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Mean growth and Taylor coefficients of some topological algebras of analytic functions

by M. Stoll (Columbia, South Carolina, U.S.A.)

Abstract. Let D denote the unit disc $\{|z| < 1\}$ in C and let $H(D)$ denote the space of analytic functions on D. For each $\beta > 0$ we denote by F_β the set of all functions $f \in H(D)$ for which $\lim_{r\to 1}(1-r)^\beta \log^+ M_\infty(r,f) = 0$, where $M_\infty(r,f) = \max_{|z|\leqslant r}|f(z)|$, and for each $\alpha > 1$ we denote by $(\mathrm{Log}^+H)^\alpha$ the Hardy–Orlicz space of functions $f \in H(D)$ for which $\sup_{0<r<1}\int_0^{2\pi}[\log^+|f(re^{it})|]^\alpha dt < \infty$. In the paper we show that for $f(z) = \sum a_n z^n \in H(D)$, the following are equivalent:

(a) $f \in F_\beta$;

(b) $|||f|||_c = \int_0^1 \exp[-c(1-r)^{-\beta}]\, M_\infty(r,f)\,dr < \infty$ for all $c > 0$;

(c) $||f||_c = \sum_{n=0}^{\infty} |a_n| \exp[-cn^{\beta/(1+\beta)}] < \infty$ for all $c > 0$.

We also show that F_β with the topology given by the seminorms $||\ ||_c$ (or $|||\ |||_c$), $c > 0$, is a Fréchet algebra, and that for each $\alpha > 1$, $(\mathrm{Log}^+H)^\alpha$ with the topology given by the translation invariant metric ϱ_α defined by

$$\varrho_\alpha(f,0) = \lim_{r\to 1}\left[\frac{1}{2\pi}\int_0^{2\pi}[\log(1+|f(re^{it})|)]^\alpha dt\right]^{1/\alpha}$$

is a dense subspace of $F_{1/\alpha}$. Furthermore, the topology in $F_{1/\alpha}$ defined by the family of seminorms $||\ ||_c$ is weaker than the topology in $(\mathrm{Log}^+H)^\alpha$ given by the metric ϱ_α.

1. Introduction. Let D denote the unit disc $\{|z| < 1\}$ in C and let $H(D)$ denote the space of analytic functions on D. A function $f \in H(D)$ belongs to the Nevanlinna class N of functions of bounded characteristic if

$$\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{it})|\,dt < \infty.$$

A function $f \in N$ is said to belong to the class N^+ if

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{it})|\,dt = \frac{1}{2\pi}\int_0^{2\pi}\log^+|f^*(e^{it})|\,dt,$$

where $f^*(e^{it}) = \lim_{r\to 1} f(re^{it})$ a.e. on $|z| = 1$.

In [9], N. Yanagihara has shown that if $f \in N^+$, then

$$\lim_{r\to 1}(1-r)\log^+ M_\infty(r,f) = 0, \tag{1.1}$$

where $M_\infty(r,f) = \max_{|z|=r}|f(z)|$. Using (1.1) as a motivation, for each $\beta > 0$, denote by F_β the space of all functions $f \in H(D)$ which satisfy

$$\lim_{r\to 1}(1-r)^\beta \log^+ M_\infty(r,f) = 0. \tag{1.2}$$

For $\beta = 1$, the space F_1 has also been denoted by F^+ and has been studied by N. Yanagihara in [7], [8]. In [7] it was shown that F^+ is the containing Fréchet space for N^+. By [2], p. 106, $N \subset F_\beta$ for all $\beta > 1$.

In Section 2 we give several necessary and sufficient conditions for a function $f \in H(D)$ to be in F_β. Here we show that $f(z) = \sum a_n z^n \in F_\beta$ if and only if

$$a_n = O\left(\exp\left[o\left(n^{\beta/(1+\beta)}\right)\right]\right).$$

In Section 3 we consider some topological properties of the space F_β. Since the results are in many cases analogous to the results established by N. Yanagihara in [7], the proofs will be kept brief and in some cases omitted.

In Section 4 we consider the Hardy–Orlicz spaces $(\mathrm{Log}^+ H)^a$ defined for all $a > 1$. A function $f \in H(D)$ is said to belong to $(\mathrm{Log}^+ H)^a$, $a > 1$, if

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} [\log^+ |f(re^{it})|]^a\, dt < \infty.$$

Here we show that $(\mathrm{Log}^+ H)^a$, $a > 1$, is contained in $F_{1/a}$, but is not contained in F_β for any $\beta < 1/a$. We will also define a translation invariant metric ϱ_a on $(\mathrm{Log}^+ H)^a$ and show that with respect to ϱ_a, $(\mathrm{Log}^+ H)^a$ is an F-algebra, i.e., a topological vector space with a complete translation invariant metric in which multiplication is continuous. Some other properties of the spaces $(\mathrm{Log}^+ H)^a$ are also given.

Finally in Section 5 we consider the Bergmann algebra $\mathcal{N}^+(D)$, which is defined to be the space of all $f \in H(D)$ for which $\log^+|f|$ is integrable on D with respect to the area measure $dA = \frac{1}{\pi}\, dx\, dy$. Here we show that $\mathcal{N}^+(D)$ is a dense subspace of F_2 and that $\mathcal{N}^+(D)$ is not contained in F_β for any $\beta < 2$. Also, we conclude by showing for all $\beta < 1$, $F_\beta \subset \mathcal{N}^+(D)$.

2. Mean growth and Taylor coefficients. For $f \in H(D)$, we write

$$M_p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt, \qquad 0 < p < \infty,$$

$$M_\infty(r,f) = \max_{|z|=r} |f(z)|.$$

For each $\beta > 0$, F_β denotes the space of all analytic functions f on D for which $\lim_{r\to 1}(1-r)^\beta \log^+ M_\infty(r,f) = 0$. Clearly, if $0 < \alpha < \beta$, then $F_\alpha \subset F_\beta$, and Theorem 2.2 of this section can be used to show that the containment is proper.

THEOREM 2.1. *For f analytic on D, the following are equivalent.*

(a) $f \in F_\beta$.

(b) *For all p, $0 < p \leqslant \infty$,* $\lim_{r\to 1}(1-r)^\beta \log^+ M_p(r,f) = 0$.

(c) *For all p, $0 < p \leqslant \infty$, and $c > 0$,*

$$\int_0^1 \exp[-c(1-r)^{-\beta}] M_p(r,f)\, dr < \infty. \tag{2.1}$$

(d) *For some p, $0 < p \leqslant \infty$,* $\int_0^1 \exp[-c(1-r)^{-\beta}] M_p(r,f)\, dr < \infty$ *for all $c > 0$.*

Proof. Since $M_p(r,f) \leqslant [M_\infty(r,f)]^p$, $0 < p < \infty$, (b) follows directly from (a).

