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A difference method for the non-linear partial differential equation of the first order

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§ 1. Introduction. This paper deals with the difference method for the solution of the partial differential equation

$$\frac{\partial u}{\partial \xi}=f\left(\xi, x, u, \frac{\partial u}{\partial x}\right), \tag{1.1}$$

and also the equation with an arbitrary number of independent variables.

In the two-dimensional case the corresponding difference equation is of the form

$$v^{\mu+1,m} = v^{\mu,m}+ kf\left(\xi^{\mu}, x^{m}, v^{\mu,m}, \frac{v^{\mu,m}-v^{\mu,m-1}}{h}\right), \tag{1.2}$$

$v^{\mu,m}$ being the approximate value at the nodal point $\xi^{\mu} = \mu k$, $x^{m} = mh$ $(\mu, m = 0, 1, 2, \ldots)$, cf. fig. 1.

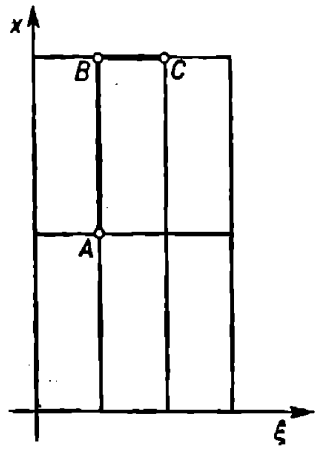

Fig. 1. The nodal points. The indices for A, B, C are $(\mu, m-1)$, (μ, m), $(\mu+1, m)$, respectively

S. Łojasiewicz [1] introduced a special system of ordinary differential equations (a difference-differential system) approximating partial differential equation (1.1), f being periodic in x. Then A. Pliś obtained similar (unpublished) results without the requirement of periodicity for the right-hand member of (1.1), and proposed the difference scheme (1.2).

In this paper we prove the convergence of the difference method in the p-dimensional case, cf. § 4 and Theorem 1, some error estimates being given.

§ 2. The mesh and the differences. Let us denote by m the sequence of p natural numbers

$$m = (m_1, m_2, \ldots, m_p), \tag{2.1}$$

and let

$$M = (\mu, m), \tag{2.2}$$

where μ is a natural number.

We shall consider the points x^m of the real p-dimensional space R^p with coordinates

$$x^m = (x_1^m, x_2^m, \ldots, x_p^m) \in R^p, \tag{2.3}$$

and also the nodal points

$$(\xi^\mu, x^m) \in R^{p+1}, \tag{2.4}$$

the corresponding values ξ^μ, x^m being defined by

$$\begin{gathered} \xi^\mu = \mu k, \quad x_j^\nu = \nu h \quad (\mu = 0, 1, \ldots;\ \nu = 0, 1, \ldots;\ j = 1, 2, \ldots, p), \\ 0 < h = \text{const}, \quad 0 < k = \text{const}, \quad \text{for} \quad (\xi^\mu, x_1^\nu, \ldots, x_p^\nu) \in E, \end{gathered} \tag{2.5}$$

where

$$E\colon\ 0 \leqslant \xi \leqslant a,\ 0 \leqslant x_j \leqslant a,\ a > 0\ (j = 1, 2, \ldots, p). \tag{2.6}$$

There is a one-to-one correspondence between the nodal points (2.4) and the indices (2.2).

We shall consider also the nodal points characterized by the following sequences of indices:

$$\omega(M) = (\mu+1, m), \qquad j(M) = \big((\mu, j(m)\big), \tag{2.7}$$

where

$$j(m) = (m_1, \ldots, m_{j-1}, m_j - 1, m_{j+1}, \ldots, m_p) \qquad (j = 1, 2, \ldots, p). \tag{2.8}$$

(2.7) and (2.8) imply the relations

$$j(\mu, m) = \big(\mu, j(m)\big), \qquad \text{for} \qquad j = 1, 2, \ldots, p. \tag{2.9}$$

Suppose that to each nodal point with indices (2.1) there corresponds a number

$$v^M. \tag{2.10}$$

We introduce the differences

$$(2.11)\qquad \begin{aligned} v^{M\sim} &= \frac{1}{k}\left(v^{\omega(M)} - v^{M}\right), \\ v^{Mj} &= \frac{1}{h}\left(v^{M} - v^{j(M)}\right) \qquad (j = 1, 2, \ldots, p), \end{aligned}$$

and also the vector $v^{M\Delta}$ with coordinates

$$(2.12)\qquad v^{M\Delta} = (v^{M1}, v^{M2}, \ldots, v^{Mp}).$$

§ 3. Throughout the rest of the paper we shall use the following assumptions H:

