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On some linear eigenvalue problems for strongly elliptic systems with an indefinite weight matrix function

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Abstract. In this paper a linear eigenvalue problem for a system of differential equations with an indefinite matrix function is considered. The main result of the paper is contained in Theorem 3. In Section 3 some properties of eigenfunctions which correspond to principal eigenvalue of consideration system are given.

Introduction. In this paper we investigate the linear eigenvalue problem for a system of differential equations of the form

$$\mathscr{L}U = \lambda PU \quad \text{in } \Omega, \tag{1}$$

where

$$\mathscr{L}U := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[A_{ij}\frac{\partial U}{\partial x_j}\right] + QU \tag{2}$$

is a strongly uniformly elliptic differential expression of second order having real-valued symmetric $n \times n$ matrices $A_{ij} = A_{ji}$ $(i, j = 1, \ldots, N)$ of class C^1 and P, Q continuous in $\bar{\Omega}$. Here Ω is a bounded domain in $\boldsymbol{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$.

We shall also consider a boundary condition

$$\frac{dU}{d\nu} - KU = 0 \quad \text{or} \quad U = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where K is a symmetric matrix which is continuous and positive definite on $\partial\Omega$; $dU/d\nu$ is the transversal derivative of U with respect to system (1), i.e.,

$$\frac{dU}{d\nu} := \sum_{i,j=1}^{N} A_{ij}\frac{\partial U}{\partial x_i}\cos(n, x_j), \tag{4}$$

n being the internal normal to $\partial\Omega$.

In the case where P in equation (1) is the positive definite matrix in the closure $\bar{\Omega}$ of the domain Ω, the problem (1), (3) was investigate in papers [1], [2] and [5].

We are interested here in the situation where the matrix P in equation (1) is indefinite in Ω. The purpose of this paper is to transfer some results of paper [4] to the case of system (1). The main result of paper [4] which we are interested is following.

Let us consider the problem

$$(5) \qquad -\sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{N} a_k \frac{\partial u}{\partial x_k} + a_0 u = \lambda m u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where the differential expression on the left-hand side of (5) is a strongly uniformly elliptic of second order with real-valued coefficient functions $a_{ij} = a_{ji}$, a_k, $a_0 \geqslant 0$ belonging to $C^{\theta}(\bar{\Omega})$ $(0 < \theta < 1)$, $m \in C(\bar{\Omega})$ a given (real-valued) function, $\lambda \in R$ the eigenvalue parameter.

If the weight function m may change sign in Ω and if $m(x_0) > 0$ for some $x_0 \in \Omega$, then (5) admits a principal eigenvalue $\lambda_1(m) > 0$ characterized by being unique positive eigenvalue having a positive eigenfunction.

Let us observe that the above result, proved by Hess and Kato in paper [4], can be expressed in the following form.

If the weight function m may change sign in Ω and if $m(x_0) > 0$ for some $x_0 \in \Omega$, then (5) admits a principal eigenvalue $\lambda_1(m) > 0$ characterized by that each eigenfunction corresponding to $\lambda_1(m)$ has the same properties as the first eigenfunction of the problem (5) in the case where the weight function m is positive in the whole Ω.

The result of Hess–Kato from paper [4], in the last version, may be transferred on the system of differential equations. In this paper we shall make it for the problem (1), (3). The method used here is conceptually different than the method of paper [4]. A weak point of this method is that it requires the assumption that the expression (2) is symmetric.

1. Let $H := L_2^n(\Omega)$ be a Hilbert space of the vector functions $U = (u_1, \ldots, u_n)$, where $u_i \in L_2(\Omega)$, $i = 1, \ldots, n$. Let L be a self-adjoint operator on H, which is a Fridrichs expansion of the operator defined by the expression $\mathscr{L}$ in domain D_L. Here $D_L \subset L_2^n(\Omega)$ is the set of the vector functions which satisfy the boundary condition (3). Let V: $H \to H$ be the multiplication operator by the matrix function P. We define $U \in D_L$ to be a solution of the problem (1), (3) for the parameter $\lambda \in R$ provided U solves

$$(6) \qquad LU = \lambda VU.$$

Let us consider, apart from (6), the following eigenvalue problem with a parameter $t \in R$:

$$(L - tV)\,U = \mu U, \tag{7}$$

where L and V are the operators defined above and $\mu \in R$ is an eigenvalue parameter.

