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On the continuous solutions of a non-linear functional equation of the first order

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Abstract. Some theorems about the existence, uniqueness and properties of the continuous solution of the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]),$$

where φ is the unknown function, are given, under the hypothesis of the existence of semicontinuous solutions of the functional inequalities

$$\varphi(x) \leqslant h(x, \varphi[f(x)])$$

and

$$h(x, \varphi[f(x)]) \leqslant \varphi(x).$$

In the present paper we are concerned with the existence, uniqueness and some properties of the continuous solutions of the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{1}$$

where φ is the unknown function.

We assume that

(i) The real function f is defined and continuous in an interval I and, for a $\xi \in I$, it fulfils the inequalities

$$0 < \frac{f(x)-\xi}{x-\xi} < 1 \quad \text{for} \quad x \in I, x \neq \xi;$$

(ii) h is a real function defined and continuous in a set $\Omega \subset R^2$ containing the point (ξ, η), where η is a solution of the equation

$$\eta = h(\xi, \eta). \tag{2}$$

Moreover, in a neighbourhood of (ξ, η) h fulfils a Lipschitz condition with respect to the second variable

$$|h(x, y_1)-h(x, y_2)| \leqslant \gamma(x)\,|y_1-y_2|, \quad (x, y_i) \in U \cap \Omega;\ i = 1, 2,$$

where

$$U = U_1 \times U_2, \quad U_1 = \{x \in R\colon |x-\xi| \leqslant c\}, \quad U_2 = \{y \in R\colon |y-\eta| \leqslant d\}$$

and c, d are positive numbers.

For a subinterval J of I we denote by $\Phi(J)$ the class of all real functions φ defined on J and fulfilling the condition $\varphi[f(x)]\in\Omega_x$ for $x\in J$, where Ω_x denotes the x-section of Ω:

$$\Omega_x = \{y\in R:\ (x,y)\in\Omega\},$$

(cf. [2], p. 68).

Our next assumptions read as follows:

(iii) The functional inequality

$$\varphi(x)\leqslant h(x,\varphi[f(x)]), \tag{3}$$

has in the class $\Phi(I)$ a lower semicontinuous solution φ_1^*, whereas the inequality

$$h(x,\varphi[f(x)])\leqslant\varphi(x), \tag{4}$$

has in this class an upper semicontinuous solution φ_2^*. Furthermore $\varphi_1^*(x)\leqslant\varphi_2^*(x)$ for $x\in I$;

(iv) For every fixed $x\in I$ the set Ω_x is an interval, $h(f(x),\Omega_{f(x)})\subset\Omega_x$ and h is an increasing function with respect to the second variable in the interval $I(x) = \langle\varphi_1^*[f(x)],\varphi_2^*[f(x)]\rangle$.

Put

$$G_n(x) = \prod_{i=0}^{n-1}\gamma[f^i(x)],\qquad n=1,2,\ldots \tag{5}$$

We have the following

THEOREM 1. *Suppose that hypotheses* (i)–(iv) *are fulfilled. If equation* (2) *has in the interval* $I(\xi)$ *exactly one solution* η *and the sequence* $\{G_n\}$ *defined by* (5) *is bounded in a vicinity of the point* ξ:

$$G_n(x)\leqslant M(x),\qquad x\in I\cap\langle\xi-\delta,\xi+\delta\rangle,\ x\neq\xi;\ n=1,2,\ldots, \tag{6}$$

then equation (1) *has exactly one solution* $\varphi\in\Phi(I)$ *continuous at the point* ξ *and such that* $\varphi(\xi)=\eta$. *This solution is a continuous function, fulfils the condition*

$$\varphi_1^*(x)\leqslant\varphi(x)\leqslant\varphi_2^*(x),\qquad x\in I, \tag{7}$$

and is given by the formula

$$\varphi(x) = \lim_{n\to\infty}\varphi_n(x), \tag{8}$$

where

$$\varphi_{n+1}(x) = h(x,\varphi_n[f(x)]),\qquad n=0,1,2,\ldots$$

and φ_0 *is an arbitrary function belonging to* $\Phi(I)$, *continuous at the point* ξ, *and such that* $\varphi_0(\xi)=\eta$.