Assumptions H. (1) Assume that the scalar function $f(\xi, x, u, q)$, $x = (x_1, \ldots, x_p)$, $q = (q_1, \ldots, q_p)$, is of the class C^1 for $(\xi, x, u, q) \in D$, where

$$(3.1)\qquad D\colon\ 0 \leqslant \xi \leqslant a,\ 0 \leqslant x_j \leqslant a,\ -\infty < u < +\infty,\ -\infty < q_j < +\infty,$$
$$a > 0 \ (j = 1, 2, \ldots, p).$$

(2) The derivatives f_u and f_{q_j} fulfil conditions

$$(3.2)\qquad |f_u| \leqslant L, \quad f_{q_j} \leqslant 0 \quad (j = 1, 2, \ldots, p),$$

the mesh size h and k (cf. (2.5)) being defined so as to obtain

$$(3.3)\qquad \sum_{j=1}^{p} f_{q_j} + \frac{h}{k} \geqslant 0 \quad \text{for} \quad (\xi, x, u, q) \in D.$$

(3) The scalar function $u(\xi, x)$ of the class C^1 satisfies the partial differential equation

$$(3.4)\qquad \frac{\partial u}{\partial \xi} = f\left(\xi, x, u, \frac{\partial u}{\partial x}\right),$$

for $(\xi, x) \in E$ (cf. (2.6)), $\frac{\partial u}{\partial x}$ being defined as

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right),$$

and the boundary conditions

$$(3.5)\qquad \begin{aligned} &u(0, x) = \varphi_0(x), \\ &u(\xi, x) = \varphi_j(\xi, x), \quad \text{for} \quad (\xi, x) \in E,\ x_j = 0 \ (j = 1, 2, \ldots, p). \end{aligned}$$

Remark 1. We call attention to the fact that, according to (3.3) and (3.2), the choice of the space interval h places a restriction on the size of the time interval k.

§ 4. The approximate solution v^M of the partial differential equation. We accept the following boundary conditions for the numbers v^M (cf. § 2):

$$(4.1)\qquad v^M = \begin{cases} \varphi_0(x^m) & \text{for} \quad M=(0,m), \\ \varphi_j(\xi^\mu, x_1^{m_1}, \dots, x_j^0, \dots, x_p^{m_p}) & \text{for} \quad \mu = 0,1,\dots; \\ & j=1,2,\dots,p;\ M=(\mu, m_1, \dots, 0, \dots, m_p), \end{cases}$$

the values v^M at the remaining nodal points being defined successively with the aid of the difference equation

$$(4.2)\qquad v^{M\sim} = f(\xi^\mu, x^m, v^M, v^{M\Delta}).$$

We denote by u^M the value of the solution $u(\xi, x)$ of equation (3.4) at the nodal point (2.2), and we define the corresponding differences as in the case of numbers v^M, cf. (2.11) and (2.12).

The boundary conditions (3.5) imply the boundary conditions for u^M. They are of the following form:

$$(4.3)\qquad u^M = \begin{cases} \varphi_0(x^m) & \text{for} \quad M=(0,m), \\ \varphi_j(\xi^\mu, x_1^{m_1}, \dots, x_j^0, \dots, x_p^{m_p}) & \text{for} \quad \mu = 0,1,\dots; \\ & j=1,2,\dots,p;\ M=(\mu, m_1, \dots, 0, \dots, m_p). \end{cases}$$

The values u^M satisfy the equation

$$(4.4)\qquad u^{M\sim} = f(\xi^\mu, x^m, u^M, u^{M\Delta}) + \eta^M,$$

where

$$\max_M |\eta^M| \to 0, \qquad \text{as} \qquad h \to 0,$$

at all nodal points $M=(\mu, m)$ in the region E (cf. (2.6)), for $m_j \geqslant 1$ $(j = 1, 2, \dots, p)$.

Equation (4.4) is the consequence of (3.4) since $u(\xi, x)$ is of the class C^1.

Remark 2. Procedure (4.2) can be applied to digital computers. To explain this possibility we observe that (4.2) can be rewritten in the form

$$v^{\mu+1,m} = v^M + kf\left(\xi^\mu, x^m, v^M, \frac{v^M - v^{1(M)}}{h}, \dots, \frac{v^M - v^{p(M)}}{h}\right),$$

where $M = (\mu, m)$, because of (2.11) and (2.12). This formula enables one to calculate $v^{\mu+1,m}$ with the aid of the preceding values only (cf. fig. 2).