We define eigenvalues and eigenfunctions of the problem (7) in the following way (variationally): the first eigenvalue $\mu_1 = \mu_1(t)$ of the problem (7) for fixed $t \in R$ is

$$\mu_1(t) := \min\{(L\Phi, \Phi) - t(V\Phi, \Phi)\colon\ \Phi \in D_L,\ \|\Phi\| = 1\}, \tag{8}$$

and the first eigenfunction U_t is a function Φ at which minimum (8) is attained.

Since for each $t \in R$, the operator $L - tV$ is self-adjoint and bounded below in H, the problem (7) has the smallest eigenvalue $\mu_1(t)$ and associated eigenfunction $U_t \in D_L$. In paper [1] we proved that the function U_t satisfies equation (7) with $\mu = \mu_1(t)$.

In the sequel we shall need the following result.

THEOREM 1. *For every $t \in R$ the problem* (7) *has the first eigenvalue $\mu_1 = \mu_1(t)$ such that the function $t \to \mu_1(t)$ is continuous for $t \in R$ and is differentiable for $t \in R$ except at most countably many points, and*

$$\mu_1'(t) = -(VU_t, U_t). \tag{9}$$

Proof (see [3]). Let $t \in R$ be a fixed number and let $h \in R$, $h \neq 0$. For every vector function $\Phi \in D_L$ we have the equality

$$(L\Phi, \Phi) - (t+h)(V\Phi, \Phi) = (L\Phi, \Phi) - t(V\Phi, \Phi) - h(V\Phi, \Phi). \tag{10}$$

Putting in (10) $\Phi = U_{t+h}$, by (8) we get

$$\mu_1(t+h) = (LU_{t+h}, U_{t+h}) - t(VU_{t+h}, U_{t+h}) - h(VU_{t+h}, U_{t+h}).$$

From this, again by (8), we have

$$\mu_1(t+h) \geqslant \mu_1(t) - h(VU_{t+h}, U_{t+h})$$

or

$$\mu_1(t) - \mu_1(t+h) \leqslant h(VU_{t+h}, U_{t+h}). \tag{11}$$

Analogously from (10) we get

$$\mu_1(t) - \mu_1(t+h) \geqslant h(VU_t, U_t). \tag{12}$$

Since the operator V is bounded on H, from (11) and (12) we have

$$|\mu_1(t+h) - \mu_1(t)| \leqslant \|V\|\,|h|. \tag{13}$$

From (13) follows the continuity of the function $t \to \mu_1(t)$ for $t \in R$.

Let us observe that owing to inequalities (11) and (12) we get

$$-h(VU_{t+h}, U_{t+h}) \leqslant \mu_1(t+h) - \mu_1(t) \leqslant -h(VU_t, U_t). \tag{14}$$

Let h in inequality (14) be a positive number; then

$$-(VU_{t+h}, U_{t+h}) \leqslant \frac{\mu_1(t+h)-\mu_1(t)}{h} \leqslant -(VU_t, U_t). \tag{15}$$

If $h < 0$, then

$$-(VU_t, U_t) \leqslant \frac{\mu_1(t+h)-\mu_1(t)}{h} \leqslant -(VU_{t+h}, U_{t+h}). \tag{16}$$

From inequality (14) follows in particular that the function $t \to (VU_t, U_t)$ is increasing for $t \in R$. By monotonicity it is continuous except at most countably many points. Therefore, by (15) and (16) we get the assertion of Theorem 1.