Suppose $\lim_{r\to 1}(1-r)^\beta \log^+ M_p(r,f) = 0$, $0 < p \leqslant \infty$. Then there exists a function $w(r) \downarrow 0$ as $r \to 1$ such that

$$M_p(r,f) \leqslant \exp\left[\frac{w(r)}{(1-r)^\beta}\right].$$

Therefore, for any $c > 0$, $\int_0^1 \exp[-c(1-r)^{-\beta}]\, M_p(r,f)\, dr < \infty$, which proves (2.1). Clearly (c) $\Rightarrow$ (d).

We now show that (d) $\Rightarrow$ (a). Suppose that for some $p > 0$, $I_p(c) = \int_0^1 \exp[-c(1-r)^{-\beta}]\, M_p(r,f)\, dr < \infty$ for all $c > 0$. Since $M_p(r,f)$ is a non-decreasing function of r, for any $R < 1$,

$$I_p(c) \geqslant M_p(R,f) \int_R^1 \exp[-c(1-r)^{-\beta}]\, dr. \tag{2.2}$$

By the change of variable $t = (1-r)^{-\beta}$,

$$\int_R^1 \exp[-c(1-r)^{-\beta}]dr = \frac{1}{\beta}\int_T^1 \exp(-ct)t^{-\gamma}dt,$$

where $T = (1-R)^{-\beta}$ and $\gamma = (\beta+1)/\beta$. The function $t^{\gamma}\exp[-ct]$ for $t \in (0, \infty)$ has a maximum at $t = \gamma/c$. Therefore

$$t^{-\gamma}\exp[-ct] \geqslant \left(\frac{ce}{\gamma}\right)^{\gamma}\exp[-2ct],$$

and

$$\frac{1}{\beta}\int_T^{\infty} \exp[-ct]t^{-\gamma}dt \geqslant \frac{1}{\beta}\left(\frac{ce}{\gamma}\right)^{\gamma}\frac{1}{2c}\exp[-2cT].$$

Combining this with (2.2) gives

$$M_p(R,f) \leqslant 2\beta c\left(\frac{\gamma}{ce}\right)^{\gamma}\exp[2c(1-R)^{-\beta}]I_p(c), \tag{2.3}$$

valid for all R, $0 < R < 1$, with $\gamma = (\beta+1)/\beta$.

If $p = \infty$, then by (2.3), $\overline{\lim\limits_{R\to 1}}(1-R)^{\beta}\log^+ M_\infty(R,f) \leqslant 2c$, for all $c > 0$, from which it follows that $f \in F_\beta$.

Suppose $0 < p < \infty$. Since $|f(z)|^p$ is subharmonic in $|z| < \varrho$ and continuous in $|z| \leqslant \varrho$, where $0 < \varrho < 1$ is arbitrary,

$$|f(z)|^p \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{\varrho^2-|z|^2}{|\varrho e^{i\theta}-z|^2}|f(\varrho e^{i\theta})|^p d\theta,$$

which is valid for all z, $|z| < \varrho$. Hence if $z = re^{i\theta}$, $0 < r < 1$ and $\varrho = (1+r)/2$,

$$[M_\infty(r,f)]^p \leqslant \frac{\varrho+r}{\varrho-r}M_p(\varrho,f) \leqslant \frac{4}{1-r}M_p(\varrho,f).$$

Combining this with (2.3), with $R = (1+r)/2$, gives

$$[M_\infty(r,f)]^p \leqslant A(\beta,c)\frac{\exp[c2^{\beta+1}(1-r)^{-\beta}]}{(1-r)}I_p(c), \tag{2.4}$$

where $A(\beta, c)$ is a constant depending only on β and c. Hence,

$$\overline{\lim_{r\to 1}}(1-r)^{\beta}\log^+ M_\infty(r,f) \leqslant \frac{1}{p}c2^{\beta+1}$$

for all $c > 0$, from which it follows that $f \in F_\beta$.

Remark 1. If $f \in F_\beta$, then there exists a constant $A(\beta, c)$, depending only on β and c, such that

$$|f(z)| \leqslant A(\beta, c)\exp[2c(1-|z|)^{-\beta}]\, I_\infty(f, c) \tag{2.5}$$

and for $0 < p < \infty$,

$$|f(z)|^p \leqslant 4A(\beta, c)\,(1-|z|)^{-1}\exp[2^{\beta+1}c(1-|z|)^{-\beta}]\, I_p(f, c), \tag{2.6}$$

where for $0 < p \leqslant \infty$, $I_p(f, c) = \int\limits_0^1 \exp[-c(1-r)^{-\beta}]\, M_p(r, f)\,dr$, and $A(\beta, c) = 2\beta c(\gamma/ce)^\gamma$, $\gamma = (\beta+1)/\beta$. These follow directly from inequality (2.3) for $p = \infty$ and from (2.4) for $0 < p < \infty$.

Suppose $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. In [7], N. Yanagihara has shown that $f \in F_1$ (F^+ in notation of [7]) if and only if

$$a_n = O\left[\exp\left(o(\sqrt{n})\right)\right],$$

i.e., $\overline{\lim\limits_{n\to\infty}} \frac{1}{\sqrt{n}} \log|a_n| \leqslant 0$. We obtain the following generalization.

THEOREM 2.2. *Suppose* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *is analytic in* D. *Then the following are equivalent.*

(a) $f \in F_\beta$.

(b) *There exists a sequence* $\{\lambda_n\}$ *of positive real numbers with* $\lambda_n \downarrow 0$ *such that*

$$|a_n| \leqslant \exp[\lambda_n n^{\beta/(1+\beta)}]. \tag{2.7}$$

(c) *For any* $c > 0$,

$$\sum_{n=0}^{\infty} |a_n| \exp[-cn^{\beta/(1+\beta)}] < \infty. \tag{2.8}$$

Proof. (a) $\Rightarrow$ (b). Suppose $f \in F_\beta$. Then by Theorem 2.1 (b), there exists $\omega(r)\downarrow 0$ as $r\to 1$ such that $M_1(r, f) \leqslant \exp[\omega(r)\,(1-r)^{-\beta}]$. Therefore, for all $n \geqslant 0$ and $0 < r < 1$,

$$|a_n| \leqslant \frac{M_1(r, f)}{r^n} \leqslant \frac{\exp[\omega(r)(1-r)^{-\beta}]}{r^n}.$$

Let $\varepsilon_k \downarrow 0$ (assume $\varepsilon_1 < 2^{-(\beta+1)}$) and choose $\varrho_k \uparrow 1$ such that $\omega(r) \leqslant \varepsilon_k$ for $r \geqslant \varrho_k$. Choose a sequence of integers n_k, $n_{k+1} > n_k$, such that

$$1 - \left(\frac{\varepsilon_k}{n_k}\right)^{1/(\beta+1)} \geqslant \varrho_k.$$

For $n_k \leqslant n < n_{k+1}$, set

$$r_n = 1 - \left(\frac{\varepsilon_k}{n}\right)^{1/(\beta+1)}. \tag{2.9}$$

Then $r_n \geqslant \varrho_k$ and

$$\exp[\omega(r_n)(1-r_n)^{-\beta}] \leqslant \exp[\varepsilon_k^{1/(1+\beta)} n^{\beta/(1+\beta)}].$$

Furthermore, by the inequality $1 - x > e^{-2x}$ which is valid for all x with $0 < x \leqslant .796$,

$$(r_n)^n = \left(1 - \left(\frac{\varepsilon_k}{n}\right)^{1/(\beta+1)}\right)^n \geqslant \exp[-2\varepsilon_k^{1/(\beta+1)} n^{\beta/(\beta+1)}].$$

Therefore,

$$|a_n| \leqslant \exp[3\varepsilon_k^{1/(\beta+1)} n^{\beta/(\beta+1)}],$$

from which the result follows with $\lambda_k = 3\varepsilon_k^{1/(\beta+1)}$, $n_k \leqslant n < n_{k+1}$.