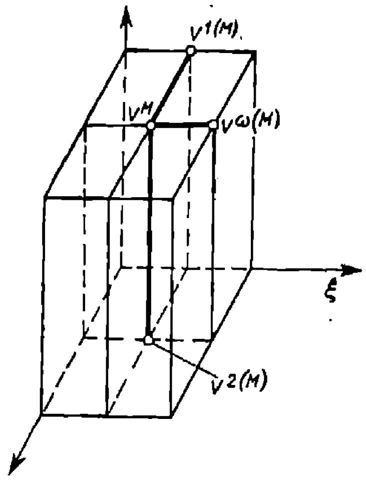

Fig. 2. $M = (1, 1, 1)$, $\omega(M) = (2, 1, 1)$, $1(M) = (1, 0, 1)$, $2(M) = (1, 1, 0)$

§ 5. Lemma 1. *Suppose that the numbers s^μ satisfy the non-homogeneous linear difference inequality*

(5.1) $$s^{\mu\sim} \leqslant Ls^\mu + \varepsilon \quad (\mu = 0, 1, \ldots),$$

and the initial condition $s^0 = 0$, where

$$s^{\mu\sim} = \frac{1}{H}(s^{\mu+1} - s^\mu), \quad 0 < H = \text{const}, \quad 0 < L = \text{const}, \quad 0 < \varepsilon = \text{const}.$$

Under these assumptions

(5.2) $$s^\mu \leqslant \frac{\varepsilon}{L}(e^{LH\mu} - 1) \quad (\mu = 0, 1, \ldots).$$

(5.2) can be proved with the aid of finite induction.

Remark 3. If the numbers z^μ satisfy the inequality

(5.3) $$z^{\mu\sim} \geqslant Lz^\mu - \varepsilon \quad (\mu = 0, 1, \ldots),$$

and the initial condition $z^0 = 0$, then

(5.4) $$z^\mu \geqslant -\frac{\varepsilon}{L}(e^{LH\mu} - 1) \quad (\mu = 0, 1, \ldots).$$

(5.4) can be proved also with the aid of finite induction.

§ 6. Lemma 2. *Suppose that the assumption* H *are fulfilled, and the values u^M and v^M* (cf. § 4) *satisfy* (4.3), (4.4) *and* (4.1), (4.2), *respectively, at the nodal points in the region E* (cf. (2.6)).

Let us write

(6.1) $$r^M = u^M - v^M,$$

(6.2) $$s^\mu = \max_m r^{\mu,m}, \quad z^\mu = \min_m r^{\mu,m} \quad (\mu = 0, 1, \ldots),$$

in the region E. Obviously $s^\mu \geqslant 0$ and $z^\mu \leqslant 0$.

Under these assumptions the numbers s^μ and z^μ satisfy respectively the non-homogeneous linear difference inequalities

(6.3) $$s^{\mu\sim} \leqslant Ls^\mu + \varepsilon(h), \quad z^{\mu\sim} \geqslant Lz^\mu - \varepsilon(h) \quad (\mu = 0, 1, \dots),$$

and the initial conditions $s^0 = 0$, $z^0 = 0$, where $0 \leqslant \varepsilon(h) \to 0$, as $h \to 0$.

Proof. There exist an $a = (a_1, \dots, a_p)$ and a $b = (b_1, \dots, b_p)$ such that

(6.4) $$s^{\mu+1} = \max_m r^{\mu+1,m} = r^{\mu+1,a},$$

(6.5) $$s^\mu = \max_m r^{\mu,m} = r^{\mu,b},$$

whence

(6.6) $$s^{\mu\sim} = \frac{1}{k}(s^{\mu+1} - s^\mu) = \frac{1}{k}(r^{\mu+1,a} - r^{\mu,b}).$$

The right-hand member of (6.6) can be written in the form

(6.7) $$s^{\mu\sim} = \frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}).$$

Now we obtain

(6.8) $$\frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) = \frac{1}{k}(u^{\mu+1,a} - u^{\mu,a}) - \frac{1}{k}(v^{\mu+1,a} - v^{\mu,a}),$$

because of (6.1).

If for some j: $1 \leqslant j \leqslant p$, $a_j = 0$, inequalities (6.3) are evident, and for $a_j \geqslant 1$ $(j = 1, 2, \dots, p)$ we have

(6.9) $$\frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) = f(\xi^\mu, x^a, u^{\mu,a}, u^{(\mu,a)\Delta}) + \eta^{\mu,a} - f(\xi^\mu, x^a, v^{\mu,a}, v^{(\mu,a)\Delta}),$$

because of (4.2) and (4.4).