Since the function $t \to (VU_t, U_t)$ is increasing for $t \in R$, from (9) we get the following corollary.

COROLLARY 1. *The function $t \to \mu_1'(t)$ is decreasing for $t \in R$.*

LEMMA 1. *If $\mu_1'(t_*) = 0$, then $\mu_1(t_*) \geqslant \mu_1(0)$.*

Proof. The assumption $\mu_1'(t_*) = 0$ follows by (9), because $(VUt_*, Ut_*) = 0$. On the other hand, using this equality, we have

$$\begin{aligned}\mu_1(t_*) &= (LUt_*, Ut_*) - t_*(VUt_*, Ut_*) = (LUt_*, Ut_*)\\ &\geqslant \min\{(L\Phi, \Phi),\ \Phi \in D_L,\ \|\Phi\| = 1\} = \mu_1(0).\end{aligned}$$

So $\mu_1(t_*) \geqslant \mu_1(0)$. This yields Lemma 1.

Using Theorem 1 and Lemma 1, we shall prove the following theorem.

THEOREM 2. *Suppose that the matrix $P(x_0)$ is positive definite for some $x_0 \in \Omega$. Then the problem (7) admits the only one $t = t_0 > 0$ such that $\mu_1(t_0) = 0$.*

Proof. Since L is a positively defined operator, by (8) we have

$$\mu_1(0) > 0. \tag{17}$$

By continuity of the matrix function P in $\bar{\Omega}$, there exist $\varrho > 0$ and $\delta > 0$ such that $B_\varrho(x_0) \subset \Omega$ and $(V\Phi_0, \Phi_0) \geqslant \delta\|\Phi_0\|^2$ for all $x \in B_\varrho(x_0)$, where $B_\varrho = B_\varrho(x_0)$ denotes the open ball in R^N with center x_0 and radius ϱ, $\Phi_0 \in C_0^{\infty,n}(\Omega)$, $\Phi_0 = 0$ in $\Omega \setminus B_\varrho$, $\Phi_0 \neq 0$. Here $C_0^{\infty,n}(\Omega)$ is the space of vector functions $\Phi = (\Phi_1, \ldots, \Phi_n)$, $\Phi_i \in C_0^\infty(\Omega)$, $i = 1, \ldots, n$. Since $(L\Phi_0, \Phi_0) > 0$ and

$(V\Phi_0, \Phi_0) \geqslant \delta\|\Phi_0\|^2$, we may choose such a large $t = t_1 > 0$ that

$$(L\Phi_0, \Phi_0) - t_1(V\Phi_0, \Phi_0) < 0.$$

From this by (8) we get

$$\mu_1(t_1) < 0. \tag{18}$$

By continuity of the function $t \to \mu_1(t)$ from (17) and (18) follows the existence of a $t_0 \in (0, t_1)$ such that $\mu_1(t_0) = 0$. Let t_0 be such that $\mu_1(t) > 0$ for $t \in (0, t_0)$. Since the function $t \to \mu_1'(t)$ is decreasing, we have $\mu_1'(t_0 - 0) \leqslant 0$. From this by Lemma 1 we get $\mu_1'(t_0+0) < 0$. This inequality implies that the function $t \to \mu_1(t)$ is decreasing in interval $(t_0, +\infty)$ and so $\mu_1(t) < 0$ for $t > t_0$. Therefore the point $t_0 \in (0, t_1)$ is the only point such that $\mu_1(t_0) = 0$. Theorem 2 is proved.

