) = \pm D(1, a_2)$, and $g(K) = T_{21} = \pm D(1, a_1)$, whence $|D(1, a_1)| \geqslant 2^{n-1}$ and $|D(1, a_2)| \geqslant 2^{n-1}$. We obtain $|D(1, -a_3)| \geqslant |D(1, a_1) \cap D(1, a_2)| \geqslant 2^{n-1} \cdot 2^{n-1}/2^n = 2^{n-2} \geqslant 8$, which contradicts the hypothesis $|D(\beta_i)| < 8$, $i = 1, 2, 3$.

Thus the lemma is proved.

LEMMA 2.2. *Let K be a field with $q = 32$ and $I^3 \cap W_t = 0$. Let $\varphi \in I^2 \cap \cap W_t$ be 6-dimensional and let $\varphi \simeq \beta_1 \perp x\beta_2 \perp y\beta_3$ be the β-decomposition of φ, where $\beta_i = (1, -a_i)$, $a_i \in D(1, 1)$, $i = 1, 2, 3$, $a_1 a_2 = a_3$. If $|D(\beta_1)| = |D(\beta_2)| = 8$, then φ is isotropic.*

Proof. First observe that $D(\beta_1) \cdot D(\beta_2) = g(K)$ implies that φ is isotropic (Lemma 1.3(i)). So we can assume that $D(\beta_1) \cdot D(\beta_2)$ is a proper subgroup of $g(K)$ and in this case $D = D(\beta_1) \cap D(\beta_2)$ has at least 4 elements. If $|D| = 8$, then $D(\beta_1) = D(\beta_2)$ and $\beta_i \in W_t$. Hence by Lemma 1.3(ii), we obtain that φ is isotropic. So we can assume that $|D| = 4$. Now we shall consider three cases:

1° $a_2 \notin D(\beta_1)$, 2° $a_2 \in D(\beta_1)$ and $s \geqslant 2$, 3° $a_2 \in D(\beta_1)$ and $s = 1$.

1° If $a_2 \notin D(\beta_1)$, then also $a_1 \notin D(\beta_2)$ and $a_1 a_2 \notin D(\beta_1) \cup D(\beta_2)$. Hence

$$D \cap \{1, a_1, a_2, a_1 a_2\} = 1.$$

Now, $a_i \in D(\beta_i)$, and so

$$D(\beta_i) = \{1, a_i\} \cdot D, \quad i = 1, 2. \tag{2.2.1}$$

As regards β_3, we have $D(\beta_3) \supset \{1, a_3\} \cdot D$, since

$$D(\beta_3) = D(1, -a_1 a_2) \supset D(1, -a_1) \cap D(1, -a_2) = D.$$

If $D(\beta_3)$ contains something else, then $|D(\beta_3)| \geqslant 16$, and by Lemma 1.3(iii), φ is isotropic. Thus we may assume that

$$D(\beta_3) = \{1, a_3\} \cdot D. \tag{2.2.2}$$

From (2.2.1) we conclude that $D(\beta_1) \cdot D(\beta_2) = \{1, a_1\} \cdot \{1, a_2\} \cdot D$ consists of 16 elements. Now, $x \notin D(\beta_1) \cdot D(\beta_2)$, since otherwise $\beta_1 \perp x\beta_2$ is isotropic. Consequently,

$$\{1, x\} \cdot \{1, a_1\} \cdot \{1, a_2\} \cdot D = g(K). \tag{2.2.3}$$

From (2.2.1) we obtain

$$\{1, a_2\} \cdot D(\beta_1) = \{1, a_1\} \cdot D(\beta_2) = \{1, a_1\} \cdot \{1, a_2\} \cdot D,$$

and this combined with (2.2.3) gives

$$\begin{aligned} g(K) &= \{1, a_2\} \cdot D(\beta_1) \cup x\{1, a_1\} \cdot D(\beta_2) \\ &= D(\beta_1) \cup xD(\beta_2) \cup a_2 D(\beta_1) \cup x a_1 D(\beta_2). \end{aligned}$$

We shall show that for every $y \in g(K)$ the form φ is isotropic. Indeed, if $y \in D(\beta_1)\cup xD(\beta_2)$, then also $-y$ belongs to the same set, and φ is isotropic. The other possibility is $y \in a_2D(\beta_1)\cup xa_1D(\beta_2) = a_2\cdot D\cup a_1a_2D\cup \cup xa_1D\cup xa_1a_2D$, we have $a_1a_2D\cup xa_1a_2D \subset D(\beta_3)\cup xD(\beta_3)$; hence, if $y \in a_1a_2D\cup xa_1a_2D$, we may take $y = 1$ or $y = x$ and so $\beta_1 \perp y\beta_3$ or $x\beta_2 \perp y\beta_3$ is isotropic; hence so is φ.