(b) $\Rightarrow$ (c). Suppose $|a_n| \leqslant \exp[\lambda_n n^{\beta/(\beta+1)}]$, where $\{\lambda_n\}$ is a positive sequence decreasing to zero. Then, it is an easy consequence that the series

$$\sum_{n=0}^{\infty} |a_n| \exp[-cn^{\beta/(\beta+1)}]$$

converges for all $c > 0$.

(c) $\Rightarrow$ (a). Consider $I(c) = \int_0^1 \exp[-c(1-r)^{-\beta}] M_\infty(r, f)\,dr$. Then

$$I(c) \leqslant \sum_{n=0}^{\infty} |a_n| \int_0^1 r^n \exp[-c(1-r)^{-\beta}]\,dr. \tag{2.10}$$

Define

$$r_n' = 1 - \left(\frac{c}{n}\right)^{1/(\beta+1)}, \quad r_n'' = 1 - 2\left(\frac{c}{n}\right)^{1/(\beta+1)}. \tag{2.11}$$

Then for $r_n'' \leqslant r \leqslant r_n'$,

$$\begin{aligned} r^n \exp[-c(1-r)^{-\beta}] &\leqslant \exp[-(1+2^{-\beta})c^{1/(\beta+1)} n^{\beta/(\beta+1)}] \\ &\leqslant \exp[-c^{1/(\beta+1)} n^{\beta/(\beta+1)}]. \end{aligned}$$

For $r \leqslant r_n''$

$$\begin{aligned} r^n \exp[-c(1-r)^{-\beta}] \leqslant r^n &\leqslant \left(1 - 2\left(\frac{c}{n}\right)^{1/(\beta+1)}\right)^n \\ &\leqslant \exp[-2c^{1/(\beta+1)} n^{\beta/(\beta+1)}] \leqslant \exp[-c^{1/(\beta+1)} n^{\beta/(\beta+1)}], \end{aligned}$$

and for $r \geqslant r_n'$,

$$r^n \exp[-c(1-r)^{-\beta}] \leqslant \exp[-c^{1/(\beta+1)} n^{\beta/(\beta+1)}].$$

Therefore, for all r, $0 < r < 1$,

$$r^n \exp[-c(1-r)^{-\beta}] \leqslant \exp[-c^{1/(\beta+1)} n^{\beta/(\beta+1)}],$$

and consequently, by (2.10),

$$I(c) \leqslant \sum_{n=0}^{\infty} |a_n| \exp[-c_1^{1/(\beta+1)} n^{\beta/(\beta+1)}], \tag{2.12}$$

which is finite for all $c > 0$. Therefore by Theorem 2.1 (d), $f \in F_\beta$ which proves the result.

Remark 2. The method of proof used in this theorem is similar to that used by N. Yanagihara in Theorems 1 and 2 of [7] for $\beta = 1$. The key difference is in the definition of the sequences r_n, r'_n and r''_n given by (2.9) and (2.11) respectively.

3. F_β as a Fréchet algebra. As in [7], for $f \in F_\beta$, $\beta > 0$, we define for each $c > 0$

$$|||f|||_{\beta,c} = \int_0^1 \exp[-c(1-r)^{-\beta}]\, M_\infty(r,f)\, dr, \tag{3.1}$$

and

$$\|f\|_{\beta,c} = \sum_{n=0}^{\infty} |a_n| \exp[-cn^{\beta/(\beta+1)}]. \tag{3.2}$$

Clearly both $\{|||\ |||_{\beta,c}\}_{c>0}$ and $\{\|\ \|_{\beta,c}\}_{c>0}$ define a family of (semi) norms on F_β, with respect to which F_β is a locally convex topological vector space. The following proposition shows that the topology given by the two families of seminorms is equivalent.

PROPOSITION 3.1. *For each $c > 0$, there exists a constant $A = A(\beta, c)$ depending only on β and c, such that*

$$|||f|||_{\beta,c} \leqslant \|f\|_{\beta,c_1}, \qquad \|f\|_{\beta,c} \leqslant A\, |||f|||_{\beta,c_2} \tag{3.3}$$

with $c_1 = c^{1/(\beta+1)}$ and $c_2 = \left(\dfrac{c}{12}\right)^{1/(\beta+1)}$.

Proof. By (2.12), $|||f|||_{\beta,c} \leqslant \|f\|_{\beta,c_1}$ with $c_1 = c^{1/(\beta+1)}$. As in the proof of Theorem 2 of [4], we set

$$u(\theta) = \int_0^1 \exp[-\lambda(1-r)^{-\beta}] f(re^{i\theta})\, dr, \qquad \lambda > 0.$$

Then $|u(\theta)| \leqslant |||f|||_{\beta,\lambda}$ and

$$\frac{1}{2\pi}\int_0^{2\pi} |u(\theta)|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 \left(\int_0^1 \exp[-\lambda(1-r)^{-\beta}] r^n dr\right)^2.$$

With r_n' and r_n'' as defined in (2.11),

$$\int_0^1 \exp[-\lambda(1-r)^{-\beta}]r^n dr \geqslant \int_{r_n''}^{r_n'} \exp[-\lambda(1-r)^{-\beta}]r^n dr$$

$$\geqslant \left(\frac{\lambda}{n}\right)^{1/(\beta+1)} \exp[-5\lambda^{1/(\beta+1)}n^{\beta/(\beta+1)}] \geqslant \exp[-6\lambda^{1/(\beta+1)}n^{\beta/(\beta+1)}]$$

for n sufficiently large. Therefore

$$|||f|||_{\beta,\lambda}^2 \geqslant K\sum_{n=0}^{\infty}|a_n|^2\exp[-12\lambda^{1/(\beta+1)}n^{\beta/(\beta+1)}].$$

But

$$\|f\|_{\beta,c}^2 = \Big(\sum_{n=0}^{\infty}|a_n|\exp[-cn^{\beta/(\beta+1)}]\Big)^2$$

$$\leqslant \Big(\sum_{n=0}^{\infty}|a_n|^2\exp[-cn^{\beta/(\beta+1)}]\Big)\Big(\sum_{n=0}^{\infty}\exp[-cn^{\beta/(\beta+1)}]\Big).$$

