ANNALES
POLONICI MATHEMATICI
XVI (1965)

The cosine equation method for the eigenproblem of bounded normal operators in Hilbert space

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1. Some special cases of the cosine equation method are investigated in papers [1] and [2]. This method is actually an iterative method of finding eigenvalues and eigenvectors of linear operators in Hilbert space. The general idea of the method consists in reducing the problem to finding zero elements of a non-linear functional. In papers [1] and [2] we considered the real case of a symmetric matrix or a self-adjoint linear bounded operator, respectively. This paper contains a generalization of the method mentioned above. This generalization gives an iterative method of finding eigenvalues and eigenvectors of a bounded normal operator in a real or complex Hilbert space. If the operator in question is self-adjoint, then the generalized method coincides with the method presented in [1] and [2]. The proof of the convergence is similar to that given in [2]. The method, of course, can be also applied to the eigenproblem of a normal matrix. Besides, a generalization of the method is also given by introducing a parameter.

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a real or complex Hilbert space H. We shall assume operator A to be bounded and normal, i.e.

$$A^*A = AA^*,$$

where A^* is the adjoint of A.

The problem is to find eigenvalues λ and eigenvectors x of H satisfying the following equation

$$Ax = \lambda x, \qquad x \neq 0 . \tag{1}$$

It follows from (1) that

$$\lambda(x) = \frac{(Ax, x)}{(x, x)} . \tag{2}$$

Hence, we get the following equation

$$Ax = \lambda(x)x, \quad x \neq 0$$

instead of equation (1).

The last equation can be written in the form

$$\|Ax - \lambda(x)x\|^2 = 0,$$

or, equivalently,

$$F(x) = \|x\|^2\|Ax\|^2 - |(Ax, x)|^2 = 0, \quad x \neq 0. \tag{3}$$

As in [2] we see that our eigenproblem is equivalent to that of finding the solutions of equation (3). This equation means that the modul of the cosine of the angel between the eigenvector x and its image Ax is equal to 1. For this reason equation (3) is called the *cosine equation* for operator A. Hence, the eigenproblem of the operator A is equivalent to that of finding the solutions of the cosine equation for operator A.

We shall give an iterative method of solving equation (3). Let x_0 be the given initial approximate solution of equation (3). Then the approximate solution x_n of (3) is determined by the following formula

$$x_{n+1} = x_n - \frac{F(x_n)}{2\|y_n\|^2} y_n, \quad n = 1, 2, \ldots, \tag{4}$$

where $F(x_n)$ is defined by relation (3) in which x should be replaced by x_n, and where

$$y_n = \|Ax_n\|^2 x_n + \|x_n\|^2 A^* A x_n - (Ax_n, x_n) A^* x_n - \overline{(Ax_n, x_n)} A x_n. \tag{5}$$

We shall now establish some relations needed in the sequel. First of all let us remark by a simple calculation that the vector y_n defined by (5) can be written in the following form

$$y_n = \|x_n\|^2 \left(A^* - \frac{\overline{(Ax_n, x_n)}}{\|x_n\|^2} I\right)\left(A - \frac{(Ax_n, x_n)}{\|x_n\|^2} I\right) x_n + \left\|\left(A - \frac{(Ax_n, x_n)}{\|x_n\|^2} I\right) x_n\right\|^2 x_n, \tag{6}$$

where I denotes the identity mapping.

Let us put

$$B_x = A - \frac{(Ax, x)}{\|x\|^2} I. \tag{7}$$

Then we get instead of (6), dropping the index n,

$$y = \|x\|^2 B_x^* B_x x + \|B_x x\|^2 x. \tag{8}$$

We have, by (7)

$$F(x) = \|x\|^2 \|B_x x\|^2. \tag{9}$$

Hence, it follows in virtue of (8) that

$$(y, x) = 2F(x). \tag{10}$$

It is easy to verify that

$$B_x A = A B_x. \tag{11}$$

Since operator A is normal we get also

$$B_x^* A = A B_x^*. \tag{12}$$

Hence, we obtain, by (9), (11) and (12), the following inequalities

$$|(Ay, x)| \leqslant 2\|A\| F(x) \quad \text{and} \quad |(Ax, y)| \leqslant 2\|A\| F(x). \tag{13}$$

We can now prove the following

THEOREM. *Let x_0 be an arbitrary element of H such that the following condition is not satisfied*

$$\|x_0\|^2 = \frac{7}{4} \sum_{i=0}^{\infty} \frac{F^2(x_i)}{\|y_i\|^2}. \tag{14}$$

Let x_n be defined by process (4). *Then the sequence of numbers*

$$\left\{ \frac{(Ax_n, x_n)}{\|x_n\|^2} \right\}$$

converges to a number λ and the following condition is satisfied

$$Ax_n - \frac{(Ax_n, x_n)}{\|x_n\|^2} x_n \to 0, \quad as \quad n \to \infty.$$

Remark. If in addition operator A is completely continuous then λ is an eigenvalue of A. The sequence of x_n converges strongly to an eigenvector x^* corresponding to λ, provided that x_0 is not orthogonal to the subspace of eigenvectors corresponding to λ.