Thus it remains to consider $y \in a_2D\cup xa_1D$. Since $D \subset D(\beta_3)$, we may take $y = a_2$ or $y = xa_1$, and then $y\beta_3 = a_2(1, -a_1a_2)$ represents $-a_1$, or $y\beta_3 = xa_1(1, -a_1a_2)$ represents $-xa_2$. But β_1 represents a_1 and $x\beta_2$ represents xa_2; thus in any case, φ is isotropic.

2° Now assume $a_2 \in D(\beta_1)$ and $s \geqslant 2$. Since -1 is not a square in K, $D(\beta_1) = \pm\{1, a_1, a_2, a_1a_2\}$. Also $a_2 \in D(\beta_1)$ implies $a_1 \in D(\beta_2)$, whence $D(\beta_1) = D(\beta_2)$, and we obtain $|D| = 8$, the case considered at the begining of the proof.

3° If $a_2 \in D(\beta_1)$ and $s = 1$, then all the three groups $D(\beta_i) = D(1, a_i)$, $i = 1, 2, 3$, contain $\{1, a_1, a_2, a_1a_2\}$. In view of Lemma 1.5, we may assume that every binary form over the field represents at least 8 elements of $g(K)$. Thus the form $(1, x)$ represents at least 8 elements of $g(K)$, and if, moreover, $D(1, x)\cap\{1, a_1, a_2, a_1a_2\} = 1$, then $g(K) = \{1, a_1, a_2, a_1a_2\}\cdot \cdot D(1, x)$ and the form $\beta_1 \perp x\beta_2 = (1, a_1) \perp x(1, a_2)$ represents all the elements of the set

$$D(1, x)\cup D(a_1, xa_1)\cup D(a_2, xa_2)\cup D(a_1a_2, xa_1a_2) = \{1, a_1, a_2, a_1a_2\}\cdot D(1, x) = g(K),$$

Hence $\beta_1 \perp x\beta_2$ is universal, and φ is isotropic. Also if $D(1, y)\cap\{1, a_1, a_2, a_1a_2\} = 1$, then the form $\beta_1 \perp y\beta_3$ is universal, and φ is isotropic. Thus we may assume that $D(1, x)\cap\{1, a_1, a_2, a_1a_2\} \neq 1$, and $D(1, y)\cap \cap\{1, a_1, a_2, a_1a_2\} \neq 1$. If $(1, x)$ represents a_1 or a_2, then $\beta_1 \perp x\beta_2 \cong (1, x) \perp (a_1, xa_2)$ would be isotropic; hence we may assume that $a_1a_2 \in D(1, x)$, and, similarly, $a_2 \in D(1, y)$. But then $x \in D(1, a_1a_2)$ and $y \in D(1, a_2)$, and we obtain

$$x\beta_2 \perp y\beta_3 = xy(1, a_2) \perp xy(1, a_1a_2) = xy(1, 1) \perp xy(a_2, a_1a_2),$$

which is isotropic, since $s = 1$. Thus in any case φ is isotropic and the Lemma is proved.

LEMMA 2.3. *Let K be a field with $q = 32$, $s \geqslant 4$ and $I^3\cap W_t = 0$. If φ is a 6-dimensional form in $I^2\cap W_t$, if $\varphi = \beta_1 \perp \beta_2 \perp \beta_3$ is the β-decomposition of φ and if, moreover, $|D(\beta_1)| = 8$, $|D(\beta_2)| = |D(\beta_3)| = 4$, then φ is isotropic.*