Hence, if we set $\lambda = \left(\frac{c}{12}\right)^{\beta+1}$, there exists a constant A, depending only on β and c such that

$$\|f\|_{\beta,c} \leqslant A\,|||f|||_{\beta,\lambda}.$$

Remark 3. If in definition (3.1), one had used $M_1(r,f)$ instead of $M_\infty(r,f)$ to define a family of seminorms $P_{\beta,c}$ by

$$P_{\beta,c}(f) = \int_0^1 \exp[-c(1-r)^{-\beta}]M_1(r,f)\,dr, \qquad c > 0,$$

then $P_{\beta,c}(f) \leqslant |||f|||_{\beta,c}$, and by using the inequality $M_\infty(r,f) \leqslant \frac{\varrho+r}{\varrho-r} \times M_1(r,f)$, where $\varrho = \frac{1+r}{2}$, one can show that for every $c > 0$, there exists c_1 depending on c and a constant A depending only on β and c such that

$$|||f|||_{\beta,c} \leqslant AP_{\beta,c_1}(f).$$

THEOREM 3.2. *For all $\beta > 0$, F_β with the topology given by the (semi) norms (3.1) or (3.2) is a countably normed Fréchet algebra with*

$$\|fg\|_{\beta,c} \leqslant \|f\|_{\beta,c'}\|g\|_{\beta,c'}, \tag{3.4}$$

where $c' = c2^{-1/(\beta+1)}$, $f, g \in F_\beta$. Furthermore, if $f \in F_\beta$, then $f_r \to f$ in the topology of F_β, where for $0 < r < 1$, $f_r(z) = f(rz)$.

Proof. The proof that F_β is a countably normed Fréchet space is similar to the proof of Theorem 3 in [7] and consequently is omitted. Likewise for the result that $f_r \to f$ as $r \to 1$ in the topology of F_β. Continuity of multiplication will follow from inequality (3.4) which we now prove.

Suppose $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$. For $\lambda > 0$

$$(3.5)\qquad \|f\|_{\beta,\lambda}\|g\|_{\beta,\lambda} = \Big(\sum_{n=0}^{\infty}|a_n|\exp[-\lambda n^{\beta/(\beta+1)}]\Big)\Big(\sum_{n=0}^{\infty}|b_n|\exp[-\lambda n^{\beta/(\beta+1)}]\Big)$$
$$= \sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}|a_j|\,|b_{n-j}|\exp\big[-\lambda\big(j^{\beta/(\beta+1)}+(n-j)^{\beta/(\beta+1)}\big)\big]\Big).$$

By the inequality $(a^p+b^p)^{1/p} \leqslant 2^{(1-p)/p}(a+b)$, valid for $0<p<1$, $a, b \geqslant 0$,

$$j^{\beta/(\beta+1)}+(n-j)^{\beta/(\beta+1)} \leqslant 2^{1/(1+\beta)}n^{\beta/(\beta+1)}.$$

Therefore by (3.5)

$$\|f\|_{\beta,\lambda}\|g\|_{\beta,\lambda} \geqslant \sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}|a_j|\,|b_{n-j}|\Big)\exp[-cn^{\beta/(\beta+1)}]$$

with $c = \lambda 2^{1/(1+\beta)}$, from which (3.4) follows. Hence F_β is a Fréchet algebra for all $\beta > 0$.

Using the same method of proof as in [7], Theorem 5, one obtains the following.

THEOREM 3.3. *If γ is a continuous linear functional on F_β, $\beta > 0$, then there exists a sequence $\{b_n\}$ of complex numbers with*

$$(3.6)\qquad b_n = O(\exp[-\eta n^{\beta/(\beta+1)}])$$

for some $\eta > 0$ such that

$$(3.7)\qquad \gamma(f) = \sum_{n=0}^{\infty} a_n b_n,$$

where $f(z) = \sum a_n z^n \in F_\beta$, with convergence being absolute. Conversely, if $\{b_n\}$ is a sequence of complex numbers satisfying (3.6), then (3.7) defines a continuous linear functional on F_β.

4. The Hardy-Orlicz space $(\mathrm{Log}^+H)^\alpha$, $\alpha > 1$. As in [3], [4], for each strongly convex function φ on $(-\infty, \infty)$ we define the Hardy–Orlicz space H_φ as the space of all $f \in H(D)$ for which

$$\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\varphi\big(\log^+|f(re^{it})|\big)\,dt < \infty.$$

Recall that a convex function φ on $(-\infty, \infty)$ is strongly convex if φ is non-negative, non-decreasing, and $\varphi(t)/t \to \infty$ as $t \to \infty$. It is well known (see [3]) that $H_\varphi \subset N^+$ for all strongly convex φ and that

$$N^+ = \bigcup \{H_\varphi \mid \varphi \text{ strongly convex}\}.$$

For $0 < p < \infty$, the space H_φ with $\varphi(t) = e^{pt}$ coincides with the usual Hardy space H^p. If for each $\alpha > 1$ we define $\varphi_\alpha(t)$ on $(-\infty, \infty)$ by $\varphi_\alpha(t) = t^\alpha$ for $t \geqslant 0$, and equal to zero for $t < 0$, we obtain the spaces $(\mathrm{Log}^+ H)^\alpha$.

Let T denote the boundary of D and for $1 \leqslant p < \infty$, we denote by L^p the space of measurable functions f on T for which $|f|^p$ is integrable, with the norm given by

$$\|f\|_p = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p dt\right]^{1/p}. \tag{4.1}$$

For a function $f \in N$, we will denote by f^* the function on T given by $f^*(e^{it}) = \lim_{r\to 1} f(re^{it})$, which exists a.e. on T.

The following results for functions in $(\mathrm{Log}^+ H)^\alpha$ will be needed.

PROPOSITION 4.1. *Suppose $f \in N^+$. Then $f \in (\mathrm{Log}^+ H)^\alpha$, $\alpha > 1$, if and only if $\log^+ |f^*| \in L^\alpha$. If this is the case, then*

$$[\log^+ |f(z)|]^\alpha \leqslant \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{it})[\log^+ |f^*(e^{it})|]^\alpha dt, \tag{4.2}$$

and

$$[\log(1+|f(z)|)]^\alpha \leqslant \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{it})[\log(1+|f^*(e^{it})|)]^\alpha dt, \tag{4.3}$$

where $P(z, e^{it}) = \dfrac{1-|z|^2}{|e^{it}-z|^2}$ is the Poisson kernel. Furthermore,

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} [\log(1+|f(re^{it})|)]^\alpha dt = \frac{1}{2\pi}\int_0^{2\pi} [\log(1+|f^*(e^{it})|)]^\alpha dt. \tag{4.4}$$

Proof. For functions $f \in N^+$, the result that $f \in (\mathrm{Log}^+ H)^\alpha$, $\alpha > 1$, if and only if $(\log^+|f^*|)^\alpha \in L^1$ and inequality (4.2) are true for any strongly convex function φ and the proofs may be found in [3], [4].