Proof. We shall show the convergence of the sequence of numbers $\|x_n\|^2$. It results from (4) and (10) that

$$\|x_{n+1}\|^2 = \|x_n\|^2 - \frac{7}{4} \cdot \frac{F^2(x_n)}{\|y_n\|^2}. \tag{15}$$

Hence, it follows that the sequence of $\|x_n\|^2$ is decreasing and bounded and, consequently, convergent. Relation (15) implies

$$\|x_{n+1}\|^2 = \|x_0\|^2 - \frac{7}{4} \sum_{i=0}^{n} \frac{F^2(x_i)}{\|y_i\|^2}. \tag{16}$$

Hence, it follows the convergence of the series

$$\sum_{i=0}^{\infty} \frac{F^2(x_i)}{\|y_i\|^2}. \tag{17}$$

Relation (16) implies that $x_n \to 0$ as $n \to \infty$ if and only if condition (14) is satisfied.

We shall now prove the convergence of the sequence of numbers (Ax_n, x_n). We have by (4)

$$(Ax_{n+1}, x_{n+1}) = \left(Ax_n - \frac{F(x_n)}{2\|y_n\|^2} Ay_n, x_n - \frac{F(x_n)}{2\|y_n\|^2} y_n\right).$$

Hence, we get

$$(18) \qquad (Ax_{n+1}, x_{n+1}) - (Ax_n, x_n) = \frac{F^2(x_n)}{4\|y_n\|^2}\left[\frac{(Ay_n, y_n)}{\|y_n\|^2} - 2\,\frac{(Ay_n, x_n) + (Ax_n, y_n)}{F(x_n)}\right].$$

We shall show that the expression in square brackets in (18) is bounded. In fact, we have

$$(19) \qquad \frac{|(Ay_n, y_n)|}{\|y_n\|^2} \leqslant \|A\| \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Further, it follows from (13) that

$$(20) \qquad \frac{|(Ay_n, x_n)|}{F(x_n)} \leqslant 2\|A\|.$$

In the same way we get the following inequality

$$(21) \qquad \frac{|(Ax_n, y_n)|}{F(x_n)} \leqslant 2\|A\|.$$

We have, by (8),

$$\|y\|^2 = \|x\|^4\|B_x^* B_x x\|^2 + 3\|x\|^2\|B_x x\|^4.$$

Hence, it follows, by (7), (9) and (10), that if $\|y_n\|^2$ vanishes, then x_n is an eigenvector of operator A. Thus, it follows from (19)-(21) that the expression in square brackets in (18) is bounded. In virtue of (18) the convergence of the series (17) implies the same for the following series

$$\sum_{n=0}^{\infty} (Ax_{n+1}, x_{n+1}) - (Ax_n, x_n) = \lim_{k\to\infty} (Ax_k, x_k) - (Ax_0, x_0).$$

Thus, the convergence of the sequence of numbers (Ax_n, x_n) has been proved.

Since element x_0 is so chosen that condition (14) is not satisfied, the sequence of $\|x_n\|^2$ has a limit which is different from zero. Hence, it follows that the sequence

$$\left\{\frac{(Ax_n, x_n)}{\|x_n\|^2}\right\}$$

converges to a limit, say λ.

It is easy to see that the sequence of $\|y_n\|^2$ is bounded. Hence, it follows from (15) that

$$F(x_n) \to 0 \quad \text{as} \quad n \to \infty,$$

or, equivalently,

$$Ax_n - \frac{(Ax_n, x_n)}{\|x_n\|^2} x_n \to 0 \quad \text{as} \quad n \to \infty.$$

If $x_{n+1} = 0$, then multiplying (4) by y_n we get $F(x_n) = 0$, in virtue of (10), i.e. x_n is an eigenvector.

Suppose now in addition that the operator A is completely continuous. Then there exists a subsequence of $\{x_n\}$ strongly convergent to an eigenvector x^* corresponding to the eigenvalue λ. It is easy to see that if λ is a simple eigenvalue of A, then the sequence of x_n converges strongly to the eigenvector x^*. One can prove that also in general case, if the eigenvalue λ is not simple, then the sequence of x_n converges also to an eigenvector corresponding to λ, provided that x_0 is not orthogonal to the subspace of eigenvectors corresponding to λ.

2. We shall now introduce a parameter a in the process defined by (4). Thus, we get the following formula instead of (4)

$$x_{n+1} = x_n - a\frac{\|x_n\|^2\|Ax_n\|^2 - |(Ax_n, x_n)|^2}{2\|y_n\|^2} y_n, \tag{22}$$

where y_n is determined by relation (5). Then we obtain the following relation instead of (15)

$$\|x_{n+1}\|^2 = \|x_n\|^2 - 2a\left(1 - \tfrac{1}{8}a\right)\frac{F^2(x_n)}{\|y_n\|^2},$$

where $F(x_n)$ is defined by (3). Hence, it follows that for $0 < a < 8$ the sequence of $\|x_n\|^2$ is convergent and so is the series (17).

The same argument as in Section 1 is also applicable here provided that the number $\frac{1}{4}$ in relation (14) will be replaced by $2a(1-\frac{1}{8}a)$. The theorem of Section 1 can be proved without change for the process defined by (22).

Let us remark that $\|x_{n+1}\|^2$ is minimized for $a = 4$.

References

[1] M. Altman, *An iterative method for the eigenvalues and eigenvectors of matrices*, Bull. Acad. Polon. Sci. Sér. Sci. Math., Astr. et Phys. 9 (1961), pp. 639-644.

[2] — *An iterative method for the eigenproblem of linear operators*, ibid., pp. 751-755.

Reçu par la Rédaction le 10. 1. 1964