Proof. We have $\beta_1 = x(1, -a)$, $\beta_2 = y(1, -b)$, $\beta_3 = z(1, -ab)$ and the assumptions imply that $a, b \in D(1, 1)$, $D(1, -a) = \{\pm 1, \pm a, \pm c, \pm ac\}$ for a suitable $c \in g(K)$, $D(1, -b) = \{\pm 1, \pm b\}$, $D(1, -ab)$

$= \{\pm 1, \pm ab\}$. Here $b \notin D(1, -a)$, since otherwise $a \in D(1, -b)$ and $|D(1, -b)| > 4$, contrary to the assumption. Hence we can write (for a suitable $d \in g(K)$):

$$\begin{aligned} g(K) &= D(1, -a)\cdot\{1, b\}\cdot\{1, d\} = D(1, -a)\cdot\{1, b\} \cup dD(1, -a)\cdot\{1, b\} \\ &= D(1, -a)\cdot D(1, -ab) \cup dD(1, -a)\cdot D(1, -ab). \end{aligned}$$

On scaling φ, we may assume that $z = 1$, i.e., $\beta_3 = (1, -ab)$. Now, if $x \in D(1, -a)\cdot D(1, -ab)$, then also $-x \in D(1, -a)\cdot D(1, -ab)$ and $\beta_1 \perp \beta_3 = -x_1x_2(1, -a) \perp (1, -ab)$, where $x_1 \in D(1, -a), x_2 \in D(1, -ab)$. Now, $x_1(1, -a) = (1, -a)$; hence $\beta_1 \perp \beta_3 = -x_2(1, -a) \perp (1, -ab)$ is isotropic (since $x_2 \in D(1, -ab)$). Thus we may assume that $x \in dD(1, -a)\cdot$ $\cdot D(1, -ab)$.

Consider y. If $y \in dD(1, -a)\cdot D(1, -ab) = dD(1, -a)\cdot D(1, -b)$, then $\beta_1 \perp \beta_2 = dx_1(1, -a) \perp dy_1(1, -b)$, where x_1 and y_1 belong to $D(1, -a)\cdot D(1, -b)$. We can write $\beta_1 \perp \beta_2 = dy_1\big(x_1y_1(1, -a) \perp (1, -b)\big)$, and the form is isotropic, since here x_1, y_1 can be chosen from $D(1, -b)$, the other factors being absorbed by $(1, -a)$. Thus we may assume that $y \in D(1, -a)\cdot D(1, -b)$, and, in fact, it suffices to take $y \in D(1, -a)$, since the other factor can be absorbed by the form $(1, -b)$. Now, the forms $\pm a(1, -b) \perp (1, -ab)$ are isotropic; hence, without loss of generality, we may assume that $y = c$ or $y = ac$.

We have reduced the β-decomposition of the form φ to either of the following two diagonalizations:

$$\varphi_1 \cong x(1, -a) \perp c(1, -b) \perp (1, -ab) \quad \text{or} \quad \varphi_2 \cong x(1, -a) \perp ac(1, -b) \perp (1, -ab).$$

Rearranging the diagonal entries one can also obtain the following diagonalizations for φ_1 and φ_2 (but these are not necessarily β-decompositions):

$$\varphi_1 \cong x(1, -a) \perp (1, c) \perp -b(1, ac), \tag{2.3.1}$$

$$\varphi_1 \cong x(1, -a) \perp (1, -c) \perp -b(1, -ac), \tag{2.3.2}$$

$$\varphi_2 \cong x(1, -a) \perp -b(1, c) \perp (1, ac), \tag{2.3.3}$$

$$\varphi_2 \cong x(1, -a) \perp -b(1, -c) \perp (1, -ac). \tag{2.3.4}$$

Now we show that each of the forms $(1, \pm c)$, $(1, \pm ac)$ appearing in the above decompositions can be supposed to represent exactly 8 square classes. Indeed,

$$\{1, a, c, ac\} \subset D(1, c) \cap D(1, ac) \quad \text{and} \quad \{1, a, -c, -ac\} \subset D(1, -c) \cap D(1, -ac) \tag{2.3.5}$$

so that each of the four forms represents at least 4 elements of $g(K)$. If $|D(1, ec)| = 4$ for an $e \in \{\pm 1, \pm a\}$, then from the above inclusions we obtain $D(1, ec) \subset D(1, aec)$, and since $(1, -b) \in W_t$, by Lemma 1.3(ii), each of the forms (2.3.1)–(2.3.4) is isotropic. If, on the other hand, $|D(1, ec)| \geqslant 16$, then φ is isotropic by Lemma 1.3(iii). Thus we may assume that every form $(1, ec)$, $e = \pm 1, \pm a$, represents exactly 8 square classes. Further, we may assume that $-1 \notin D(1, ec)$, since otherwise $-ec \in D(1, 1)$ and one of the decompositions (1.3.1)–(1.3.4) is, in fact, a β-decomposition for φ containing two binary subforms, each of which represents exactly 8 square classes. According to Lemma 2.2, φ is isotropic.