COROLLARY 2. *If the matrix P in equation* (1) *is indefinite in Ω, i.e., there exist two points x_0, $x_* \in \Omega$ such that the matrix $P(x_0)$ is positive definite and $P(x_*)$ is negative definite, then there exist two numbers $t_0 > 0$ and $t_* < 0$ such that $\mu_1(t_0) = \mu_1(t_*) = 0$. Moreover, $\mu_1(t) > 0$ for $t \in (t_*, t_0)$ and $\mu_1(t) < 0$ for $t \in (-\infty, t_*) \cup (t_0, +\infty)$.*

THEOREM 3. *Under the assumptions of Theorem 2, the problem* (1), (3) *admits a positive principal eigenvalue λ_1 characterized by being the unique positive eigenvalue having an eigenfunction which is the first eigenfunction of the problem* (7). *Moreover, if $\lambda \in R$ is an eigenvalue of* (1), (3) *and $\lambda \geqslant 0$, then $\lambda \geqslant \lambda_1$.*

Proof. Let $t_0 > 0$ be a number such that $\mu_1(t_0) = 0$, where $\mu_1(t)$ is the first eigenvalue of problem (7). Let $U_1 \in D_L$ be an eigenfunction of the problem (7) associated with eigenvalue $\mu_1(t_0) = 0$. From this we have

$$LU_1 - t_0 VU_1 = 0. \tag{19}$$

Since $U_1 \in D_L$, equality (19) implies that U_1 satisfies equation (1) with $\lambda = t_0$, and the boundary condition (3). This means that t_0 is an eigenvalue of problem (1), (3) with the eigenfunction U_1. Suppose now that $\lambda = \lambda_0 \geqslant 0$ is the eigenvalue of the problem (1), (3) and let U_0 be an associated eigenfunction. Since $U_0 \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and U_0 satisfies the boundary condition (3), we have $U_0 \in D_L$. From this it follows that

$$LU_0 - \lambda_0 VU_0 = 0.$$

This equality may be written in the form

$$LU_0 - \lambda_0 VU_0 = 0 \cdot U_0. \tag{20}$$

Equality (20) states that U_0 is an eigenfunction of the problem (7) with t

$= \lambda_0$, which corresponds to the eigenvalue $\mu_0 = \mu_0(\lambda_0) = 0$. If $\lambda_0 \neq t_0$ then $\mu_0(t) \neq \mu_1(t)$ for $t \geqslant 0$. On the other hand, $\mu_1(t)$ is the smallest eigenvalue of the problem (7). Therefore we get $\mu_0(t) > \mu_1(t)$ for every $t \geqslant 0$. From this we infer that $\lambda_0 > t_0$. Writing $\lambda_1 := t_0$, we have the hypothesis of Theorem 3.

COROLLARY 3. *Under the assumptions of Corollary 2, the problem* (1), (3), *admits two principal eigenvalues* $\lambda_1 > 0$ *and* $\lambda_{-1} < 0$ *and this problem has no eigenvalue* $\lambda \in R$ *with* $\lambda_{-1} < \lambda < \lambda_1$.

THEOREM 4. *If the matrix* P *is non-positive* (*non-negative*) *in* Ω, *then the problem* (1), (3), *has no positive* (*negative*) *principal eigenvalue.*

Proof. We consider the case where the matrix P is non-positive in Ω. The case when the matrix P is non-negative is analogous. Suppose that there exists $\lambda_0 \geqslant 0$ which is principal eigenvalue of the problem (1), (3). As we know, the number $\lambda_0 \in R$ is the principal eigenvalue of the problem (1), (3) if and only if $\mu_1(\lambda_0) = 0$, where $\mu_1(t)$ is the first eigenvalue of the problem (7). On the other hand, by (9) and by assumption we have $\mu_1'(t) \geqslant 0$ for every $t \in R$. From this follows that $t \to \mu_1(t)$ is the increasing function on R. Since $\mu_1(0) > 0$, we get $\mu_1(t) > 0$ for $t \geqslant 0$. We arrive at a contradiction.

