ANNALES
POLONICI MATHEMATICI
XLVI (1985)

On some classes of infinitely decomposable Sobolev–Schwartz test functions

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Abstract. There are considered three different subspaces of Schwartz's space $\mathscr{D}$ such that each element splits to a convolution of two elements of the same subspace.

We are going to define some subclasses $\mathscr{D}_0$ of $\mathscr{D}$ such that each function $\varphi \in \mathscr{D}$ is decomposable in $\mathscr{D}_0$, i.e. $\varphi = \varphi_1 * \varphi_2$, where $\varphi_1 \in \mathscr{D}_0$ and $\varphi_2 \in \mathscr{D}_0$. We shall first construct the Fourier transforms of functions belonging to $\mathscr{D}_0$.

Let $G(z)$ $(z = x+iy)$ be a function of exponential type, not identically 0, such that

$$|G(z)| \leqslant e^{\alpha|y|}, \tag{1}$$

$$|1-G(z)| \leqslant A|z| \quad \text{for } |z| < 1, \tag{2}$$

$$|G(t)| \leqslant B/|t| \quad \text{for } t \leqslant -1 \text{ and } t \geqslant 1, \tag{3}$$

where A, B and α are positive constants.

THEOREM. *If $\lambda_n \in \boldsymbol{R}$ and $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, then the product*

$$F(z) = \prod_{n=1}^{\infty} G(\lambda_n z) \tag{4}$$

represents an entire function, not identically 0 *such that*

$$|F(z)| \leqslant e^{\beta|y|} \tag{5}$$

and

$$\int_{-\infty}^{\infty} |t^k F(t)|\, dt \leqslant M_k, \tag{6}$$

where β and M_k are positive constants and $k = 1, 2, \ldots$

Proof. By (2), we have

$$|1-G(\lambda_n z)| \leqslant A|\lambda_n|\cdot|z| \quad \text{for } |z| < 1/|\lambda_n|.$$

Since $1/|\lambda_n| \to \infty$, the series

$$\sum_{n=1}^{\infty} |1-G(\lambda_n z)|$$

is uniformly convergent on compact sets and thus the product in (4) represents an entire function which is not identically 0.

By (1), we obtain

$$|G(\lambda_n z)| \leqslant \exp(\beta|y|),$$

where $\beta \in \alpha(\sum_{n=1}^{\infty} |\lambda_n|)$, so (5) holds.

Since, by (1),

$$|G(t)| \leqslant 1 \quad \text{for real } t,$$

we have

$$|t^k F(t)| \leqslant \left|t^k \prod_{n=1}^{k+2} G(\lambda_n t)\right|.$$

Hence, in view of (3), we get

$$|t^k F(t)| \leqslant C_k/t^2 \quad \text{for } |t| \geqslant a_k \tag{7}$$

and

$$|t^k F(t)| \leqslant |t|^k \quad \text{for } |t| < a_k, \tag{8}$$

where

$$C_k = B^{k+2}|\lambda_1, \ldots, \lambda_{k+2}|^{-1} \quad \text{and} \quad a_k = \max\left(\frac{1}{|\lambda_1|}, \ldots, \frac{1}{|\lambda_{k+2}|}\right).$$

Relations (7) and (8) imply (6) and the proof is completed.

As a function G satisfying conditions (1)–(3) we can take $\frac{\sin z}{z}$. Another function of that type is $G(z) = (\exp iz - 1)/z$.

By the Paley–Wiener theorem, the function $F(z)$ in (2) is the Fourier transform of the function

$$f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} F(t)\, dt,$$

whose support is contained in $[-\beta, \beta]$. Moreover, condition (6) guarantees that $f(s) \in C^{\infty}$ and

$$f^{(k)}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (it)^k e^{-ist} F(t)\, dt.$$

Therefore to each function G satisfying (1)–(3) there corresponds a class $\mathscr{D}_0 = \mathscr{D}_G \subset \mathscr{D}$ of functions, depending on the sequence λ_n, whose Fourier transforms are equal F.

Evidently, if we split the sequence λ_n in two subsequences λ_n' and λ_n'', then $\sum_{n=1}^{\infty} |\lambda_n'| < \infty$ and $\sum_{n=1}^{\infty} |\lambda_n''| < \infty$ and

$$F(z) = F_1(z) \cdot F_2(z),$$

where

$$F_1(z) = \prod_{n=1}^{\infty} G(\lambda_n' z) \quad \text{and} \quad F_2(z) = \prod_{n=1}^{\infty} G(\lambda_n'' z).$$

That means, we have $\varphi = \varphi_1 * \varphi_2$, where φ, φ_1, φ_2 are the functions of the class $\mathscr{D}_0$, corresponding to the functions F, F_1, F_2, respectively. Thus we have

$$\mathscr{D}_0 = \mathscr{D}_0 * \mathscr{D}_0. \tag{9}$$

Another class $\mathscr{D}_0$ with this property was given in the paper: L. A. Rubel, W. A. Squires and B. A. Taylor, *Irreducibility of certain entire functions with applications to harmonic functions*, Ann. of Math. 108 (1976), 560–561. The method of the authors consisted in considering zeros of the Fourier transforms of $\varphi \in \mathscr{D}$.

Piotr Mikusiński has remarked that a class $\mathscr{D}_0 \in \mathscr{D}$ with property (9) can be easily constructed directly, without any help of Fourier transform. In fact, there are functions $\delta_n \in \mathscr{D}$ such that $\delta_n \geqslant 0$, $\int \delta_n = 1$ and the support of δ_n is in $[-\lambda_n, \lambda_n]$, where $\sum_{n=1}^{\infty} \lambda_n < \infty$. We put $\varphi \in \mathscr{D}_0$ if $\varphi = \delta_1 * \delta_2 * \ldots$ Then and (9) holds, because we may set $\varphi_1 = \delta_1 * \delta_3 * \ldots$, $\varphi_2 = \delta_2 * \delta_4 * \ldots$

The class of Piotr Mikusiński is the simplest, but does not include the preceding classes. The variety of possible subclasses satisfying (9) suggests that we possibly have $\mathscr{D} = \mathscr{D} * \mathscr{D}$ for the whole class $\mathscr{D}$.

Reçu par la Rédaction le 3.07.1983