The right-hand member of (6.9) becomes

(6.10) $$\frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) = \eta^{\mu,a} + f_u(\sim) r^{\mu,a} + \frac{1}{h}\sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a} - r^{\mu,j(a)}),$$

in view of (6.1), (2.12) and the mean value theorem, the derivative being taken at some point $(\sim)$. Therefore (6.10) and (6.7) imply

(6.11) $$s^{\mu\sim} = \eta^{\mu,a} + f_u(\sim) r^{\mu,a} + \frac{1}{h}\sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a} - r^{\mu,j(a)}) + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}).$$

Now we majorize the right-hand member of (6.11). From the definition (6.5) of $r^{\mu,a}$ follows $r^{\mu,j(a)} \leqslant r^{\mu,b}$, whence

(6.12) $$r^{\mu,a} - r^{\mu,j(a)} \geqslant r^{\mu,a} - r^{\mu,b},$$

and

$$(6.13)\qquad \sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a}-r^{\mu,j(a)}) \leqslant \sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a}-r^{\mu,b}),$$

since $f_{q_j} \leqslant 0$ by assumptions H.

Therefore (6.13) and (6.11) imply

$$(6.14)\qquad s^{\mu\sim} \leqslant \eta^{\mu,a}+f_u(\sim)r^{\mu,a}+\frac{1}{h}(r^{\mu,a}-r^{\mu,b})\left[\sum_{j=1}^{p} f_{q_j}(\sim)+\frac{h}{k}\right].$$

The last term of the right-hand member of (6.14) is non-positive because of (3.3) and condition $r^{\mu,a}-r^{\mu,b} \leqslant 0$, cf. (6.5). Hence by (6.5)

$$s^{\mu\sim} \leqslant Ls^{\mu}+\varepsilon(h) \qquad (\mu = 0, 1, \ldots),$$

where

$$\varepsilon(h) = \max_{M} |\eta^{M}|,$$

which completes the proof of the first inequality of (6.3).

The second inequality of (6.3) for z^{μ} can be proved in a similar way. The initial conditions for s^0 and z^0 follow from the identity of the boundary values for u^M and v^M.

This completes the proof of Lemma 2.

§ 7. THEOREM 1. *Suppose that the function $f(\xi, x, u, q)$ fulfils assumptions* H, *the values u^M and v^M at the nodal points being defined by* (3.4), (3.5) *and* (4.1), (4.2), *respectively* (cf. § 4).

Let $\varepsilon(h) = \max_{M} |\eta^{M}|$ for all nodal points in the region E, η^M being taken from (4.4).

Under these assumptions

(i) *the error estimate*

$$(7.1)\qquad |r^{\mu,m}| \leqslant \frac{\varepsilon(h)}{L}(e^{Lk\mu}-1),$$

holds for $\mu = 0, 1, \ldots$

(ii) *the difference method* (4.2) *is convergent, i.e.*

$$(7.2)\qquad \lim_{h\to 0} r^{M} = 0.$$

Remark 4. The expression $\varepsilon(h) = \max_{M} |\eta^{M}|$ can be evaluated in terms of constants estimating the derivatives of $f(\xi, x, u, q)$ and $\varphi_0, \ldots, \varphi_p$ (up to second order).

Proof of the theorem. (7.2) follows immediately from (7.1) and $\lim_{h\to 0} \varepsilon(h) = 0$; consequently, all that remains to be proved is that relations (7.1) are satisfied.

From (6.2) and Lemma 2 it follows that s^μ and z^μ satisfy difference inequalities (6.3) and conditions $s^0 = z^0 = 0$.

Replacing H in Lemma 1 by k we get by (6.3), (5.2) and (5.4)

$$\begin{aligned} r^{\mu,m} &\leqslant s^\mu \leqslant \frac{\varepsilon(h)}{L}(e^{Lk\mu}-1), \\ r^{\mu,m} &\geqslant z^\mu \geqslant -\frac{\varepsilon(h)}{L}(e^{Lk\mu}-1) \end{aligned} \qquad (\mu = 0, 1, \ldots),$$

which implies (7.1).

This completes the proof of Theorem 1.

Reference

[1] S. Łojasiewicz, *Sur le problème de Cauchy pour les systèmes d'équations aux dérivées partielles du premier ordre dans le cas hyperbolique de deux variables indépendentes*, Ann. Polon. Math. 3 (1956), pp. 87-117.

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