By the inequality $\log(1+|x|) \leqslant \log 2 + \log^+|x|$, it follows that

$$\sup_{0<r<1}\int_0^{2\pi} [\log(1+|f(re^{it})|)]^\alpha dt < \infty.$$

Therefore, since $\log(1+|f|)$ is subharmonic, (4.3) and (4.4) follow by Theorem 2 of [4].

For $f, g \in (\mathrm{Log}^+ H)^\alpha$, $\alpha > 1$, define

$$\varrho_\alpha(f, g) = \|\log(1+|f^*-g^*|)\|_\alpha, \tag{4.5}$$

where $\|\ \|_\alpha$ is given by 4.1 and f^*, g^* denote the boundary values of f and g respectively. By (4.4),

$$\varrho_\alpha(f, g) = \lim_{r\to 1}\left[\frac{1}{2\pi}\int_0^{2\pi}[\log(1+|f(re^{it})-g(re^{it})|)]^\alpha dt\right]^{1/\alpha}.$$

The above definition of ϱ_α has been motivated by the metric ϱ on N^+ given by $\varrho(f, g) = \frac{1}{2\pi}\int_0^{2\pi}\log(1+|f^*-g^*|)\,dt$, which was introduced by N. Yanagihara in [8] in his study of the space N^+.

By the inequality $\log(1+|x+y|)\leqslant\log(1+|x|)+\log(1+|y|)$ and Minkowski's inequality it follows that ϱ_α satisfies the triangle inequality and by (4.3) $\varrho_\alpha(f, 0) = 0$ if and only if $f(z) = 0$ for all $z\in D$. Hence ϱ_α defines a translation invariant metric on $(\mathrm{Log}^+H)^\alpha$. Furthermore, if f, $g\in(\mathrm{Log}^+H)^\alpha$, then, since $\log^+|fg|\leqslant\log^+|f|+\log^+|g|$, $fg\in(\mathrm{Log}^+H)^\alpha$, i.e., $(\mathrm{Log}^+H)^\alpha$ is an algebra. In fact, we obtain the following.

THEOREM 4.2. *The space* $(\mathrm{Log}^+H)^\alpha$, $\alpha>1$, *with the topology given by the metric* ϱ_α *is an F-algebra, that is, a topological vector space whose topology is given by a complete, translation invariant metric in which multiplication is continuous. Furthermore, if* $f\in(\mathrm{Log}^+H)^\alpha$, *then*

$$\lim_{r\to 1}\varrho_\alpha(f_r, f) = 0 \tag{4.6}$$

where $f_r(z) = f(rz)$, $0<r<1$.

Proof. Clearly $(\mathrm{Log}^+H)^\alpha$ is a vector space. If $\{f_n\}$ is a Cauchy sequence in $(\mathrm{Log}^+H)^\alpha$, then by (4.3) $f_n(z)$ converges uniformly on compact subsets of D to an analytic function $f(z)$. Furthermore, since $\{f_n\}$ is a Cauchy sequence $\{\varrho_\alpha(f_n, 0)\}$ is bounded, say by C. Therefore, for each r, $0<r<1$,

$$\frac{1}{2\pi}\int_0^{2\pi}[\log(1+|f(re^{it})|)]^\alpha dt = \lim_{n\to\infty}\int_0^{2\pi}[\log(1+|f_n(re^{it})|)]^\alpha dt\leqslant C^\alpha,$$

from which it follows that $f\in(\mathrm{Log}^+H)^\alpha$. Similarly, for each r, $0<r<1$,

$$\frac{1}{2\pi}\int_0^{2\pi}[\log(1+|f_n(re^{it})-f(re^{it})|)]^\alpha dt$$
$$\leqslant\lim_{m\to\infty}\frac{1}{2\pi}\int_0^{2\pi}[\log(1+|f_n(re^{it})-f_m(re^{it})|)]^\alpha dt\leqslant\lim_{m\to\infty}[\varrho_\alpha(f_n, f_m)]^\alpha.$$

Therefore by (4.4),

$$\varrho_\alpha(f_n, f)\leqslant\lim_{m\to\infty}\varrho_\alpha(f_n, f_m)$$

which shows that $f_n \to f$ with respect to ϱ_a. Note, in the above we have used the fact that $[\log(1+|f|)]^a$ is subharmonic and hence $\frac{1}{2\pi}\int_0^{2\pi} [\log(1+ + |f(re^{it})|)]^a dt$ is a non-decreasing function of r.

We now proceed to show that multiplication is continuous. Suppose $f_n \to f$, $g_n \to g$, $f_n, g_n, f, g \in (\mathrm{Log}^+ H)^a$. Since

$$(f_n g_n - fg) = (f_n - f)(g_n - g) + (fg_n - fg) + (gf_n - gf),$$

and since $\log(1+|xy|) \leqslant \log(1+|x|) + \log(1+|y|)$, by the triangle inequality,

$$\varrho_a(f_n g_n, fg) \leqslant \varrho_a(f_n, f) + \varrho_a(g_n, g) + \varrho_a(fg_n, fg) + \varrho_a(gf_n, gf).$$

Therefore, it suffices to show that if $f_n \to f$, then $gf_n \to gf$ for all $g \in (\mathrm{Log}^+ H)^a$. Since

$$\|\log(1+|f_n^*|) - \log(1+|f^*|)\|_a \leqslant \|\log(1+|f_n^* - f^*|)\|_a,$$

$f_n^* \to f^*$ in measure and consequently $\log(1+|g^* f_n^* - g^* f^*|)$ converges to zero in measure. Furthermore, since

$$[\log(1+|g^* f_n^* - g^* f^*|)]^a \leqslant 2^a\{[\log(1+|g^*|)]^a + [\log(1+|f_n^* - f^*)]^a\},$$

by a standard argument (e.g., proof of Theorem 1 in [5]), $\lim_{n\to\infty} \|\log(1+ + |g^* f_n^* - g^* f^*|)\|_a = 0$. Hence $(\mathrm{Log}^+ H)^a$ is a topological algebra.

Suppose $f \in (\mathrm{Log}^+ H)^a$. By Theorem 4 of [4],

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} [\log^+ |f(re^{it}) - f^*(e^{it})|]^a dt = 0.$$

Also, since $f(re^{it}) \to f^*(e^{it})$ a.e. and

$$[\log(1+|f(re^{it}) - f^*(e^{it})|)]^a \leqslant 2^a\{[\log 2]^a + [\log^+ |f(re^{it}) - f^*(e^{it})|]^a\},$$

a straight forward argument using Egorov's theorem shows that $\lim_{r\to 1} \varrho_a(f_r, f) = 0$, which proves the result.

We now make the connection between the spaces $(\mathrm{Log}^+ H)^a$, $a > 1$, and F_β.

THEOREM 4.3.

(a) $(\mathrm{Log}^+ H)^a$, $a > 1$, *is a dense subspace of* $F_{1/a}$.

(b) *The topology in* $F_{1/a}$, *defined by the family of seminorms* (3.1) *or* (3.2) *is weaker than the topology in* $(\mathrm{Log}^+ H)^a$ *given by the metric* (4.5).