2. In this section we consider the following equation (cf. [4])

$$LU = \lambda(V-s)U, \tag{21}$$

where L and V are the operators from equation (6) and $s \in I := (-\infty, \bar{s})$, $\bar{s} := \sup_{x \in \Omega} \{\inf(V\xi, \xi) \colon \xi \in \mathbf{R}^n;\ |\xi| = 1\}$. By the definition of the number $\bar{s}$, we have that for every $s \in I$ the matrix $(P - sE)$ is positive definite in some $x_0 \in \Omega$ and the matrix $(P - \bar{s}E)$ is non-positive in Ω; here E is the unit matrix. The purpose of this section is to determine the dependence of the positive principal eigenvalue of the problem (21) on the parameter $s \in I$. In this purpose let us consider apart from (21) the following eigenvalue problem with two parameters

$$LU - t(V-s)U = \mu U, \tag{22}$$

where $t \in R$, $s \in I$, and $\mu \in R$ is the eigenvalue parameter.

THEOREM 5. *The function* $s \to \lambda_1(s)$ *is continuous for* $s \in I$, *where* $\lambda_1(s)$ *is the principal positive eigenvalue of the problem* (21).

Proof. First we prove that the function $(t, s) \to \mu_1(t, s)$ is continuous for $t \in R$, $s \in I$, where $\mu_1(t, s)$ is the first eigenvalue of the problem (22). Reasoning analogously as in the proof of Theorem 1, we get the inequality

$$\begin{aligned} h\big((V-s)U, U\big) - (t+h)k &\leqslant \mu_1(t, s) - \mu_1(t+h, s+k) \\ &\leqslant h\big((V-s)U_0, U_0\big) - (t+h)k, \end{aligned} \tag{23}$$

where U_0 and U are the first eigenfunctions of the problem (22) associated with the eigenvalues $\mu_1(t, s)$ and $\mu_1(t+h, s+k)$, respectively, such that $\|U_0\| = \|U\| = 1$, s, $(s+k) \in I$. From (23) follows the continuity of the function $(t, s) \to \mu_1(t, s)$.

As we know the principal eigenvalue of the problem (21), $\lambda_1(s)$ for $s \in I$, is defined by equality $\mu_1(\lambda_1(s), s) = 0$. Let $(t_0, s_0) \in R \times I$ be such that $\mu_1(t_0, s_0) = 0$. Since $\partial\mu_1/\partial t(t_0, s_0) \neq 0$, due to Lemma 1, from the theorem on implicit function we get the continuity of the function $s \to \lambda_1(s)$ in some surrounding of the point $s_0 \in I$. Since s_0 is an arbitrary point of the interval I, we obtain the hypothesis of Theorem 5.

THEOREM 6. *The function $s \to \lambda_1(s)$ is increasing in interval I and $\lambda_1(s) \to +\infty$ as $s \nearrow \bar{s}$.*

Proof. Let us remark that for every fixed $t > 0$ the function $s \to \mu_1(t, s)$ is increasing in interval I. Since the positive principal eigenvalue $\lambda_1(s)$ is defined by $\mu_1(\lambda_1(s), s) = 0$, it follows that the function $s \to \lambda_1(s)$ is increasing in I. By monotonicity the limit $\lambda_1(s)$ exists, as $s \nearrow \bar{s}$. Suppose $0 < \bar{\lambda} := \lim_{s \nearrow \bar{s}} \lambda_1(s) < +\infty$.

From the assumptions on the operator L follows that L^{-1} exists and L^{-1} is compact operator on H. By compactness of the operator L^{-1} in H (see [4], proof of Lemma 4) follows the existence of a function U, $\|U\| = 1$ such that

$$U = \bar{\lambda} L^{-1}(V - \bar{s})\, U.$$

This equality may be written in the form

$$LU - \bar{\lambda}(V - \bar{s})\, U = 0. \tag{24}$$

However, $(P - \bar{s}E)$ is non-positive matrix in Ω, and (24) contradics Theorem 4.