For the rest of the proof we consider the diagonalizations (2.3.1) and (2.3.3) of φ. On scaling φ, we can assume that

$$\varphi \cong (1, -a) \perp x(1, c) \perp y(1, ac), \tag{2.3.6}$$

where x any y are appropriate elements of K^* and $|D(1, -a)| = |D(1, c)| = |D(1, ac)| = 8$, $-1 \notin D(1, c)$, $-1 \notin D(1, ac)$ and $|D(1, c) \cdot D(1, ac)| = 16$ (the latter follows from (2.3.5)).

First consider the case where $-1 \in D(1, c) \cdot D(1, ac)$. Then

$$D(1, c) \cdot D(1, ac) = \{1, -1\} \cdot D(1, c) = \{1, -1\} \cdot D(1, ac)$$

and

$$g(K) = \{1, -1\} \cdot D(1, c) \cdot \{1, k\} = \{1, -1\} \cdot D(1, ac) \cdot \{1, k\}, \tag{2.3.7}$$

for a suitable $k \in g(K)$. Since $(1, -a)$ represents ± 1, and φ has diagonalization (2.3.6), we may assume that $x \notin \pm D(1, c)$ and $y \notin \pm D(1, ac)$. Hence, by (2.3.7), we must have $x = kx_1$, $y = ky_1$, where $x_1 \in \pm D(1, c)$ and $y_1 \in \pm D(1, ac)$. But now $x(1, c) \perp y(1, ac) \cong kx_1(1, c) \perp ky_1(1, ac) \cong (\pm k)(1, c) \perp (\pm k)(1, ac)$, and this is isotropic if the signs are opposite. If the signs coincide, we take $d \in D(1, c)$ such that $-d \in D(1, ac)$, which is possible since $-1 \in D(1, c) \cdot D(1, ac)$. Then

$$x(1, c) \perp y(1, ac) \cong (\pm k)d(1, c) \perp (\mp k)d(1, ac).$$

and again the form is isotropic.

Now consider the case where $-1 \notin D(1, c) \cdot D(1, ac)$. In this case

$$g(K) = \{1, -1\} \cdot D(1, c) \cdot D(1, ac), \tag{2.3.8}$$

and the form (2.3.6) can be written as follows:

$$(1, -a) \perp \pm x_1 x_2 (1, c) \perp \pm y_1 y_2 (1, ac),$$

where $x_1, y_1 \in D(1, c)$ and $x_2, y_2 \in D(1, ac)$.

On scaling, we obtain

$$x_2 y_1 (1, -a) \perp \pm (1, c) \perp \pm (1, ac).$$

Hence we may assume that

$$\varphi = z(1, -a) \perp (1, c) \perp (1, ac),$$

or, rearranging the diagonal entries,

$$\varphi = (1, 1) \perp c(1, a) \perp z(1, -a). \tag{2.3.9}$$

Here the case $|D(1, 1)| \geqslant 16$ is ruled out by Lemma 1.3(iii), since $(1, -a) \in W_t$, and if $|D(1, 1)| = 8$, we get $g(K) = D(1, 1) \cdot D(1, -a)$ (since $D(1, 1) \cap D(1, -a) = \{1, a\}$, according to the fact that $-1 \notin D(1, ec)$, for $e = \pm 1, \pm a$). By Lemma 1.3(i) we obtain that φ is isotropic.