(c) *Given* $a > 1$, *for each* $\beta > a$ *there exists a function* $f_\beta \in (\mathrm{Log}^+ H)^a$ *such that*

$$\overline{\lim_{r\to 1}}\,(1-r)^{1/\beta} \log^+ M_\infty(r, f_\beta) > 0,$$

i.e., $(\mathrm{Log}^+ H)^a$ *is not contained in* $F_{1/\beta}$ *for any* $\beta > a$.

Proof. (a) Suppose $f \in (\mathrm{Log}^+ H)^a$, $a > 1$. By (4.2) $[\log^+ |f(z)|]^a \leqslant F(z)$, where $F(z)$ is a non-negative harmonic function and is given by the Poisson integral of the integrable function $[\log^+ |f^*|]^a$. Using the fact that the Poisson kernel is an approximate identity, a straightforward argument (e.g., proof of Theorem 1 in [9], or the Lemma in [6]) shows that $\lim\limits_{r\to 1}(1-r) \times \times M_\infty(r, F) = 0$. Therefore,

$$\lim_{r\to 1}(1-r)^{1/a}\log^+ M_\infty(r,f) = 0,$$

and hence $(\mathrm{Log}^+ H)^a \subset F_{1/a}$. If $f \in F_{1/a}$, then for each r, $0 < r < 1$, f_r, given by $f_r(z) = f(rz)$, is in $(\mathrm{Log}^+ H)^a$ and converges to f in the topology of $F_{1/a}$, i.e. $(\mathrm{Log}^+ H)^a$ is dense in $F_{1/a}$.

(b) Suppose $\{f_n\} \subset (\mathrm{Log}^+ H)^a$ and $f_n \to 0$ in the topology of $(\mathrm{Log}^+ H)_a$ given by the metric ϱ_a. Then by (4.3), $f_n \to 0$ uniformly on compact subsets of U, and by (4.2),

$$M_\infty(r, f_n) \leqslant \exp\left[\left(\frac{2}{1-r}\right)^{1/a} \varrho_a(f_n, 0)\right].$$

Let $\varepsilon > 0$ be given and let $c > 0$ be arbitrary. Choose ϱ, $0 < \varrho < 1$, such that

$$\int_\varrho^1 \exp\left[-\frac{c}{2}(1-r)^{-1/a}\right] dr < \frac{\varepsilon}{2}.$$

Also, choose an integer N such that for all $n \geqslant N$, $2^{1/a}\varrho_a(f_n, 0) < c/2$ and

$$\int_0^\varrho \exp[-c(1-r)^{-1/a}] M_\infty(r, f_n)\, dr < \frac{\varepsilon}{2}.$$

Then for $n \geqslant N$,

$$|||f_n|||_{1/a,c} \leqslant \int_0^\varrho \exp[-c(1-r)^{-1/a}] M_\infty(r, f_n)\, dr + \int_\varrho^1 \exp\left[-\frac{c}{2}(1-r)^{-1/a}\right] dr$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, since $\varepsilon > 0$ was arbitrary, $\lim\limits_{n\to\infty} |||f_n|||_{1/a,c} = 0$. Since this is true for all $c > 0$, $f_n \to f$ in the topology of $F_{1/a}$ given by the (semi) norms (3.1). Hence the topology of $F_{1/a}$ restricted to $(\mathrm{Log}^+ H)^a$ is weaker than the topology of $(\mathrm{Log}^+ H)^a$ given by the metric (4.5)

(c) Fix an $a > 1$. For each $\beta > a$, define

$$f_\beta(z) = \exp\left[\left(\frac{1+z}{1-z}\right)^{1/\beta}\right].$$

Since $[\log^+|f_\beta(z)|]^\alpha \leqslant \left|\frac{1+z}{1-z}\right|^{\alpha/\beta}$ and $\frac{1+z}{1-z} \in H^p$ for all $p<1$, $f_\beta \in (\mathrm{Log}^+ H)^\alpha$. Write

$$\frac{1+z}{1-z} = \left|\frac{1+z}{1-z}\right| e^{i\varphi(z)}, \quad |\varphi(z)| < \frac{\pi}{2}.$$

Then

$$u(z) = \mathrm{Re}\left(\frac{1+z}{1-z}\right)^{1/\beta} = \left|\frac{1+z}{1-z}\right|^{1/\beta} \cos\frac{1}{\beta}\varphi(z)$$

and hence $u(z) > 0$. Furthermore, for all z, $|z| \leqslant r < 1$,

$$u(z) \leqslant \left(\frac{1+r}{1-r}\right)^{1/\beta}$$

with equality at $z = r$. Therefore,

$$\log^+ M_\infty(r, f_\beta) = \max_{|z|\leqslant r} u(z) = \left(\frac{1+r}{1-r}\right)^{1/\beta}$$

and $\lim\limits_{r\to 1}(1-r)^{1/\beta}\log^+ M_\infty(r, f_\beta) = 2^{1/\beta}$, which shows that $(\mathrm{Log}^+ H)^\alpha$ is not contained in $F_{1/\beta}$ for any $\beta > \alpha$.

Corollary 4.4.

(a) *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in (\mathrm{Log}^+ H)^\alpha$, *then* $a_n = O(\exp[o(n^{1/(\alpha+1)})])$.

(b) *If* $\{b_n\}$ *is a sequence of complex numbers with* $b_n = O(\exp[-\eta\cdot n^{1/(\alpha+1)}])$ *for some* $\eta > 0$, *then*

$$\gamma(f) = \sum_{n=0}^{\infty} a_n b_n \tag{4.7}$$

with $f(z) = \sum a_n z^n$ *defines a continuous linear functional on* $(\mathrm{Log}^+ H)^\alpha$, *the series converging absolutely.*

Remark 4. In [7], [8], N. Yanagihara has shown that if γ is a continuous linear functional on N^+, then there exists a sequence $\{b_n\}$ of complex numbers with $b_n = O(\exp[-\eta\sqrt{n}])$ for some $\eta > 0$ such that

$$\gamma(f) = \sum_{n=0}^{\infty} a_n b_n,$$

where $f(z) = \sum a_n z^n \in N^+$. Using classical methods (e.g. [1], p. 115) one can show that if γ is a continuous linear functional on $(\mathrm{Log}^+ H)^\alpha$, then

$$\gamma(f) = \lim_{r\to 1}\sum_{n=0}^{\infty} a_n b_n r^n, \tag{4.8}$$

where $f(z) = \sum a_n z^n \in (\mathrm{Log}^+ H)^a$ and $b_n = \gamma(z^n)$. Furthermore, it is also possible to show that the function $h(z)$ given by $h(z) = \sum_{n=0}^{\infty} b_n z^n$ is analytic in D and continuous on $\bar{D}$. However, we have been unable at this point to show that $b_n = O(\exp[-\eta n^{1/(1+a)}])$ for some $\eta > 0$, i.e., that γ is continuous on $F_{1/a}$.