3. In this section we shall give some properties of the eigenfunctions which correspond to principal eigenvalue of the problem (1), (3). To this purpose we make some assumptions concerning the coefficients of the problem (1), (3). Suppose that $A_{ij} := a_{ij}E$, $Q := qE$, $P := mM$, $K := kE$, where $a_{ij} = a_{ji}$ $(i, j = 1, \ldots, N)$ are real-valued functions of class $C^1(\bar{\Omega})$, $q, m, k \in C(\bar{\Omega})$, $k > 0$, and M is a symmetric positive definite $n \times n$ matrix with a real constant elements (see [2]).

Under these assumptions the problem (1), (3) takes the form

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_p}{\partial x_j}\right) + qu = \lambda m \sum_{l=1}^{n} \varrho_{pl} u_l \quad \text{in } \Omega \tag{25}$$

with boundary condition

$$\frac{du_p}{d\nu} - ku_p = 0 \quad \text{or} \quad u_p = 0 \quad \text{on } \partial\Omega,\ p = 1, \ldots, n, \tag{26}$$

where

$$\frac{du_p}{d\nu} = \sum_{i,j=1}^{N} a_{ij} \frac{\partial u_p}{\partial x_j} \cos(n, x_i), \qquad M = \{\varrho_{pl}\}.$$

Since by definition the matrix M is symmetric and positive definite, there exists an orthogonal transformation

$$W = ZU \tag{27}$$

such that $ZMZ^{-1} := \bar{M} = \{\delta_{pl}\varrho_p\}$, where $\varrho_p > 0$ $(p = 1, \ldots, n)$. We see easily that the transformation (27) reduces the system (25) to a system of n independent equations and the boundary condition (26) to n independent conditions; it can be written in the form

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w_p}{\partial x_j}\right) + qw_p = \lambda m \varrho_p w_p \quad \text{in } \Omega, \tag{28}$$

$$\frac{dw_p}{d\nu} - kw_p = 0 \quad \text{or} \quad w_p = 0 \quad \text{on } \partial\Omega\ (p = 1, \ldots, n). \tag{29}$$

The problem (28), (29) can be treated as n independent problems which coincide with the problem considered in papers [4] and [3]. Using the results of these papers, we have the following result for the problem (28), (29).

THEOREM 7. *Suppose that the weight function m may change sign in Ω and that $m(x_0) > 0$ for some $x_0 \in \Omega$. Then the problem* (25), (26) *admits n eigenvalues λ_{1p} $(p = 1, \ldots, n)$ such that the corresponding eigenfunctions U_{1p} $(p = 1, \ldots, n)$ do not vanish in the domain Ω.*

Proof. Let us denote by λ_{1p} $(p = 1, \ldots, n)$ the principal positive eigenvalue of the problem (28), (29) and the corresponding positive in Ω eigenfunction by w_{1p} $(p = 1, \ldots, n)$. The existence of these eigenvalues and eigenfunctions follows from papers [4] and [3]. On the other hand, to the eigenvalue λ_{1p} there corresponds a vector eigenfunction $W_{1p} := (0, \ldots, w_{1p}, \ldots, 0)$ with all components except the pth equal to zero. Under transformation (27), to the function W_{1p} there corresponds the function $U_{1p} = Z^{-1} W_{1p}$. We easily see that the function U_{1p} may be written in the form $U_{1p} = A_p w_{1p}$, where A_p denotes the vector whose components are elements of the pth row of matrix Z^{-1}. Theorem 7 is proved.

Remark. 1. In the case of the problem (25), (26) the positive principal eigenvalue of this problem is in the sense of Theorem 3 defined by

$$\lambda_1 := \min(\lambda_{11}, \ldots, \lambda_{1n})$$

where λ_{1p} $(p = 1, \ldots, n)$ are defined in Theorem 7.

Remark 2. As we know (cf. [4]), the positive principal eigenvalue of one equation of second order, if it exists of course, is the eigenvalue with geometric and algebraic multiplicity one. Whereas in the case of a system of equations the multiplicity of positive principal eigenvalue in general is greater than one. Indeed, it sufices that in system (27) is $\varrho_1 = \ldots = \varrho_n > 0$, then the multiplicity of λ_1 is at least n.

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