So let us assume that $|D(1, 1)| = 4$, i.e., $D(1, 1) = \{1, a, b, ab\}$. Suppose the field is non-real; then $s \geqslant 4$ and $I^3 = 0$ imply $s = 4$. Thus we must have $-1 \in D(a, b)$ for appropriate $a, b \in D(1, 1)$, and we obtain $-b \in D(1, a)$ and $-b \in D(1, ab)$. Since $I^3 = 0$, every 2-fold Pfister form is universal; hence $D((1, -b) \perp a(1, -b)) = g(K)$ and we obtain

$$\begin{aligned} g(K) &= \bigcup \{D(\alpha, \beta) \colon \alpha \in D(1, -b), \beta \in aD(1, -b)\} \\ &= D(1, -a) \cup bD(1, -a) \cup \pm D(1, a) \cup D \pm (1, ab), \end{aligned}$$

since $-b \in D(1, a)$ implies $\pm bD(1, a) = \pm D(1, a)$, and $-b \in D(1, ab)$ implies $\pm bD(1, ab) = \pm D(1, ab)$, and the other subsets are easily seen to be contained in the union.

If now $|D(1, a)| = 4$, then $\pm D(1, a) = \pm \{1, a, -b, -ab\} \subset \pm D(1, ab)$, and

$$g(K) = D(1, -a) \cdot \{1, b\} \cup \pm D(1, ab),$$

i.e., $g(K)$ is the union of two subgroups, and $D(1, -a) \cdot \{1, b\}$ is known to have 16 elements. Hence $g(K) = \pm D(1, ab)$ and $|D(1, ab)| \geqslant 16$. Now, $(1, -ab) \cdot \varphi = 0$ in $W(K)$ and $D(1, -ab) = \{\pm 1, \pm ab\}$. Whence, by Lemma 1.2, we must have (up to a scalar multiple)

$$\varphi = (1, 1) \perp x(1, ab) \perp y(1, -ab), \qquad x, y \in K^*.$$

But here $|D(1, ab)| \geqslant 16$, and according to Lemma 1.3(iii), φ is isotropic. The other possibility is $|D(1, a)| \geqslant 8$. But then from $D(1, -a) \cap D(1, a) \subset D(1, 1)$ (use (2.1.1)) it follows that $D(1, -a) \cap D(1, a) = \{1, a\}$, whence $g(K) = D(1, -a) \cdot D(1, a)$. Relation (2.3.9) and Lemma 1.3(i) show that φ is isotropic.

It remains to consider the case where K is a formally real field. We have $a, b \in D(1, 1)$ and $D(1, a) \cap D(1, b) \subset D(1, -ab) = \{\pm 1, \pm ab\}$. If $ab \in D(1, a) \cap D(1, b)$, then also a and b belong to $D(1, a) \cap D(1, b)$, which is impossible. Hence $D(1, a) \cap D(1, b) = \{1\}$. If $|D(1, a)| = 2$ or $|D(1, b)| = 2$, Lemma 1.4 applies. Otherwise we have $|D(1, a) \cdot D(1, b)| = |D(1, a)| \cdot |D(1, b)| \geqslant 16$ and since $q = 32$, the subgroup $D(1, a) \cdot D(1, b)$ consists of all totally positive square classes. In particular, K has a unique ordering, and we can assume that c is positive. Then from (2.3.8) it follows that $D(1, c) \cdot D(1, ac) = D(1, a) \cdot D(1, b)$, and hence $b \in D(1, c) \cdot D(1, ac)$. But the latter implies that the forms (2.3.1) and (2.3.3) are isotropic. Thus in all cases φ turns out to be isotropic and the lemma is proved.

References

[1] J. K. Arason und A. Pfister, *Beweis des Krullschen Durchschnittsatzes für den Wittring*, Invent. Math. 12 (1971), p. 173–176.

[2] R. Elman and T. Y. Lam, *Quadratic forms over formally real fields and pythagorean fields*, Amer. J. Math. 94 (1972), p. 1155–1194.

[3] — — *Quadratic forms and the u-invariant. I*, Math. Z. 131 (1973), p. 283–304.

[4] T.Y. Lam, *The algebraic theory of quadratic forms*, Reading, 1973.

[5] A. Pfister, *Quadratische Formen in beliebigen Körpern*, Invent. Math. 1 (1966), p. 116–132.

[6] K. Szymiczek, *Universal binary quadratic forms*, Uniw. Śląski w Katowicach, Prace Mat. 5 (1974), p. 49–57.

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