We conclude this section by giving some other properties of the spaces $(\mathrm{Log}^+ H)^a$. For any a, β, $1 < a < \beta < \infty$ and for all $p > 0$, the following holds:

$$H^p \subset (\mathrm{Log}^+ H)^\beta \subset (\mathrm{Log}^+ H)^a \subset N^+$$

and the containment is proper. The fact that $H^p \subset (\mathrm{Log}^+ H)^\beta$ for all $p > 0$ and all $\beta > 0$ is a consequence of the following inequality:

$$\log^+ x \leqslant \frac{1}{re} x^r, \quad x \geqslant 1,\ r > 0.$$

The following theorem characterizes the invertible elements in $(\mathrm{Log}^+ H)^a$.

THEOREM 4.4. *A function $f \in (\mathrm{Log}^+ H)^a$, $a > 1$, is invertible if and only if $f(z) = \exp g(z)$, where $g(z) \in H^a$.*

Proof. Suppose $f(z) = \exp g(z)$, $g \in H^a$. Then $\left|\log|f(z)|\right| \leqslant |g(z)|$ and consequently both f and $\frac{1}{f} \in (\mathrm{Log}^+ H)^a$.

Conversely, suppose $f \in (\mathrm{Log}^+ H)^a$ is invertible. Since $(\mathrm{Log}^+ H)^a \subset N^+$, f is invertible in N^+ and hence is an outer function in N^+, i.e., $f(z) = \exp g(z)$, where

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f^*(e^{it})|\, dt, \quad [1], \text{ p. } 25.$$

Since both f and $\frac{1}{f} \in (\mathrm{Log}^+ H)^a$, by Proposition 4.1, $(\log^+|f^*|)^a$ and $(\log^+|1/f^*|)^a$ are integrable. But $\log^+|1/f^*| = \log^-|f^*| = \max\{0, -\log|f^*|\}$. Therefore, $\left|\log|f^*|\right| \in L^a$. Let

$$u(z) = \mathrm{Re}\, g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} \log|f^*(e^{it})|\, dt.$$

Since $\left|\log|f^*|\right|^a$ is integrable and $a > 1$, by the M. Riesz theorem [1], p. 54, $g(z) \in H^a$, which proves the result.

5. The Bergman algebra $\mathcal{N}^+(D)$. We denote by $\mathcal{N}^+(D)$ the space of analytic functions f for which $\log^+|f(z)|$ is integrable with respect to the area measure $dA(z) = \frac{1}{\pi} dx\,dy$ over the disc D. If $f(z)$ is not identically zero, then since $\log|f|$ is subharmonic, $\log|f|$ is integrable with respect to the area measure dA if and only if $\log^+|f|$ is integrable. For $f, g \in \mathcal{N}^+(D)$, define

$$d(f,g) = \int_{|z|<1} \log\bigl(1+|f(z)-g(z)|\bigr)\,dA(z). \tag{5.1}$$

Clearly $d(f,g) < \infty$ for all $f, g \in \mathcal{N}^+(D)$ and defines a translation invariant metric on $\mathcal{N}^+(D)$.

The following proposition will be needed.

PROPOSITION 5.1. *Let u be a non-negative subharmonic function on D which is integrable with respect to the area measure dA; then for all $z \in D$,*

$$u(z) \leqslant \int_D u(\xi)\left(\frac{1-|z|^2}{|1-\bar{z}\xi|^2}\right) dA(\xi) \tag{5.2}$$

and

$$u(z) \leqslant \left(\frac{1+|z|}{1-|z|}\right)^2 \int_D u(\xi)\,dA(\xi). \tag{5.3}$$

Proof. For any subharmonic function u which is integrable with respect to area measure dA,

$$u(0) \leqslant \frac{1}{\pi}\int_0^1\int_0^{2\pi} u(re^{i\theta})\,r\,dr\,d\theta = \int_D u(\xi)\,dA(\xi). \tag{5.4}$$

Let $z \in D$ be arbitrary and define $\gamma\colon D \to D$ by $\gamma(\xi) = (z-\xi)/(1-\bar{z}\xi)$. Then $u(z) = u\circ\gamma^{-1}(0)$ and $u\circ\gamma^{-1}$ is subharmonic. Since

$$\int_D u\circ\gamma^{-1}(\xi)\,dA(\xi) = \int_D u(\xi)\,|\gamma'(\xi)|^2\,dA(\xi)$$

and

$$|\gamma'(\xi)|^2 = \left(\frac{1-|z|^2}{|1-\bar{z}\xi|^2}\right)^2 \leqslant \left(\frac{1+|z|}{1-|z|}\right)^2, \tag{5.5}$$

$u\circ\gamma^{-1}$ is integrable on D and by (5.4), and the above, (5.2) and (5.3) follow.

COROLLARY 5.2. *If $f \in \mathcal{N}^+(D)$, then*

$$\log^+|f(z)| \leqslant \int_D \log^+|f(\xi)|\left(\frac{1-|z|^2}{|1-\bar{z}\xi|^2}\right)^2 dA(\xi) \tag{5.6}$$

and

$$\log(1+|f(z)|) \leqslant \left(\frac{1+|z|}{1-|z|}\right)^2 d(f, 0). \tag{5.7}$$

Using (5.7) and standard techniques one obtains the following analogue of Theorem 4.2.

THEOREM 5.3. *$\mathcal{N}^+(D)$ with the topology given by the metric d is an F-algebra. Furthermore, if $f \in \mathcal{N}^+(D)$, then $\lim_{r\to 1} d(f_r, f) = 0$.*

THEOREM 5.4.

(a) *$\mathcal{N}^+(D)$ is a dense subspace of F_2.*

(b) *The topology in F_2, defined by the family of (semi) norms* (3.1) *or* (3.2) *is weaker than the topology in $\mathcal{N}^+(D)$ given by the metric* (5.1).

(c) *For each $\beta < 2$, there exists $f_\beta \in \mathcal{N}^+(D)$ such that*

$$\overline{\lim_{r\to 1}}\, (1-r)^\beta \log^+ M_\infty(r, f_\beta) > 0,$$

i.e., $\mathcal{N}^+(D)$ is not contained in F_β for any $\beta < 2$.

Proof. (a) Let $f \in \mathcal{N}^+(D)$ and let $\varepsilon > 0$ be given. Since $\log^+|f|$ is integrable on D, there exists $\delta > 0$ such that

$$\int_E \log^+|f|\, dA < \varepsilon$$

for all measurable subsets E of D with $A(E) < \delta$. For each r, $0 < r < 1$, let $D_r = \{z \mid |z| < r\}$. Choose R sufficiently close to 1 such that $A(D - D_R) < \delta$. Hence by (5.5) and (5.6), for all $w \in D$,

$$\begin{aligned}\log^+|f(w)| &\leqslant \int_{D_R} \log^+|f(z)| \left(\frac{1-|w|^2}{|1-w\bar{z}|^2}\right)^2 dA + \int_{D-D_R} \log^+|f(z)| \left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^2 dA \\ &\leqslant \left(\frac{1-|w|^2}{(1-R|w|)^2}\right)^2 \int_{D_R} \log^+|f|\, dA + \frac{4\varepsilon}{(1-|w|)^2}.\end{aligned}$$

Therefore, for all r, $0 < r < 1$,

$$\log^+ M_\infty(r, f) \leqslant \left(\frac{1-r^2}{(1-Rr)^2}\right)^2 d(f, 0) + \frac{4\varepsilon}{(1-r)^2}.$$

Consequently, $\overline{\lim_{r\to 1}}(1-r)^2 \log^+ M_\infty(r, f) \leqslant 4\varepsilon$, from which it follows that $\mathcal{N}^+(D) \subset F_2$. If $f \in F_2$, then $f_r \in \mathcal{N}^+(D)$ for all r, $0 < r < 1$, and $f_r \to f$ in the topology of F_2. Therefore $\mathcal{N}^+(D)$ is a dense subspace of F_2.