Proof. Let Φ_1 be the class of all lower semicontinuous functions φ belonging to $\Phi(I_0)$ and fulfilling the condition

$$\varphi_1^*(x) \leqslant \varphi(x) \leqslant \varphi_2^*(x); \quad x \in I_0, \tag{9}$$

where $I_0 = U_1 \cap \langle \xi - \delta, \xi + \delta \rangle \cap I$; and similarly, let Φ_2 be the class of all upper semicontinuous functions φ belonging to $\Phi(I_0)$ and fulfilling condition (9). Put

$$\varphi_{i,0}(x) = \varphi_i^*(x), \quad \varphi_{i,n+1}(x) = h(x, \varphi_{i,n}[f(x)]), \quad x \in I_0;\ i = 1,2;\ n = 0, 1, 2, \ldots \tag{10}$$

It follows from the continuity of the functions f and h and from hypotheses (i), (iii) and (iv) that

$$\varphi_{i,n} \in \Phi_i, \quad i = 1, 2;\ n = 0, 1, 2, \ldots \tag{11}$$

and

$$\varphi_{1,n}(x) \leqslant \varphi_{1,n+1}(x), \quad \varphi_{2,n+1}(x) \leqslant \varphi_{2,n}(x), \quad x \in I_0;\ n = 0, 1, 2, \ldots \tag{12}$$

Hence the sequences $\{\varphi_{i,n}\}$, $i = 1, 2$, are convergent. Let

$$\varphi_i(x) = \lim_{n\to\infty} \varphi_{i,n}(x), \quad x \in I_0;\ i = 1, 2. \tag{13}$$

On account of (11), (12) and (13)

$$\varphi_i \in \Phi_i, \quad i = 1, 2. \tag{14}$$

Moreover, by (10) and (13) we have

$$\varphi_i(x) = h(x, \varphi_i[f(x)]), \quad x \in I_0;\ i = 1, 2. \tag{15}$$

Putting $x = \xi$ in (15) and making use of the facts that $f(\xi) = \xi$ and that η is the unique solution of equation (2) in $I(\xi)$, we obtain

$$\varphi_i(\xi) = \eta, \quad i = 1, 2. \tag{16}$$

We shall show that

$$\limsup_{x\to\xi} \varphi_1(x) \leqslant \eta, \quad \liminf_{x\to\xi} \varphi_2(x) \geqslant \eta. \tag{17}$$

Indeed, write $\limsup_{x\to\xi} \varphi_1(x) = a_0$. Evidently $a_0 \in I(\xi)$. There exists a sequence $\{x_n\}$, $x_n \in I_0$, such that $\lim_{n\to\infty} x_n = \xi$ and $\lim_{n\to\infty} \varphi_1(x_n) = a_0$. From the sequence $\{\varphi_1[f(x_n)]\}$ we can choose a convergent subsequence $\{\varphi_1[f(x_{n_k})]\}$. Since $\lim_{k\to\infty} f(x_{n_k}) = \xi$, we have $\lim_{k\to\infty} \varphi_1[f(x_{n_k})] = b \leqslant a_0$, $b \in I(\xi)$. Hence and from (15) we obtain $a_0 = h(\xi, b) \leqslant h(\xi, a_0)$. Let $a_{n+1} = h(\xi, a_n)$, $n = 0, 1, 2, \ldots$ The sequence $\{a_n\}$ is increasing and bounded from above ($a_n \in I(\xi)$, $n = 0, 1, 2, \ldots$), and so it is convergent. Its limit belongs to $I(\xi)$ and fulfils equation (2) and hence it must be equal to η. By the monotonicity of $\{a_n\}$ we have $a_0 \leqslant \eta$. Similarly we can prove the second inequality in (17). Thus in view of (14), (16) and (17) we have the continuity of φ_i, for $i = 1, 2$, at the point ξ. A similar argument as in proof of Theorem 1 in [1] shows that

$$\varphi_1(x) = \varphi_2(x), \quad x \in I_0,\ x \neq \xi.$$

Hence and from (16) $\varphi_1 = \varphi_2 = \bar{\varphi}$. By (14) $\bar{\varphi} \in \Phi_1 \cap \Phi_2$ turns out to be a continuous solution of equation (1) in I_0. This solution can be uniquely extended onto the whole interval I and the extension φ is continuous ([2], p. 70, Theorem 3.2). We shall show that φ fulfils (7). For the indirect proof suppose that there is an $x_0 \in I$ such that the condition

$$\varphi_1^*(x_0) \leqslant \varphi(x_0) \leqslant \varphi_2^*(x_0)$$

is not fulfilled. Since $\lim_{n\to\infty} f^n(x_0) = \xi$, there is a non-negative integer n such that the condition

$$\varphi_1^*[f^n(x_0)] \leqslant \varphi[f^n(x_0)] \leqslant \varphi_2^*[f^n(x_0)]$$