(b) Suppose $f_n \in \mathcal{N}^+(D)$ and $f_n \to 0$ in $\mathcal{N}^+(D)$. Then by (5.7), $f_n \to 0$ uniformly on compact subsets of D and by (5.3)

$$M_\infty(r, f_n) \leqslant \exp[4(1-r)^{-2} d(f_n, 0)].$$

Let $\varepsilon > 0$ be given and let $c > 0$ be arbitrary. As in the proof of Theorem 4.3 (b), choose ϱ, $0 < \varrho < 1$, such that

$$\int_{\varrho}^{1} \exp\left[-\frac{c}{2}(1-r)^{-2}\right] dr < \frac{\varepsilon}{2}$$

and choose an integer N such that for all $n \geqslant N$, $4d(f_n, 0) < c/2$ and

$$\int_{0}^{\varrho} \exp[-c(1-r)^{-2}]\, M_\infty(r, f_n)\, dr < \frac{\varepsilon}{2}.$$

Then for all $n \geqslant N$, $|||f_n|||_{2,c} < \varepsilon$. Therefore $f_n \to 0$ in F_2.

For the proof of (c) we need the following lemma.

Lemma 5.5. *For all $a < 2$, $f(z) = (1-z)^{-\frac{a}{2}}$ is integrable on D with respect to the area measure.*

Proof. Let $0 < \varrho < 1$ be arbitrary and let $D_\varrho = \{z| \; |z| \leqslant \varrho\}$. Then

$$\int_{D_\varrho} |f|\, dA = \frac{1}{\pi} \int_{0}^{\varrho} \int_{0}^{2\pi} \frac{1}{|1-re^{i\theta}|^{a}}\, r dr d\theta$$

$$= \frac{1}{\pi} \int_{0}^{\varrho} (1-r^2)^{-a/2} \int_{0}^{2\pi} \left[\frac{(1-r^2)}{|1-re^{i\theta}|^2}\right]^{a/2} d\theta r dr.$$

Since $\frac{1}{2}a < 1$,

$$\frac{1}{2\pi} \int_{0}^{2\pi} \left[\frac{1-r^2}{|1-re^{i\theta}|^2}\right]^{a/2} d\theta \leqslant \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1-r^2}{|1-re^{i\theta}|^2}\, d\theta = 1.$$

Therefore,

$$\int_{D_\varrho} |f|\, dA \leqslant \int_{0}^{\varrho} 2r(1-r^2)^{-a/2} dr \leqslant \frac{2}{2-a}$$

from which the result follows.

Proof of (c). Suppose $1 \leqslant \beta < 2$. Let $f_\beta(z) = \exp\left(\dfrac{1+z}{1-z}\right)^\beta$. By the lemma, $f_\beta \in \mathcal{N}^+(D)$ and as in Theorem 4.3, $\log^+ M_\infty(r, f_\beta) = \left(\dfrac{1+r}{1-r}\right)^\beta$. Therefore, $\lim\limits_{r\to 1}(1-r)^\beta \log^+ M_\infty(r, f_\beta) = 2^\beta$. For $0 < \beta < 1$, $f(z) = \exp\left(\dfrac{1+z}{1-z}\right) \in \mathcal{N}^+(D)$ and $\overline{\lim\limits_{r\to 1}}\,(1-r)^\beta \log^+ M_\infty(r, f) = +\infty$. Consequently, for each $\beta < 2$, there exists $f_\beta \in \mathcal{N}^+(D)$ such that $\overline{\lim\limits_{r\to 1}}(1-r)^\beta \log^+ M_\infty(r, f_\beta) > 0$.

COROLLARY 5.6. *If* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{N}^+(D)$, *then* $a_n = O\big(\exp[o(n^{2/3})]\big)$ *and*

$$\sum_{n=0}^{\infty} |a_n| \exp[-cn^{2/3}] < \infty$$

for all $c > 0$.

Remark 5. In analogy with the Hardy–Orlicz spaces $(\mathrm{Log}^+ H)^a$, one might also consider for each $a > 1$ the space of functions for which $(\log^+|f|)^a$ is integrable with respect to area measure. Let $\big(\mathrm{Log}^+ H(D)\big)^a$ denote the space of $f \in H(D)$ for which

$$\int_D (\log^+|f|)^a dA < \infty.$$

Then by (5.2)

$$[\log^+|f(z)|]^a \leqslant \int_D [\log^+|f(\xi)|]^a \left(\frac{1-|z|^2}{|1-\bar{z}\xi|^2}\right)^2 dA(\xi), \tag{5.8}$$

and by the same method of proof as Theorem 5.4 (a)

$$\lim_{r\to 1}(1-r)^{2/a}\log^+ M_\infty(r,f) = 0. \tag{5.9}$$

THEOREM 5.7. *For all* $\beta < 1$, $F_\beta \subset \mathcal{N}^+(D)$.

Proof. Suppose $f \in F_\beta$, $0 < \beta < 1$. Then there exists $\omega(r) \downarrow 0$ as $r \to 1$ such that

$$\log^+ M_\infty(r,f) \leqslant \frac{\omega(r)}{(1-r)^\beta}.$$

Therefore, for all $\varrho < 1$,

$$\int_{D_\varrho} \log^+|f(z)|\, dA(z) \leqslant 2\int_0^\varrho \frac{\omega(r)}{(1-r)^\beta} r dr \leqslant \frac{2\omega(0)}{1-\beta},$$

where $D_\varrho = \{z \mid |z| \leqslant \varrho\}$. Hence $\int_D \log^+|f|\, dA < \infty$ and consequently $f \in \mathcal{N}^+(D)$.

Remark 7. For $\beta > 2$, using Theorem 2.2 it is easy to construct an example of a function $f \in F_\beta$ but $f \notin F_2$, and hence F_β is not contained in $\mathcal{N}^+(D)$ for any $\beta > 2$. In view of Theorem 5.7, it would be interesting to know if for any values of β, $1 \leqslant \beta \leqslant 2$, $F_\beta \subset \mathcal{N}^+(D)$.

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UNIVERSITY OF SOUTH CAROLINA
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
COLUMBIA, SOUTH CAROLINA

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