is not fulfilled, whereas the condition

$$\varphi_1^*[f^{n+1}(x_0)] \leqslant \varphi[f^{n+1}(x_0)] \leqslant \varphi_2^*[f^{n+1}(x_0)]$$

is fulfilled. Hence, and from hypotheses (iii) and (iv) we obtain

$$\begin{aligned}\varphi_1^*[f^n(x_0)] &\leqslant h\big(f^n(x_0), \varphi_1^*[f^{n+1}(x_0)]\big) \leqslant h\big(f^n(x_0), \varphi[f^{n+1}(x_0)]\big)\\ &\leqslant h\big(f^n(x_0), \varphi_2^*[f^{n+1}(x_0)]\big) \leqslant \varphi_2^*[f^n(x_0)].\end{aligned}$$

This together with (1) contradicts our assumption. Recalling the proofs of Theorems 2 and 1 in [1] and the fact that $\varphi(\xi) = \varphi_n(\xi) = \eta$ for $n = 0, 1, 2, \ldots$, we get formula (8) and the required uniqueness, respectively.

The next two theorems say something about the properties of the solutions obtained.

THEOREM 2. *Suppose that the hypotheses of Theorem 1 are fulfilled. If f is increasing and h is increasing with respect to each variable in the set $\bigcup\{\{x\}\times I(x)\colon x \in I\}$ and, moreover, φ_1^* or φ_2^* is increasing, then the solution obtained in Theorem 1 is an increasing function.*

THEOREM 3. *Suppose that the hypotheses of Theorem 1 are fulfilled. If f is increasing and convex and h is increasing with respect to each variable and convex in the set $\bigcup\{\{x\}\times I(x)\colon x \in I\}$, whereas φ_1^* is increasing and convex and φ_2^* is decreasing and concave, then the solution obtained in Theorem 1 is an increasing and convex function.*

In fact, in the case of Theorem 2, for every $n = 0, 1, 2, \ldots, \varphi_{1,n}$ defined by (10) is an increasing function or for every $n = 0, 1, 2, \ldots, \varphi_{2,n}$ defined by (10) is an increasing function. In the case of Theorem 3, for every $n = 0, 1, 2, \ldots, \varphi_{1,n}$ is an increasing and convex function. Furthermore, the extension of an increasing solution is increasing and the extension of an increasing and convex solution is increasing and convex (cf. [4]).

Now we shall show that the above theorems imply the following results.

THEOREM 4. *Let hypotheses* (i), (ii) *and condition* (6) *be fufilled and suppose that Ω is an open set* (¹). *If, for every fixed $x \in U_1$, h is an increasing function with respect to the second variable in U_2 and*

$$|h(\xi, y)-\eta| < |y-\eta|, \quad y \in U_2, \ y \neq \eta, \tag{18}$$

then in a neighbourhood of ξ there exists a unique solution φ of equation (1) *which is continuous at ξ and such that $\varphi(\xi) = \eta$. This solution is a continuous function.*

THEOREM 5. *Suppose that the hypotheses of Theorem 4 are fulfilled. If f is increasing [and convex] in I and h is increasing with respect to each variable [and convex] in $U \cap \Omega$, then the solution obtained in Theorem 4 is an increasing [and convex] function.*

Indeed, it follows from (18) and from the continuity of h that there exist positive numbers c_0, d_0 such that $I_0 \subset U_1 \cap I$, $I_0 \times \langle \eta - d_0; \eta + d_0\rangle \subset U \cap \Omega$ and $|h(x, y)-\eta| \leqslant d_0$ for $(x, y) \in I_0 \times \langle \eta - d_0, \eta + d_0\rangle$, where $I_0 = \langle \xi, \xi + c_0\rangle$ if ξ is the left endpoint of the interval I, $I_0 = \langle \xi - c_0, \xi + c_0\rangle$ if ξ is an interior point of I, and $I_0 = \langle \xi - c_0, \xi\rangle$ if ξ is the right endpoint of I. Hence, the functions φ_i^*, $i = 1, 2$, defined by

$$\varphi_1^*(x) = \eta - d_0, \quad \varphi_2^*(x) = \eta + d_0; \quad x \in I_0$$

are continuous solutions of inequalities (3) and (4), respectively, and η is the unique solution of equation (2) in the interval $\langle \eta - d_0, \eta + d_0\rangle$.

References

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(¹) Instead of this assumption we may require that Ω should contain a rectangle Q (possibly closed) such that $(\xi, \eta) \in Q$ and $(x, h(x, y)) \in Q$ for $(x, y) \in Q$ (cf. [3]).